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Research Article

On the Generalized Order-k Jacobsthal and Jacobtshal-Lucas Numbers

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Abstract: The classic Jacobsthal numbers were generalized to k sequences of the generalized order-k Jacobsthal numbers and then have been studied by several authors. In this paper, we explain that all of these studies used an incorrect version of order-k Jacobsthal numbers for reasons and give the correct definition of order-k Jacobsthal numbers. Further, we introduce the compatible generalized order-k Jacobsthal-Lucas numbers with the generalized order-k Jacobsthal numbers. Next, we give some properties of order-k Jacobsthal numbers and order-k Jacobsthal-Lucas numbers, including generating matrix, generalized Binet's formula, and elementary matrix identities. Further, we investigate specific examples for our results and give special identities, i.e., sum formula and interrelationships between these sequences.

Keywords: Generalized order-k sequence, Jacobsthal sequence, trace of matrix, Binet's formula, Jacobsthal-Lucas sequence.

1. Introduction

In modern science and daily mathematical practices, a great number of researchers have investigated many integer sequences and their generalizations for a long time, e.g., Fibonacci numbers or Lucas p-numbers. There are many papers and monographs devoted to the subject in the current literature. For example, the readers can read the references in [13, 20]. The main framework of the paper is carved out from the usual Pell and Jacobsthal numbers.

This paper deals with the well-known Jacobsthal $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal-Lucas $\{j_n\}_{n=0}^{\infty}$ sequences, which are defined recursively as

$$J_0 = 0,\ J_1 = 1 \text{ and } J_{n+1} = J_n + 2J_{n-1} \text{ for } n \geqslant 2 \tag{1}$$

and

$$j_0 = 2,\ j_1 = 1 \text{ and } j_{n+1} = j_n + 2j_{n-1} \text{ for } n \geqslant 2, \tag{2}$$

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respectively. It should be noted that, in [10], Horadam investigated extensively and attracted attention to the mentioned sequences. These can be given in the following ways, named Binet's formulas:

$$J_n = \frac{2^n - (-1)^n}{3} \tag{3}$$

and

$$j_n = 2^n + (-1)^n, \tag{4}$$

respectively. In addition, the author of [10] presented some properties for these sequences as follows:

$$j_n J_n = J_{2n}, \tag{5}$$

$$j_n = J_{n+1} + 2J_{n-1}, \tag{6}$$

$$J_{n+1}J_{n-1} - {J_n}^2 = (-1)^n 2^{n-1}, \tag{7}$$

and

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2} - 5}{2}. \tag{8}$$

In [11, 12], Koken and Bozkurt showed that the terms of the mentioned sequences can also be obtained via a generating matrix as follows:

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} \text{ and } E^n = \begin{cases} 3^n \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} & if\ n\ even \\ 3^{n-1} \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix} & if\ n\ odd \end{cases}, \tag{9}$$

where $F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$. It should be noted that the following interesting property is satisfied:

$$tr\left(F^n\right) = J_{n+1} + 2J_{n-1}, \tag{10}$$

which is the right-hand side of Equation (6). Here, $tr\left(.\right)$ denotes the trace of an n-square matrix. Using Equation (1), we get

$$tr\left(F^n\right) = J_n + 4J_{n-1}. \tag{11}$$

Today, there are many systematic investigations regarding the Jacobsthal and Jacobsthal-Lucas sequences. The references given in [4, 5] can be read in this scope.

It should be noted that the main field of studies regarding the second-order sequences is to consider obtaining their generalized versions. These processes were made in various ways. Some of them can be summarized as follows. The second-order sequence can be defined with more general initial conditions, e.g., Horadam [8]; the coefficients of sequence can be chosen from more general

Table 1: Some values of the generalized order-4 Jacobsthal numbers

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
J_n^3	1	2	4	9	20	44	97	214	472	1041	2296	5064	11169	24634	54332
J_{-n}^3	0	1	0	-2	1	4	-4	-7	12	10	-31	-8	72	-15	-152
J_n^4	1	1	3	6	14	30	67	147	325	716	1580	3484	7685	16949	37383
J_{-n}^4	0	0	1	-1	-1	2	2	6	-1	13	-3	-28	20	52	-67

terms, e.g., Horadam [9]; each term of the sequence can be defined as a linear combination of the preceding arbitrary two terms e.g. Stakhov [16]; each term is a linear combination of k preceding terms with k initial conditions, e.g., Miles [15]; or the Binet's formula of the sequence can be considered in the general form, e.g., Stakhov and Rozin [17]. There are also many papers as in the references [1–3, 7, 18, 19] devoted to the subject.

In particular, we would like to mention a paper herein. In [21], Yilmaz and Bozkurt presented a new generalization of the second-order Jacobsthal numbers, inspired by Miles [15], as follows:

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + \cdots + J_{n-k+1}^i + J_{n-k}^i \tag{12}$$

for $n > 0$ and $1 \leqslant i \leqslant k$, with initial terms

$$J_n^i = \begin{cases} 1, & \text{if } i+n=1 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1-k \leqslant n \leqslant 0, \tag{13}$$

where J_n^i is the nth term of the ith sequence. Clearly, when $k = 2$ and $i = 1$, this definition reduces to the the famous Jacobsthal sequence. The mentioned paper also displays some properties of the sequence, generalized Binet's formula and summation formula.

However, based on some reasons, we think that this definition, unfortunately, is not proper, namely not a generalization of the Jacobsthal numbers. It does not satisfy many chronic and genetic identities of the Jacobsthal sequence. For example, considering the definition in Equation (12) and initial conditions in (13), the terms of the generalized sequence with the negative subscripts are also an integer. But, as known from the results of Daşdemir [6], the terms with negative subscripts should not be integers. Moreover, one positive – one negative, consecutively, order of terms with negative subscripts is also not provided. Besides, there are also many issues similar to these situations. For a concrete example, letting $i = 3$ or $i = 4$ and $k = 4$, we can, thus, obtain the terms with negative subscripts of the generalized order-4 Jacobsthal numbers as in Table 1.

Based on the justification briefly explained above, in this paper, we give the true definition for generalization of the usual Jacobsthal sequence, i.e., generalized order-k Jacobsthal sequence, and then summarize elementary properties, including the generating matrix and generalized Binet's formula. One of the most important highlights of this study is that k-sequences of the generalized order-k Jacobsthal-Lucas sequence are defined. To do this, we will make use of the miscellaneous properties of matrices. Further, for this new definition, appropriate initial conditions that are of two different forms are also given. In particular, it is stated that the same integer sequences are obtained in both cases but the order of the sequences is permutationally changed.

2. On the Generalized Order- k Jacobsthal Numbers

In this section, we present the results regarding the generalized order- k Jacobsthal numbers. Note that since all the conclusions can be proved easily by using the known ways in the current literature, we will omit the proof processes in order not to bore the readers. For this purpose, the following definition is the starting point of the paper.

Definition 2.1 *k sequences of the generalized order-k Jacobsthal numbers are defined as*

$$J_n^i = J_{n-1}^i + J_{n-2}^i + \cdots + J_{n-k+1}^i + 2J_{n-k}^i \tag{14}$$

for $n > 0$ and $1 \leqslant i \leqslant k$, with initial terms

$$J_n^i = \begin{cases} 1, & \text{if } i+n=1 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1-k \leqslant n \leqslant 0, \tag{15}$$

where J_n^i is the nth term of the ith sequence.

For the case where $i = 1$ and $k = 2$, our definition is reduced directly to the usual Jacobsthal numbers, i.e., $J_n^1 = J_n$, and when $i = k = 2$, it is reduced to two times of the Jacobsthal numbers, namely $J_n^2 = 2J_n$. In particular, the sequence J_n^k is called the generalized k-Jacobsthal numbers in the case of $i = k$. By the way, Table 2 displays some values of the generalized order- k Jacobsthal numbers, including the related initial conditions.

After this point, we will summarize the results regarding the generalized order- k Jacobsthal numbers. Let us define the following matrices:

$$\mathbf{A}_k = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } \mathbf{J}_{\tilde{k},n} = \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^{k-1} & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^{k-1} & J_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{n-k-2}^1 & J_{n-k-2}^2 & \cdots & J_{n-k-2}^{k-1} & J_{n-k-2}^k \\ J_{n-k-1}^1 & J_{n-k-1}^2 & \cdots & J_{n-k-1}^{k-1} & J_{n-k-1}^k \end{bmatrix}. \tag{16}$$

It is clear from Definition 2.1 that $\mathbf{J}_{\tilde{k},n} = \mathbf{A}_k \mathbf{J}_{\tilde{k},n-1}$. Expanding the right-hand side of this equation yields the following theorem.

Theorem 2.2 *The matrix equation*

$$\mathbf{J}_{\tilde{k},n} = {\mathbf{A}_k}^n \tag{17}$$

holds for a positive integer n.

Table 2: Some values of the generalized order-k Jacobsthal numbers

	$k=2$		$k=3$			$k=4$			
n/i	1	2	1	2	3	1	2	3	4
-7	$-\frac{21}{64}$	$\frac{43}{64}$	$-\frac{9}{32}$	$\frac{23}{32}$	$-\frac{9}{32}$	$-\frac{1}{16}$	$-\frac{1}{16}$	$-\frac{1}{16}$	$\frac{15}{16}$
-6	$\frac{11}{32}$	$-\frac{21}{32}$	$\frac{7}{16}$	$-\frac{9}{16}$	$-\frac{9}{16}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{7}{8}$	$-\frac{1}{8}$
-5	$-\frac{5}{6}$	$\frac{11}{16}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{7}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
-4	$\frac{3}{8}$	$-\frac{5}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
-3	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1
-2	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	0	0	1	0
-1	0	1	0	1	0	0	1	0	0
0	1	0	1	0	0	1	0	0	0
1	1	2	1	1	2	1	1	1	2
2	3	2	2	3	2	2	2	3	2
3	5	6	5	4	4	4	5	4	4
4	11	10	9	9	10	9	8	8	8
5	21	22	18	19	18	17	17	17	18
6	43	42	37	36	36	34	34	35	34
7	85	86	73	73	74	68	69	68	68

	$k=5$					$k=6$					
n/i	1	2	3	4	5	1	2	3	4	5	6
-7	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{7}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
-6	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
-5	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	1
-4	0	0	0	0	1	0	0	0	0	1	0
-3	0	0	0	1	0	0	0	0	1	0	0
-2	0	0	1	0	0	0	0	1	0	0	0
-1	0	1	0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0
1	1	1	1	1	2	1	1	1	1	1	1
2	2	2	2	3	2	2	2	2	2	3	2
3	4	4	5	4	4	4	4	4	5	4	4
4	8	9	8	8	8	8	8	9	8	8	8
5	16	16	16	16	16	16	17	16	16	16	16
6	33	33	33	33	34	33	32	32	32	32	32
7	66	66	66	67	66	65	65	65	65	65	66

Next, the following results are satisfied.

Corollary 2.3 *Let n be any positive integer. Then, we have*

$$\det\left(\mathbf{J}_{k,n}^{\sim}\right) = \begin{cases} 2^{n}, & \text{if } k \text{ is odd} \\ (-2)^{n}, & \text{if } k \text{ is even} \end{cases}. \tag{18}$$

Lemma 2.4 *For a positive integer n, we have*

$$J_{n}^{i} = J_{n-1}^{1} + J_{n-1}^{i+1} \text{ and } 2J_{n+1}^{1} = J_{n}^{k}. \tag{19}$$

Theorem 2.2 says us that $\mathbf{J}_{k,n}^{\sim}$ is a generating matrix for the generalized order-k Jacobsthal numbers. This means that readers have a powerful tool for discovering their new identities. For instance, by taking the multiplication identities of matrices into account, we can write

$$\mathbf{J}_{k,n+m}^{\sim} = \mathbf{J}_{k,n}^{\sim}\mathbf{J}_{k,m}^{\sim} = \mathbf{J}_{k,m}^{\sim}\mathbf{J}_{k,n}^{\sim}. \tag{20}$$

We can, therefore, write the following strange result as an example.

Theorem 2.5 *Let n and m be any positive integers. Then,*

$$J_{n+m}^{i} = \sum_{j=1}^{k} J_{n}^{j} J_{m-j+1}^{i}. \tag{21}$$

Letting also

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \mathbf{A}_{k} \end{bmatrix} \text{ and } \mathbf{T}_{n} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_{n} & & & & \\ S_{n-1} & & & & \\ \vdots & & & & \\ S_{n-k+1} & & & & \mathbf{J}_{k,n}^{\sim} \end{bmatrix}, \tag{22}$$

where $S_{n} = \sum_{i=0}^{n-1} J_{i}^{1}$. Moreover, using the fact that $J_{n}^{1} = \frac{1}{2}J_{n+1}^{k}$ yields $S_{n} = \frac{1}{2}\sum_{i=0}^{n} J_{i}^{k}$ and $S_{n} = J_{n-1}^{1} + S_{n-1}$. Herein, according to all these explanations and since $S_{-i} = 0$ for $1 \leq i \leq k$ and $\mathbf{T}_{1} = \mathbf{J}$, we can write

$$\mathbf{T}_{n+1} = \mathbf{T}_{n}\mathbf{J} = \mathbf{T}_{n-1}\mathbf{J}^{2} = \ldots = \mathbf{T}_{1}\mathbf{J}^{n-1} = \mathbf{J}^{n}.$$

In this case, computing the right-hand side of the equation $\mathbf{T}_{n} = \mathbf{T}_{1}\mathbf{T}_{n-1}$, we can write

$$\begin{aligned} S_{n} = 1 + S_{n-1} + 2S_{n-2} &= 1 + S_{n} - J_{n-1}^{1} + 2\left(S_{n-1} - J_{n-2}^{1}\right) \\ &= 1 + S_{n} - J_{n-1}^{1} + 2\left(S_{n} - J_{n-1}^{1} - J_{n-2}^{1}\right) \end{aligned}$$

and the following result can be given.

Theorem 2.6 *Let n be an positive integer. Then, we have*

$$S_n = \frac{1}{2}\left(3J_{n-1}^1 + 2J_{n-2}^1 - 1\right). \tag{23}$$

It should be noted that for $k = 2$, the summation in Equation (23) is reduced to the famous formula of the usual Jacobsthal numbers as

$$\sum_{i=1}^{n-1} J_i = \frac{J_{n+1} - 1}{2}.$$

It should be noted that Equation (14) is actually the k-order linear homogeneous difference equation, with constant coefficients, in the form of

$$x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-k+1} + 2x_{n-k}.$$

We can then explore a solution to the last equation as $x_n = \lambda^n$, where λ is an unknown constant to be determined. On the substitution of this linear solution into our difference equation, we can, therefore, write

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} - \cdots - \lambda^{n-k+1} - 2\lambda^{n-k} = 0$$

or equally

$$(\lambda - 2)\left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1\right) = 0.$$

Therefore, our characteristic equation has the distinct roots such as $\lambda_1 = 2$ and $\lambda_j = e^{\frac{2\pi i j}{k}}$ for $j = 2, 3, \cdots, k$, which are the eigenvalues of the matrix $\mathbf{A}_k$. Let $\mathbf{V}$ be Vandermonde matrix as follows:

$$\mathbf{V} = \begin{bmatrix} {\lambda_1}^{k-1} & {\lambda_2}^{k-1} & \cdots & {\lambda_k}^{k-1} \\ {\lambda_1}^{k-2} & {\lambda_2}^{k-2} & \cdots & {\lambda_k}^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Also, we consider the vector

$$\mathbf{b}_k{}^i = \begin{bmatrix} {\lambda_1}^{n+k-i} & {\lambda_2}^{n+k-i} & \ldots & {\lambda_k}^{n+k-i} \end{bmatrix}^T.$$

Also, $\mathbf{V}_j{}^{(i)}$ be k-square matrix obtained from $\mathbf{V}$ by replacing the jth column of $\mathbf{V}$ with $\mathbf{b}_k{}^i$. Then, we have the generalized Binet's formula for the generalized order-k Jacobsthal numbers in the following theorem.

Theorem 2.7 *Let J_n^i be the nth term of ith Jacobsthal sequence for $1 \le i \le k$. Then, we have*

$$J_{n-i+1}^j = \frac{\det\left(\mathbf{V}_j{}^{(i)}\right)}{\det\left(\mathbf{V}\right)}.$$

We now focus on the generating functions for the generalized order-k Jacobsthal numbers. Fur this purpose, introduce the function

$$G^i{}_k(x) = \sum_{v=0}^{\infty} J_v{}^i x^v = J_0{}^i + J_1{}^i x + J_2{}^i x^2 + \cdots + J_k{}^i x^k + \cdots.$$

In contrast to the current literature, in our definition, the superscript i is arbitrary over its possible values, not fixed. In this case, the following important result can be given.

Theorem 2.8 *The generating functions for k sequences of the generalized order-k Jacobsthal numbers are*

$$G^i{}_k(x) = \frac{J_0{}^i + x\left(\sum_{v=0}^{k+n-1}{}^{*}\, x^v\right) J_n^i}{1 - x - x^2 - x^3 - \cdots - x^{k-1} - 2x^k}, \tag{24}$$

where $1-k \leqslant n \leqslant -1$ and the asterisk in summation denotes a protocol such that only the last term of the finite series is multiplied by 2.

Proof Summing $G^i{}_k(x)$, $-xG^i{}_k(x)$, $-x^2G^i{}_k(x)$, $\cdots$, $-x^{k-1}G^i{}_k(x)$, $-2x^kG^i{}_k(x)$ up, we can write

$$\begin{aligned}\left(1 - x - x^2 - x^3 - \cdots - x^{k-1} - 2x^k\right) G^i{}_k(x) &= J_0{}^i + \left(J_1{}^i - J_0{}^i\right)x + \left(J_2{}^i - J_1{}^i - J_0{}^i\right)x^2 \\ &+ \left(J_3{}^i - J_2{}^i - J_1{}^i - J_0{}^i\right)x^3 + \cdots + \left(J_{k-1}{}^i - J_{k-2}{}^i - J_{k-3}{}^i - J_0{}^i\right)x^{k-1}\end{aligned}$$

or in other situation,

$$\begin{aligned}\left(1 - x - x^2 - x^3 - \cdots - x^{k-1} - 2x^k\right) G^i{}_k(x) &= J_0{}^i + \left(J^i_{-1} + \cdots + J^i_{-k+2} + 2J^i_{-k+1}\right)x \\ + \left(J^i_{-1} + \cdots + J^i_{-k+3} + 2J^i_{-k+2}\right)x^2 &+ \left(J^i_{-1} + \cdots + J^i_{-k+4} + 2J^i_{-k+3}\right)x^3 + \cdots + 2J^i_{-1}x^{k-1}.\end{aligned}$$

The coefficients of the last equation only consists of the initial conditions. Hence, the result follows. □

As an example, when $i = 3$ and $k = 7$, the generating function for the third sequence of the generalized order-7 Jacobsthal numbers is found as follows:

$$G^3{}_7(x) = \frac{x\left(1 + x + x^2 + x^3 + 2x^4\right)}{1 - x - x^2 - x^3 - x^4 - x^5 - x^6 - 2x^7}.$$

In addition, for the cases $i = 1, k = 2$ and $i = k = 2$, Equation (24) takes the shape

$$G^1{}_2(x) = \frac{1}{1 - x - 2x^2} \text{ and } G^2{}_2(x) = \frac{x}{1 - x - 2x^2},$$

respectively. In particular the second equation is in the well-known form.

3. On the Generalized Order-k Jacobsthal-Lucas Sequence

In this section, we consider the generalized order-k Jacobsthal-Lucas numbers. For this purpose, two special cases will be handled.

3.1. Constructing Generalized Jacobsthal-Lucas Sequences by Employing Trace Operator

As remembered, Equations (10) and (11) present a wonderful link between the Jacobsthal and Jacobsthal-Lucas numbers as follows:

$$j_n = tr\left(F^n\right) = J_n + 4J_{n-1} = J_n + 2\left(2J_{n-1}\right).$$

We can, therefore, use this idea to construct higher-ordered Jacobsthal-Lucas numbers. For this purpose, we, firstly, introduce tri-Jacobsthal numbers of third-order recurrence relation

$$J_0^3 = 0,\ J_1^3 = 0,\ J_2^3 = 1 \text{ and } J_{n+1}^3 = J_n^3 + J_{n-1}^3 + 2J_{n-2}^3 \text{ for } n \geqslant 0$$

and their generating matrix is

$$F_3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that these definitions can be obtained by simple vector-matrix operations. On the basis of the induction method, one can prove

$$F_3{}^n = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{n+2}^3 & J_{n+1}^3 + 2J_n^3 & 2J_{n+1}^3 \\ J_{n+1}^3 & J_n^3 + 2J_{n-1}^3 & 2J_n^3 \\ J_n^3 & J_{n-1}^3 + 2J_{n-2}^3 & 2J_{n-1}^3 \end{bmatrix}.$$

As a result, we get

$$tr\left(F_3{}^n\right) = J_{n+2}^3 + J_n^3 + 2J_{n-1}^3 + 2J_{n-1}^3 = J_{n+1}^3 + 2J_n^3 + 6J_{n-1}^3 = J_{n+1}^3 + 2J_n^3 + 2\left(3J_{n-1}^3\right).$$

After a similar process, one can write

$$tr\left(F_4{}^n\right) = J_{n+2}^4 + 2J_{n+1}^4 + 3J_n^4 + 2\left(4J_{n-1}^4\right)$$

and

$$tr\left(F_5{}^n\right) = J_{n+3}^4 + 2J_{n+2}^4 + 3J_{n+1}^4 + 4J_n^4 + 2\left(5J_{n-1}^4\right).$$

This process regularly continues as above with minor changes for increasing values of order. Hence, we can write this observation in a more general form as in the following.

Corollary 3.1 *Let J_n^k be a generalized Jacobsthal numbers of order-k in Definition 2.1. Then, the generalized Jacobsthal-Lucas numbers $\left\{j_n^k\right\}_{n=0}^{\infty}$ of order-k satisfies the interrelationship*

$$j_n^k = J_{n+k-2}^k + 2J_{n+k-3}^k + 3J_{n+k-4}^k + \cdots + (k-2)\,J_{n-3}^k + (k-1)\,J_{n-2}^k + 2kJ_{n-1}^k.$$

[h!]

Table 3: Some values of the generalized order-2, order-3 and order-4 Jacobsthal-Lucas numbers

		$k=2$		$k=3$			$k=4$			
n	$i=$	1	2	1	2	3	1	2	3	4
-3							$-\frac{7}{8}$	$\frac{1}{8}$	$\frac{9}{8}$	$\frac{49}{8}$
-2				$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{17}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{21}{4}$	$-\frac{7}{4}$
-1		$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{7}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{9}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$
0		2	-1	3	-2	-1	4	-3	-2	-1
1		1	4	1	2	6	1	2	3	8
2		5	2	3	7	2	3	4	9	2
3		7	10	10	5	6	7	12	5	6
4		17	14	15	16	20	19	12	13	14
5		31	34	31	35	30	31	32	33	38

It should be noted that Corollary 3.1 is a general result. One can obtain the generalized Jacobsthal-Lucas numbers from the equation

$$j_n = J_{n+k-2} + 2J_{n+k-3} + 3J_{n+k-4} + \cdots + (k-2)\,J_{n-3} + (k-1)\,J_{n-2} + 2kJ_{n-1}.$$

These numbers satisfy the recurrence relation

$$j_n = j_{n-1} + j_{n-2} + j_{n-3} + j_{n-4} + \ldots + 2j_{n-k}$$

with the initial conditions $j_0 = k$ and for $1 \le i < k$, $j_{-k+i} = 2^{-k+i} - 1$.

Now, we obtain some terms of generalized order-k Jacobsthal-Lucas numbers. The easiest way of this purpose is to use the matrix $\mathbf{A}_k$ in Equation (16) with the matrix

$$\mathbf{M}_k = [1, 2, 3, \cdots, k-1, 2k].$$

Then, for any integer n, nth terms of the generalized order-k Jacobsthal-Lucas numbers can be found with the equation

$$[j_n^1, j_n^2, j_n^3, \cdots, j_n^k] = \mathbf{M}_k \times \mathbf{A}_k^{n-1}. \tag{25}$$

Using Equation (25), we can give some values of generalized order-k Jacobsthal-Lucas numbers which are given in the Table ??.

Let us define the following matrices to use matrix methods for the generalized order$-k$ Jacobsthal-Lucas numbers

$$\mathbf{B}_k = \begin{bmatrix} k & 1-k & 2-k & 3-k & 4-k & \cdots & -1 \\ 2^{-1}-1 & 2^{-1}+k & 2^{-1}-k+1 & 2^{-1}-k+2 & 2^{-1}-k+3 & \cdots & 2^{-1}-2 \\ 2^{-2}-1 & 2^{-2} & 2^{-2}+k+1 & 2^{-2}-k+2 & 2^{-2}-k+3 & \cdots & 2^{-2}-2 \\ 2^{-3}-1 & 2^{-3} & 2^{-3}+1 & 2^{-3}+k+2 & 2^{-2}-k+3 & \cdots & 2^{-3}-2 \\ 2^{-4}-1 & 2^{-4} & 2^{-4}+1 & 2^{-4}+2 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{-k}-1 & 2^{1-k} & \cdots & \cdots & 2^{1-k}+3 & \cdots & 2^{1-k}+2k-2 \end{bmatrix} \tag{26}$$

and

$$\mathbf{j}_{k,n}^{\sim} = \begin{bmatrix} j_n^1 & J_n^2 & \cdots & j_n^{k-1} & j_n^k \\ j_{n-1}^1 & j_{n-1}^2 & \cdots & j_{n-1}^{k-1} & j_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ j_{n-k-2}^1 & j_{n-k-2}^2 & \cdots & j_{n-k-2}^{k-1} & j_{n-k-2}^k \\ j_{n-k-1}^1 & j_{n-k-1}^2 & \cdots & j_{n-k-1}^{k-1} & j_{n-k-1}^k \end{bmatrix}. \tag{27}$$

It is easy to prove the following theorem.

Theorem 3.2 *The matrix equation*

$$\mathbf{j}_{k,n}^{\sim} = \mathbf{B}_k \times \mathbf{A}_k^n \tag{28}$$

holds for a positive integer n.

This preparation leads us to the definition of the generalized order$-k$ Jacobsthal-Lucas numbers as follows.

Definition 3.3 *The* k*-sequences of the generalized order*$-k$ *Jacobsthal-Lucas numbers* $(KSOKJ-L)$ *for* $n > k$ *and* $1 \leqslant i \leqslant k$ *are defined as*

$$j_n^i = j_{n-1}^i + j_{n-2}^i + j_{n-3}^i + j_{n-4}^i + \ldots + 2j_{n-k}^i$$

with initial conditions

$$j_{k,n}^i = \begin{cases} k, & \text{if } (i,n) = (1,0) \\ -1-k+i, & if\ n = 0 \\ 2^n + i - 2, & \text{if } 2-k \leqslant i+n \leqslant 0 \\ 2^n + i + k - 2, & \text{if } i+n = 1,\ i > 1 \\ 2^n + i - k - 2, & \text{if } i+n \geqslant 2 \text{ and } n \neq 0 \end{cases},$$

We can find generating function and Binet's formula for the generalized order$-k$ Jacobsthal-Lucas numbers via Theorem 3.2 similar to the generalized order$-k$ Jacobsthal numbers mentioned above. But we don't give these calculations since these repetitions may be tedious for the readers.

3.2. Constructing Generalized Jacobsthal-Lucas Sequences by Derivative of Core Polynomial

Another method to obtain the generalized order$-k$ Jacobsthal-Lucas numbers is to use core polynomial. Let's define $P(x, t_1, t_2, \cdots, t_k) = x^k - t_1 x^{k-1} - t_2 x^{k-2} - \cdots - t_k$, where $t_1, t_2, \cdots, t_k$ are constants. So, its derivative is $P'(x, t_1, t_2, \cdots, t_k) = kx^{k-1} - t_1(k-1)x^{k-2} - \cdots - t_{k-1}$. It is obvious that if we take $t_1 = t_2 = \cdots = t_{k-1} = 1$ and $t_k = 2$, this polynomial reduces to the characteristic equation of order$-k$ Jacobsthal and Jacobsthal-Lucas numbers. We define the following matrix

$$\mathbf{C}_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{29}$$

Then we can obtain the initial conditions of the generalized order$-k$ Jacobsthal-Lucas numbers by using the equation (see [14] for details)

$$\mathbf{j}'_{k,0} = -\mathbf{C}_k{}^{-k+1} - 2\mathbf{C}_k{}^{-k+2} - 3\mathbf{C}_k{}^{-k+3} - \cdots + k\mathbf{C}_k{}^{0}. \tag{30}$$

For example, if we take $k = 3$, then we have

$$\mathbf{C}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \tag{31}$$

and Equation (30) gives

$$\mathbf{j}'_{3,0} = \begin{bmatrix} \frac{17}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{7}{2} & -\frac{1}{2} \\ -1 & -2 & 3 \end{bmatrix}. \tag{32}$$

If we take $k = 4$, Equation (30) gives

$$\mathbf{j}'_{4,0} = \begin{bmatrix} \frac{49}{8} & \frac{9}{8} & \frac{1}{8} & -\frac{7}{8} \\ -\frac{7}{4} & \frac{21}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{9}{2} & -\frac{1}{2} \\ -1 & -2 & -3 & 4 \end{bmatrix}. \tag{33}$$

Examining Equations (32) and (33) with Table 3, it is clear that matrices $\mathbf{j}^{\sim}_{k,0}$ and $\mathbf{j}'_{k,0}$ give same sequences in different order. Namely, let σ be the permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & k \\ k & k-1 & k-2 & \cdots & 1 \end{pmatrix} \in S_k$. Then nth column of the matrix $\mathbf{j}^{\sim}_{k,0}$ is the $\sigma(n)-$th column of the matrix $\mathbf{j}'_{k,0}$.

Finally, we are ready to define another form of generalized order$-k$ Jacobsthal-Lucas numbers by using the matrix $\mathbf{j}'_{k,0}$.

Definition 3.4 *The $k-$ sequences of the generalized order-k Jacobsthal-Lucas numbers satisfy the following recurrence relation for $n > k$ and $1 \leqslant i \leqslant k$*

$$j_n^i = j_{n-1}^i + j_{n-2}^i + j_{n-3}^i + j_{n-4}^i + \cdots\cdots\cdots + 2j_{n-k}^i$$

with initial conditions for $1 - k \leq i \leq 0$

$$j_{k,n}^i = \begin{cases} 2^n - i - 1, & \text{if } i - n < k \\ 2^n + 2k - i - 1, & \text{if } i - n = k \\ 2^n + k - i - 1, & \text{if } i - n > k \end{cases}.$$

Example 3.5 *For $k = 8$ and $i = 5$, Definition 3.3 gives the following sequence*

$$\{j_{8,n}^5\}_{n=-7}^{\infty} = \left\{\frac{385}{128}, \frac{193}{64}, \frac{97}{32}, -\frac{39}{8}, -\frac{19}{4}, -\frac{9}{2}, -4, 5, 7, 11, 27, 27, 59, 123, 251, 515, \cdots\right\}.$$

The same sequence can be obtained by taking $k = 8$ and $i = 4$ in Definition 3.4.

4. Conclusions

Due to an upward trend and scientific importance in mathematics and other branches, the integer sequences and their generalizations become indispensable to exploring wide usage areas and applications of the sequences under consideration. As the authors, while reviewing the current literature, we have caught our attention that the definition given for $k-$sequences of the generalized order-k Jacobsthal numbers is incorrect based on several reasons. To address this issue, at first, we presented the correct definition for the mentioned generalization. Unfortunately, our definition overrides all the results of the papers produced within the scope of the study by Yilmaz and Bozkurt [21]. Then, after the presentation of the definition, some fundamental identities of the generalization under consideration were performed, e.g., generating matrix, generating functions, and summation formula. Following, we took how to generalize the usual Jacobsthal-Lucas sequence in the framework of our new definition regarding the generalized Jacobsthal numbers into account. Instead of ordinary approaches in the literature, we developed combinatorial modeling to generalize the Jacobsthal-Lucas sequence to $k-$sequences of the generalized order-k numbers and gave two new definitions for that aim. Both definitions satisfy the following recurrence relation

$$j_n^i = j_{n-1}^i + j_{n-2}^i + j_{n-3}^i + j_{n-4}^i + \ldots + 2j_{n-k}^i \quad (\text{for } n > k \text{ and } 1 \leqslant i \leqslant k),$$

where two different initial conditions

$$j_{k,n}^i = \begin{cases} k, & if \ (i,n) = (1,0) \\ -1-k+i, & if \ n = 0 \\ 2^n + i - 2, & if \ 2-k \leqslant i+n \leqslant 0 \\ 2^n + i + k - 2, & if \ i+n = 1, \ i > 1 \\ 2^n + i - k - 2, & if \ i+n \geqslant 2 \text{ and } n \neq 0 \end{cases}$$

and

$$j_{k,n}^i = \begin{cases} 2^n - i - 1, & if \ i-n < k \\ 2^n + 2k - i - 1, & if \ i-n = k \\ 2^n + k - i - 1, & if \ i-n > k \end{cases}.$$

It should be noted that both definitions are, in fact, the permutationally same. Namely, ith sequence in the one generalization implies $(k-i+1)$-th sequence in the other one.

Another remarkable relation is the following relation between $k-$sequences of the generalized order-k Jacobsthal and Jacobsthal-Lucas numbers

$$j_n^k = J_{n+k-2}^k + 2J_{n+k-3}^k + 3J_{n+k-4}^k + \cdots + (k-2)\,J_{n-3}^k + (k-1)\,J_{n-2}^k + 2kJ_{n-1}^k.$$

As a concluding remark, we imply that the novelties of this study also provide researchers with many potential research opportunities.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Ahmet Daşdemir]: Evaluation of data, contributed to the research method, solving the problem, wrote the manuscript (%40).

Author [Göksal Bilgici]: Thought and designed the research/problem, solving the problem, wrote the manuscript (%35).

Author [Hossen Mohammed Mahdi Ahmed]: Collected the data, evaluation of data (%25).

Conflicts of Interest

The authors declare no conflict of interest.

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