MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.1395051

12 (2) 93-100 (2024) - Research Article

ISSN: 2147-6268

Certain Results for Invariant Submanifolds of an Almost α-Cosymplectic (k, μ, ν)-Space

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Abstract

In this paper we present invariant submanifolds of an almost α-cosymplectic (k, μ, ν)-space. Then, we gave some results for an invariant submanifold of an almost α-cosymplectic (k, μ, ν)-space to be totally geodesic. As a result, we have discovered some interesting conclusions about invariant submanifolds of an almost cosymplectic (k, μ, ν)-space.

Keywords: *α-cosymplectic (k, μ, ν)-space, W_1^*-curvature tensor, W_7-curvature tensor*

AMS Subject Classification (2020): *53C15; 53C25; 53D25*

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1. Introduction

T. Koufogiorgos and C. Tsichlias found a new class of 3-dimensional contact metric manifolds that k and μ are non-constant smooth functions. They generalized $(k, \mu)-$contact metric manifolds on non-Sasakian manifolds for $n > 1$, where the functions k, μ are constants [1].

S. I. Goldberg and K. Yano obtained integrability conditions for almost cosymplectic structures on almost contact manifolds. The simplest examples of almost cosymplectic manifolds are these structures of almost Kaehler manifolds, the real $\mathbb{R}$ line and the circle S^1. Besides, they studied an almost cosymplectic manifold is cosymplectic only in the case it is locally flat [2].

İ. Küpeli Erken researched almost $\alpha-$cosymplectic manifolds. They studied, respectively, projectively flat, conformally flat and concircularly flat almost $\alpha-$cosymplectic manifolds (with the $\eta-$parallel tensor field ϕh). They devoted to properties of almost with the $\eta-$parallel tensor field ϕh [3].

For an almost contact metric structure to be almost cosymplectic, Z. Olszak provided a few necessary requirements. They established the absence of virtually cosymplectic manifolds in dimensions bigger than three with non-zero constant sectional curvature. Fortunately, such locally flat manifolds with zero sectional curvature do exist and were cosymplectic. Additionally, they looked at several constraints on virtually cosymplectic manifolds that had conformally flat surfaces or constant ϕ-sectional curvature [4].

Received : *23-11-2023,* ***Accepted :*** *12-01-2024,* ***Available online :*** *18-03-2024*

*(**Cite as** "P. Uygun, M. Atçeken, T. Mert, Certain Results for Invariant Submanifolds of an Almost α-Cosymplectic (k, μ, ν)-Space, Math. Sci. Appl. E-Notes, 12(2) (2024), 93-100")*

In 2022, M. Atçeken studied the invariant submanifolds of an almost α-cosymplectic (k,μ,ν)-space that satisfying certain geometric requirements so that $Q(\sigma,R)=0$,
$Q(S,\sigma)=0$, $Q(S,\widetilde{\nabla}\sigma)=0$, $Q(S,\widetilde{R}\cdot\sigma)=0$, $Q(g,C\cdot R)=0$ and $Q(S,C\cdot\sigma)=0$. He showed that under certain circumstances, these conditions are identical to totally geodesic [5]. Additionally, some geometers have worked on the almost Kenmotsu (k,μ,ν)-space [6–8].

Our article's focus is on invariant submanifolds of an almost α-cosymplectic (k,μ,ν)-space, which is inspired by the works mentioned studies. In addition, we research several conditions for an α-cosymplectic (k,μ,ν)-space's invariant submanifold to be totally geodesic. Then, some classifications and characterizations have been developed.

2. Preliminaries

An almost contact manifold is of 1-form η satisfying on M^{2n+1}, an odd-dimensional manifold, a field ϕ of endomorphisms of the tangent spaces, a characteristic or Reeb vector field, and a vector field ξ

$$\phi^2=-I+\eta\otimes\xi,\ \ \eta(\xi)=1, \tag{2.1}$$

in which $I:TM^{2n+1}\to TM^{2n+1}$ denotes an identity mapping. Because of (2.1), it follows

$$\eta\circ\phi=0,\ \phi\xi=0,\ rank(\phi)\text{=}2n. \tag{2.2}$$

An almost contact manifold $M^{2n+1}(\phi,\xi,\eta)$ is said to be normal if the tensor field $N=[\phi,\phi]+2d\eta\otimes\xi=0$, where $[\phi,\phi]$ denote the Nijenhuis tensor field of ϕ. Any almost contact manifold $M^{2n+1}(\phi,\xi,\eta)$ is known to have a Riemannian metric like that

$$g(\phi\omega_1,\phi\omega_2)=g(\omega_1,\omega_2)-\eta(\omega_1)\eta(\omega_2), \tag{2.3}$$

for all vector fields $\omega_1,\omega_2\in\Gamma(TM)$ [9]. A metric of this type, g is known as an equipped metric, and the structure (ϕ,η,ξ,g) and manifold $M^{2n+1}(\phi,\eta,\xi,g)$, associated with it, are known as an almost contact metric manifolds and denoted by as $M^{2n+1}(\phi,\eta,\xi,g)$. It is defined for $M^{2n+1}(\phi,\eta,\xi,g)$ to have a 2-form Φ. It is known as the fundamental form of $M^{2n+1}(\phi,\eta,\xi,g)$ when $\Phi(\omega_1,\omega_2)=g(\phi\omega_1,\omega_2)$. An almost contact metric manifold is referred to as a cosymplectic manifold if η and Φ are closed, that is, $d\eta=d\Phi=0$ [10].
The definition of an almost α-cosymplectic manifold for every real number α is [11]

$$d\eta=0,\ d\Phi=2\alpha\eta\wedge\Phi. \tag{2.4}$$

An $\alpha-$cosymplectic refers to a normal almost $\alpha-$cosymplectic manifold [12]. It is well known that the following equality holds for the tensor h on the contact metric manifold $M^{2n+1}(\phi,\eta,\xi,g)$, described by $2h=L_\xi\phi$,

$$\widetilde{\nabla}_{\omega_1}\xi=-\phi\omega_1-\phi h\omega_1,\ h\phi+\phi h=0,\ trh=tr\phi h=0,\ h\xi=0, \tag{2.5}$$

where,$\widetilde{\nabla}$ is the Levi-Civita connection on M^{2n+1} [6].

The following presented the notation of the $(k,\mu,\nu)-$contact metric manifold, which expands above generalized (k,μ)-spaces:

$$R(\omega_1,\omega_2)\xi=\eta(\omega_2)\left[kI+\mu h+\nu\phi h\right]\omega_1+\eta(\omega_1)\left[kI+\mu h+\nu\phi h\right]\omega_2, \tag{2.6}$$

where R is the Riemannian curvature tensor of M^{2n+1} and certain smooth functions k, μ and ν on M^{2n+1}, ω_1,ω_2 are vector fields [13].

Lemma 2.1. *Given $M^{2n+1}(\phi,\eta,\xi,g)$ is an almost $\alpha-$cosymplectic $(k,\mu,\nu)-$space, so*

$$h^2=(k+\alpha^2)\phi^2, \tag{2.7}$$

$$\xi(k)=2(k+\alpha^2)(\nu-2\alpha), \tag{2.8}$$

$$\begin{aligned}R(\xi,\omega_1)\omega_2 &= k[g(\omega_1,\omega_2)\xi-\eta(\omega_2)\omega_1]+\mu[g(h\omega_1,\omega_2)\xi-\eta(\omega_2)h\omega_1]\\ &\quad+\nu[g(\phi h\omega_1,\omega_2)\xi-\eta(\omega_2)\phi h\omega_1],\end{aligned} \tag{2.9}$$

$$(\widetilde{\nabla}_{\omega_1}\phi)\omega_2 = g(\alpha\phi\omega_1 + hu_1, \omega_2)\xi - \eta(\omega_2)(\alpha\phi\omega_1 + h\omega_1), \tag{2.10}$$

$$\widetilde{\nabla}_{\omega_1}\xi = -\alpha\phi^2\omega_1 - \phi h\omega_1, \tag{2.11}$$

for any vector fields ω_1, ω_2 on M^{2n+1} [9].

Let M be an immersed submanifold of $\widetilde{M}^{2n+1}$, which is an almost $\alpha-$cosymplectic (k,μ,ν)-space. We denote the tangent and normal subspaces of M in $\widetilde{M}$ by $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively, the Gauss and Weingarten formulas are provided, respectively, by

$$\widetilde{\nabla}_{\omega_1}\omega_2 = \nabla_{\omega_1}\omega_2 + \sigma(\omega_1, \omega_2), \tag{2.12}$$

and

$$\widetilde{\nabla}_{\omega_1}\omega_5 = -A_{\omega_5}\omega_1 + \nabla^{\perp}_{\omega_1}\omega_5 \tag{2.13}$$

for all $\omega_1, \omega_2 \in \Gamma(TM)$ and $\omega_5 \in \Gamma(T^{\perp}M)$, σ and A are referred to as the second fundamental form and shape operators of M, respectively, ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$. $\Gamma(TM)$ stands for the set of differentiable vector fields on M. They are associated by

$$g(A_{\omega_5}\omega_1, \omega_2) = g(\sigma(\omega_1, \omega_2), \omega_5). \tag{2.14}$$

The second fundamental form σ is first covariant derivative is given by

$$(\widetilde{\nabla}_{\omega_1}\sigma)(\omega_2, \omega_3) = \nabla^{\perp}_{\omega_1}\sigma(\omega_2, \omega_3) - \sigma(\nabla_{\omega_1}\omega_2, \omega_3) - \sigma(\omega_2, \nabla_{\omega_1}\omega_3), \tag{2.15}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, the second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of submanifold, then we have the Gauss formulae.

$$\begin{aligned}\widetilde{R}(\omega_1, \omega_2)\omega_3 &= R(\omega_1, \omega_2)\omega_3 + A_{\sigma(\omega_1,\omega_3)}\omega_2 - A_{\sigma(\omega_2,\omega_3)}\omega_1 + (\widetilde{\nabla}_{\omega_1}\sigma)(\omega_2, \omega_3) \\ &\quad -(\widetilde{\nabla}_{\omega_2}\sigma)(\omega_1, \omega_3),\end{aligned} \tag{2.16}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$.

$\widetilde{R} \cdot \sigma$ is given by

$$\begin{aligned}(\widetilde{R}(\omega_1, \omega_2) \cdot \sigma)(\omega_4, \omega_5) &= R^{\perp}(\omega_1, \omega_2)\sigma(\omega_4, \omega_5) - \sigma(R(\omega_1, \omega_2)\omega_4, \omega_5) \\ &\quad -\sigma(\omega_4, R(\omega_1, \omega_2)\omega_5),\end{aligned} \tag{2.17}$$

where

$$R^{\perp}(\omega_1, \omega_2) = [\nabla^{\perp}_{\omega_1}, \nabla^{\perp}_{\omega_2}] - \nabla^{\perp}_{[\omega_1, \omega_2]},$$

denote the normal bundle's Riemannian curvature tensor.

For the Riemannian manifold (M^{2n+1}, g), the W_1^* curvature tensor is determined by

$$W_1^*(\omega_1, \omega_2)\omega_3 = R(\omega_1, \omega_2)\omega_3 - \frac{1}{2n}[S(\omega_2, \omega_3)\omega_1 - S(\omega_1, \omega_3)\omega_2], \tag{2.18}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$ [14].

Similarly, the tensor $W_1^* \cdot \sigma$ is defined by

$$\begin{aligned}(W_1^*(\omega_1, \omega_2) \cdot \sigma)(\omega_4, \omega_5) &= R^{\perp}(\omega_1, \omega_2)\sigma(\omega_4, \omega_5) - \sigma(W_1^*(\omega_1, \omega_2)\omega_4, \omega_5) \\ &\quad -\sigma(\omega_4, W_1^*(\omega_1, \omega_2)\omega_5),\end{aligned} \tag{2.19}$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma(TM)$.

Furthermore, the W_7-curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$W_7(\omega_1, \omega_2)\omega_3 = R(\omega_1, \omega_2)\omega_3 - \frac{1}{2n}[S(\omega_2, \omega_3)\omega_1 - g(\omega_2, \omega_3)Q\omega_1] \tag{2.20}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$ [15].

On a semi-Riemannian manifold (M,g), for a $(0,k)-$type tensor field $(0,k)$-type tensor field T and $(0,2)$-type tensor field A, $(0,k+2)$-type tensor field Tachibana $Q(A,T)$ is defined as

$$\begin{aligned} Q(A,T)(\omega_{11},\omega_{12},...,\omega_{1k};\omega_1,\omega_2) &= -T((\omega_1\wedge_A\omega_2)\omega_{11},\omega_{12},...,\omega_{1k}) \\ &- T(\omega_{11},(\omega_1\wedge_A\omega_2)\omega_{13},...,\omega_{1k}) \\ &. \\ &. \\ &. \\ &- T(\omega_{11},\omega_{12},...,(\omega_1\wedge_A\omega_2)\omega_{1k}), \end{aligned} \tag{2.21}$$

for all $\omega_{11},\omega_{12},...,\omega_{1k},\omega_1,\omega_2\in\chi(M)$, where

$$(\omega_1\wedge_A\omega_2)\omega_3 = A(\omega_2,\omega_3)\omega_1 - A(\omega_1,\omega_3)\omega_2. \tag{2.22}$$

3. Invariant submanifolds of an almost $\alpha-$cosymplectic (k,μ,ν)-space

Now, let M be an immersed submanifold of $\widetilde{M}^{2n+1}$ and M be an almost $\alpha-\text{cosymplectic}$ $(k,\mu,\nu)-$space. If $\phi(T_{\omega_{11}}M)\subseteq T_{\omega_{11}}M$, for each point at $\omega_{11}\in M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ with respect to ϕ. Following, it will be clear that a submanifold that is invariant with respect to ϕ is also invariant with respect to h.

Proposition 3.1. *ξ is tangent to M, let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Hence, the following equalities hold on M ;*

$$\begin{aligned} R(\omega_1,\omega_2)\xi &= k[\eta(\omega_2)\omega_1-\eta(\omega_1)\omega_2]+\mu[\eta(\omega_2)h\omega_1-\eta(\omega_1)h\omega_2] \\ &\quad +\nu[\eta(\omega_2)\phi h\omega_1-\eta(\omega_1)\phi h\omega_2] \end{aligned} \tag{3.1}$$

$$(\nabla_{\omega_1}\phi)\omega_2 = g(\alpha\phi\omega_1+h\omega_1,\omega_2)\xi-\eta(\omega_2)(\alpha\phi\omega_1+h\omega_1) \tag{3.2}$$

$$\nabla_{\omega_1}\xi = -\alpha\phi^2\omega_1-\phi h\omega_1 \tag{3.3}$$

$$\phi\sigma(\omega_1,\omega_2)=\sigma(\phi\omega_1,\omega_2)=\sigma(\omega_1,\phi\omega_2), \quad \sigma(\omega_1,\xi)=0, \tag{3.4}$$

where ∇,σ and R stand for M's Levi-Civita connection, shape operator and the Riemannian curvature tensor on M, respectively.

Proof. As the proof is a consequence of straightforward, we omit it. □

We shall assume for the remainder of this work that M is an invariant submanifold of an $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. From (2.5), we have in this instance

$$\phi h\omega_1 = -h\phi\omega_1, \tag{3.5}$$

for all $\omega_1\in\Gamma(TM)$, in other words M is also invariant with respect to the tensor field h.

Theorem 3.1. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g,W_1^*\cdot\sigma)=0$ if and only if M is either totally geodesic or $\mu^2+\nu^2=0$.*

Proof. We suppose that $Q(g,W_1^*\cdot\sigma)=0$. This means that

$$(W_1^*(\omega_1,\omega_2)\cdot\sigma)((\omega_3\wedge_g\omega_6)\omega_4,\omega_5)+(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,(\omega_3\wedge_g\omega_6)\omega_5)=0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6\in\Gamma(TM)$, which implies that

$$\begin{aligned} &(W_1^*(\omega_1,\omega_2)\cdot\sigma)+(g(\omega_4,\omega_6)\omega_3-g(\omega_3,\omega_4)\omega_6,\omega_5)+(W_1^*(\omega_1,\omega_2)\cdot\sigma) \\ &+(\omega_4,g(\omega_5,\omega_6)\omega_3-g(\omega_3,\omega_5)\omega_6)=0. \end{aligned} \tag{3.6}$$

In (3.6), putting $\omega_2=\omega_4=\omega_3=\omega_5=\xi$ and using (2.18), (2.19),(3.1), we observe

$$\begin{aligned}
&(W_1^*(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi-\omega_6,\xi)=(W_1^*(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi,\xi)\\
&-(W_1^*(\omega_1,\xi)\cdot\sigma)(\omega_6,\xi)\\
=\;&R^{\perp}(\omega_1,\xi)\sigma(\eta(\omega_6)\xi,\xi)-\sigma(\eta(\omega_6)W_1^*(\omega_1,\xi)\xi,\xi)\\
&-\sigma(\eta(\omega_6)\xi,W_1^*(\omega_1,\xi)\xi)-R^{\perp}(\omega_1,\xi)\sigma(\omega_6,\xi)\\
&+\sigma(W_1^*(\omega_1,\xi)\omega_6,\xi)+\sigma(\omega_6,W_1^*(\omega_1,\xi)\xi)=0.
\end{aligned}\tag{3.7}$$

In view of (2.6) and (2.16), non-zero components of (3.7) vectors give us

$$\sigma(W_1^*(\omega_1,\xi)\xi,\omega_6)=\sigma(\omega_6,\mu h\omega_1+\nu\phi h\omega_1)=0.\tag{3.8}$$

Also taking $\phi\omega_1$ instead of ω_1 in (3.8) and by virtue of lemma 2.1 and proposition 1, we have

$$-\mu\sigma(h\omega_1,\omega_6)+\nu\sigma(h\omega_1,\omega_6)=0.\tag{3.9}$$

Equations (3.8) and (3.9) implies that

$$\mu^2+\nu^2=0 \text{ or } \sigma=0.$$

This proves our assertion. □

Theorem 3.2. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,W_1^*\cdot\sigma)=0$ if and only if M is either totally geodesic or $2nk(\mu^2+\nu^2)=0$.*

Proof. We believe that $Q(S,W_1^*\cdot\sigma)=0$, which follows that

$$Q(S,W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_5;\omega_3,\omega_6)=0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6\in\Gamma(TM)$, by virtue of (2.19) and (2.21), we obtain

$$\begin{aligned}
&S(\omega_3,\omega_4)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_6,\omega_5)-S(\omega_6,\omega_4)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_3,\omega_5)\\
&+S(\omega_3,\omega_5)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_6)\\
&-S(\omega_6,\omega_5)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_3)=0.
\end{aligned}\tag{3.10}$$

Expanding (3.10) and putting $\omega_2=\omega_4=\omega_3=\omega_5=\xi$, non-zero components is

$$2nk\sigma(\omega_6,W_1^*(\omega_1,\xi)\xi)=0.\tag{3.11}$$

As a result, by combining the previous equation and applying (2.20), we reach

$$2nk\mu\sigma(\omega_6,\mu h\omega_1)+2nk\nu\sigma(\omega_6,\phi h\omega_1)=0.\tag{3.12}$$

On the other hand, substituting $\phi\omega_1$ for ω_1 in (3.12) and taking into account (2.7) and (3.4), we conclude that $2nk\left[(\mu^2+\nu^2)\right]\sigma(h\omega_1,\omega_6)=0$,which follows that, $2nk(\mu^2+\nu^2)=0$ or $\sigma=0$.
Thus proof is completed. □

Theorem 3.3. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g,W_7\cdot\sigma)=0$ if and only if M is either totally geodesic or $\left[k^2+(k+\alpha^2)(\mu^2+\nu^2)\right]=0$.*

Proof. We suppose that $Q(g,W_7\cdot\sigma)=0$. This means that

$$(W_7(\omega_1,\omega_2)\cdot\sigma)((\omega_3\wedge_g\omega_6)\omega_4,\omega_5)+(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,(\omega_3\wedge_g\omega_6)\omega_5)=0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6\in\Gamma(TM)$, which implies that

$$\begin{aligned}
&(W_7(\omega_1,\omega_2)\cdot\sigma)+(g(\omega_4,\omega_6)\omega_3-g(\omega_3,\omega_4)\omega_6,\omega_5)+(W_7(\omega_1,\omega_2)\cdot\sigma)\\
&+(\omega_4,g(\omega_5,\omega_6)\omega_3-g(\omega_3,\omega_5)\omega_6)=0.
\end{aligned}\tag{3.13}$$

In (3.13), putting $\omega_2 = \omega_4 = \omega_3 = \omega_5 = \xi$ and by using (2.6), (2.20), we observe

$$\begin{aligned} & (W_7(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi-\omega_6,\xi) = (W_7(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi,\xi) \\ & -(W_7(\omega_1,\xi)\cdot\sigma)(\omega_6,\xi) \\ = \; & R^{\perp}(\omega_1,\xi)\sigma(\eta(\omega_6)\xi,\xi) - \sigma(\eta(\omega_6)W_7(\omega_1,\xi)\xi,\xi) \\ & -\sigma(\eta(\omega_6)\xi, W_7(\omega_1,\xi)\xi) - R^{\perp}(\omega_1,\xi)\sigma(\omega_6,\xi) \\ & +\sigma(W_7(\omega_1,\xi)\omega_6,\xi) + \sigma(\omega_6, W_7(\omega_1,\xi)\xi) = 0. \end{aligned} \tag{3.14}$$

In view of (2.17) and (2.20), non-zero components of (3.14) give us

$$\sigma(W_7(\omega_1,\xi)\xi,\omega_6) = \sigma(\omega_6, k\omega_1 + \mu h\omega_1 + \nu\phi h\omega_1) = 0. \tag{3.15}$$

Substituting $\phi\omega_1$ for ω_1 in (3.15) and considering the equations (2.1) and (2.7), then we get

$$k\sigma(\phi\omega_6,\omega_1) - \mu\sigma(\omega_6,\phi h\omega_1) + \nu\sigma(\omega_6, h\omega_1) = 0. \tag{3.16}$$

From (3.15) and (3.16), we conclude that

$$\left[k^2 + (k+\alpha^2)(\mu^2+\nu^2)\right]\sigma(\omega_6, h\omega_1) = 0$$

So, the proof is finished. □

Theorem 3.4. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S, W_7\cdot\sigma) = 0$ if and only if M is either totally geodesic or $2nk\left[k^2 + (k+\alpha^2)(\mu^2+\nu^2)\right] = 0$.*

Proof. Let us assume that $Q(S, W_7\cdot\sigma) = 0$. It follows that

$$Q(S, W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_5;\omega_3,\omega_6) = 0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6 \in \Gamma(TM)$, by virtue of (2.17) and (2.20), we deduce that

$$\begin{aligned} & S(\omega_3,\omega_4)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_6,\omega_5) - S(\omega_6,\omega_4)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_3,\omega_5) \\ & +S(\omega_3,\omega_5)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_6) - S(\omega_6,\omega_5)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_3) = 0. \end{aligned} \tag{3.17}$$

By setting $\omega_2 = \omega_4 = \omega_3 = \omega_5 = \xi$ in the last equation and it non-zero components is

$$2nk\sigma(\omega_6, W_7(\omega_1,\xi)\xi) = 0. \tag{3.18}$$

On the other hand (3.18) can be written as follows:

$$2nk\left[k\sigma(\omega_6,\omega_1) + \mu\sigma(\omega_6,h\omega_1) + \nu\sigma(\omega_6,\phi h\omega_1)\right] = 0. \tag{3.19}$$

In the same way, by using (3.15) and (3.16), we get
$2nk\left[k^2 + (k+\alpha^2)(\mu^2+\nu^2)\right]\sigma(h\omega_1,\omega_6) = 0$, this means that,
$2nk\left[k^2 + (k+\alpha^2)(\mu^2+\nu^2)\right] = 0$ or $\sigma = 0$.
This proves our assertion. □

Example 3.1. Let $M = \{(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) \in \mathbb{R}^5, \omega_5 \neq \pm 1, 0\}$ and we take

$$\begin{aligned} e_1 &= (\omega_5+1)\frac{\partial}{\partial\omega_1}, \quad e_2 = \frac{1}{\omega_5-1}\frac{\partial}{\partial\omega_2}, \quad e_3 = \frac{1}{2}(\omega_5+1)^2\frac{\partial}{\partial\omega_3}, \\ e_4 &= \frac{5}{\omega_5-1}\frac{\partial}{\partial\omega_4}, \qquad e_5 = (\omega_5-1)\frac{\partial}{\partial\omega_5} \end{aligned}$$

are linearly independent vector fields on M. We also definite $(1,1)-$type tensor field ϕ by $\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = e_4$, $\phi e_4 = -e_3$ and $\phi e_5 = 0$.

Furthermore, the Riemannian metric tensor g is given by

$$g(e_i,e_j) = \{1,\ i=j; \quad 0,\ i\neq j\}.$$

By direct computations, we can easily to see that

$$\phi^2\omega_1 = -\omega_1 + \eta(\omega_1)\xi, \;\; \eta(\omega_1) = g(\omega_1, \xi)$$

and

$$g(\phi\omega_1, \phi\omega_2) = g(\omega_1, \omega_2) - \eta(\omega_1)\eta(\omega_2).$$

Thus $M^5(\phi, \xi, \eta, g)$ is a 5-dimensional almost contact metric manifold. From the Lie-operatory, we have the non-zero components

$$\begin{aligned} [e_1, e_5] &= -(\omega_5 - 1)e_1, \;\; [e_2, e_5] = (\omega_5 + 1)e_2, \quad [e_3, e_5] = -(\omega_5 - 1)e_3, \\ [e_4, e_5] &= (\omega_5 + 1)e_4. \end{aligned}$$

Furthermore, by ∇, we denote the Levi-Civita connection on M, by using Koszul's formula, we can reach at the non-zero components

$$\begin{aligned} \nabla_{e_1}e_5 &= -(\omega_5 - 1)e_1, \quad \nabla_{e_2}e_5 = (\omega_5 + 1)e_2, \quad \nabla_{e_3}e_5 = -(\omega_5 - 1)e_3, \\ \nabla_{e_4}e_5 &= (\omega_5 + 1)e_4. \end{aligned}$$

Comparing the above relations with

$$\nabla_{\omega_1}e_5 = \omega_1 - \eta(\omega_1)e_5 - \phi h\omega_1,$$

we can observe

$$he_1 = -\omega_5 e_2, \quad he_2 = -\omega_5 e_1, \quad he_3 = -\omega_5 e_4, \quad he_4 = -\omega_5 e_3 \text{ and } \;\; he_5 = 0.$$

By direct calculations, we get

$$\begin{aligned} R(e_1, e_5)e_5 &= ke_1 + \mu he_1 + \nu\phi he_1 = 2(\omega_5 - 1)e_1, \\ R(e_2, e_5)e_5 &= ke_2 + \mu he_2 + \nu\phi he_2 = -2w_5(\omega_5 + 1)e_2, \\ R(e_3, e_5)e_5 &= ke_3 + \mu he_3 + \nu\phi he_3 = 2(\omega_5 + 1)e_3, \end{aligned}$$

and

$$R(e_4, e_5)e_5 = ke_4 + \mu he_4 + \nu\phi he_4 = -2w_5(\omega_5 + 1)e_4,$$

which imply that $k = -(\omega_5 + 1)$, $\mu = 0$ and $\nu = 2 - \frac{1}{\omega_5} + \omega_5$.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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