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A Difference Equation of Banking Loan with Nonlinear Deposit Interest Rate

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Abstract

This paper considers a banking loan model using a difference equation with a nonlinear deposit interest rate. The construction of the model is based on a simple bank balance sheet composition and a gradient adjustment process. The model produces two unstable loan equilibriums and one stable equilibrium when the parameter corresponding to the deposit interest rate is situated between its transcritical and flip bifurcations. Some numerical simulations are presented to align with the analytical findings, such as the bifurcation diagram, Lyapunov exponent, cobweb diagram, and contour plot sensitivity. The significance of our result is that the banking regulator may consider the lower and upper bounds for setting the nonlinear interest rate regulation and provide a control regulation for other banking factors to maintain loan stability.

1. Introduction

In recent years, several researchers have used difference equations to study the dynamics of banking loans with various factors appearing in the banking system. For monopoly cases, authors in [1, 2] study the effect of banking deposit and loan costs on banking loan dynamics, authors in [3, 4] investigate the influence of capital policy and dividend payment on banking loan dynamics by incorporating economies and dis-economies of scope, authors in [5] explore the dynamics of the loan with reserve requirement policy, authors in [6] inspect the banking loan dynamics with a macro-prudential policy in Indonesia in the form of reserve requirement based on loan-to-deposit ratio, authors in [7] research the impact of the amount of premium and membership contribution policy from the Indonesia deposit insurance corporation on banking loan dynamics, and authors in [8] analyze loan benchmark interest rates in banking loan dynamics. For duopoly cases, the study is initiated by Fanti [9] where capital regulation is considered under heterogeneous and homogeneous models, and it is followed by authors in [10, 11] where they study capital regulation with banking cost under the case of Italian banks.

In this paper, we consider a monopoly model to analyze the dynamics of a single bank's loan when its deposit interest rate is assumed to be nonlinear. The nonlinear interest rate is induced in paper [11], but in the paper, the nonlinear aspect arises in loan interest rate. In economic analysis, nonlinear interest rates have been studied for decades. For example, the study in [12] investigates nonlinear dynamics in the US short-term interest rate time series, authors in [13] focus on Spanish banks' nonlinear interest rate sensitivity, the study in [14] shows that a stabilizing influence on interest rates is compatible with a nonlinear process, and authors in [15, 16] evaluate nonlinear interest rate response functions for South Africa and the United Kingdom.

In the next section, we construct the banking loan model with the nonlinear deposit interest rate, and it is followed by the analysis of its equilibrium. The following section presents several numerical simulations to confirm the analysis. The last section concludes.

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2. Model Construction

Consider a bank with the following simple balance sheet components: loan (L), capital (E), and deposit (D). The bank balance sheet's identification offers

$$L = D + E$$

The capital regulation requires every bank should maintain its capital not less than a certain percentage of its assets. Since in this model, we only consider loan as the only asset, thus we have $E \geq \kappa L$, where $0 < \kappa < 1$. For the purpose of simplification of the model's analysis, suppose that $E = \kappa L$. Thus, we obtain

$$D = (1-\kappa)L$$

Usually, the loan interest is subtracted by the banking expenses to find the bank's profit. The costs contain deposit interest, capital dividend, and operating costs. In this paper, we use a simpler profit calculation as follows

$$\pi = rL - r_D D \tag{2.1}$$

where π is profit, r is the loan interest rate, and r_D is the deposit interest rate. We assume that $r > 0$ is a constant rate, meanwhile, r_D is a nonlinear rate

$$r_D = \frac{1}{a-D} \tag{2.2}$$

where $a > D > 0$. The nonlinear deposit interest rate formulation in (2.2) still meets the condition in the well-known Monti-Klein model [17, 18], that it must have a slope. We have

$$\frac{dr_D}{dD} = \frac{1}{(a-D)^2} > 0$$

Substituting (2.2) into (2.1) produces

$$\pi = rL - \frac{1}{a-D}D = rL - \frac{(1-\kappa)L}{a-(1-\kappa)L}$$

Next, we calculate the profit marginal as follows

$$\frac{\partial \pi}{\partial L} = r - \frac{(1-\kappa)a}{[a-(1-\kappa)L]^2}$$

To model the dynamics of banking loans, we follow the gradient adjustment process [19], that is

$$L_{t+1} = L_t + \alpha L_t \frac{\partial \pi_t}{\partial L_t}$$

where $\alpha > 0$. The final model is provided below

$$L_{t+1} = f(L_t) := L_t + \alpha L_t \left(r - \frac{(1-\kappa)a}{[a-(1-\kappa)L_t]^2} \right) \tag{2.3}$$

3. Analysis

The map (2.3) has three equilibrium points, namely

$$L_0^* = 0, \quad L_1^* = \frac{a + \sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}, \quad \text{and} \quad L_2^* = \frac{a - \sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}$$

It can be seen that $L_1^* > 0$. But, for the case of L_2^*, we need a condition that will guarantee $L_2^* > 0$, that is

$$a > \frac{1-\kappa}{r} \tag{3.1}$$

To study the stability of the equilibriums, we will use the following lemma. But, before that, first, we need to calculate $f'(L_t)$. We have

$$f'(L_t) = 1 + \alpha r - \alpha a(1-\kappa)\left(\frac{a + (1-\kappa)L_t}{[a-(1-\kappa)L_t]^3} \right)$$

Lemma 3.1 ([20]). *Let x^* be an equilibrium point of the difference equation*

$$x_{n+1} = f(x_n)$$

where f is continuously diffrentiable at x^*. The following statements then hold true:

i. If $\left|f'(x^*)\right| < 1$, then x^* is locally asymptotically stable.

ii. If $\left|f'(x^*)\right| > 1$, then x^* is locally unstable.

The case when $f'(L^*)=1$, there happens a transcritical bifurcation, meanwhile, a flip bifurcation happens if $f'(L^*)=-1$ [21]. The following theorems provide information about each equilibrium's stability.

Theorem 3.2. *The equilibrium point $L_0^*=0$ of the model (2.3) is unstable for all parameters.*

Proof. By substituting $L_0^*=0$ into $f'(L^*)$, we have

$$f'(0)=1+\alpha\left(r-\frac{1-\kappa}{a}\right)$$

From condition in (3.1), we have $r-\frac{1-\kappa}{a}>0$. Thus, $f'(0)>1$. This means that $L_0^*=0$ is unstable. □

Theorem 3.3. *For all parameters, the equilibrium point $L_1^*=\frac{a+\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}$ of the model (2.3) is unstable.*

Proof. Substituting $L_1^*=\frac{a+\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}$ into $f'(L^*)$ produces

$$f'\left(\frac{a+\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}\right)=1+2\alpha r\left(\sqrt{\frac{ar}{1-\kappa}}+1\right)$$

It is clear that $f'\left(\frac{a+\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}\right)>1$, since all parameters are positive and $0<\kappa<1$. Thus, L_1^* is unstable. □

Theorem 3.4. *The equilibrium point $L_2^*=\frac{a-\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}$ of the model (2.3) is locally asymptotically stable if*

$$\frac{1-\kappa}{r}<a<\left(\frac{1-\kappa}{r}\right)\left(\frac{1+\alpha r}{\alpha r}\right)^2$$

Proof. We have

$$f'\left(\frac{a-\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}\right)=1+2\alpha r\left(1-\sqrt{\frac{ar}{1-\kappa}}\right)<1$$

if $a>\frac{1-\kappa}{r}$. This condition has already appeared in (3.1). Next, we have

$$f'\left(\frac{a-\sqrt{\frac{(1-\kappa)a}{r}}}{1-\kappa}\right)=1+2\alpha r\left(1-\sqrt{\frac{ar}{1-\kappa}}\right)>-1$$

if $a<\left(\frac{1-\kappa}{r}\right)\left(\frac{1+\alpha r}{\alpha r}\right)^2$. Thus, $|f'(L_2^*)|<1$ if $\frac{1-\kappa}{r}<a<\left(\frac{1-\kappa}{r}\right)\left(\frac{1+\alpha r}{\alpha r}\right)^2$. □

From Theorem 3.4, we have the following corollary.

Corollary 3.5. *The equilibrium L_2^* might become unstable due to transcritical and flip bifurcations when $a=a^T:=\frac{1-\kappa}{r}$ and $a=a^F:=\left(\frac{1-\kappa}{r}\right)\left(\frac{1+\alpha r}{\alpha r}\right)^2$.*

Based on Theorem 3.4 and Corollary 3.5, we can say that the stable region of (α,a)-parameter space lies in between the spaces $\{(\alpha,a):a<\frac{1-\kappa}{r}\}$ and $\{(\alpha,a):a>(\frac{1-\kappa}{r})(\frac{1+\alpha r}{\alpha r})^2\}$. This result is illustrated in Figure 4.1.

4. Numerical Simulation

For simulation purposes, we use parameter values $r=0.2$, $\kappa=0.08$, and $\alpha=50$, while a varies across the simulations. The reason for choosing the parameter values is only to display the existence of transcritical and flip bifurcations, and also the complex dynamics of the map (2.3).

The bifurcation diagram of parameter a for $a\in[4.6;6.1]$ is shown in Figure 4.2a. When a gets bigger, the loan equilibrium goes up until it reaches the flip bifurcation point, and then it produces a period-doubling and leads to chaos. The confirmation of the existence of chaotic behavior can be seen also in the Lyapunov exponent of map (2.3) as shown in Figure 4.2b by the red marker, as we know that the positive Lyapunov exponent can show the existence of chaos.

Another way to depict the complex dynamics of map (2.3) is by presenting the cobweb diagram. In Figure 4.3, we simulate four scenarios for the cobweb diagram and the respected time series of L_t. First, a stable banking loan is presented in Figure 4.3a for $a=5$. Second, a display of two-period of banking loan is depicted in Figure 4.3b for $a=5.6$. Third, a four-period banking loan dynamics for $a=5.85$ as shown in Figure 4.3c. The last is for $a=6$ which shows the chaotic dynamics of banking loan, see Figure 4.3d.

The next simulation is a numerical sensitivity analysis to see the effect of changes of two parameters simultaneously on the stability of a banking loan. Before that, we define a function $S=\left(\frac{ar}{1-\kappa}\right)\left(\frac{\alpha r}{1+\alpha r}\right)^2$. Then, the stability condition in Theorem 3.4 can seen as $\left(\frac{\alpha r}{1+\alpha r}\right)^2<S<1$. We perform a contour plot S to see the impact of a combination of two parameters on banking loan stability by looking at the region where S has a value less than 1. The contour plot of S is displayed in Figure 4.4. This enables us to observe which region can guarantee a stable banking loan, and also it can be used for controlling the stability of banking loan.

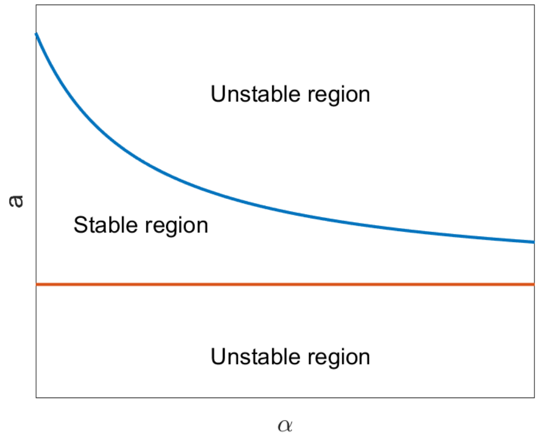


Figure 4.1: Stability region of (α, a)-parameter space for map (2.3).

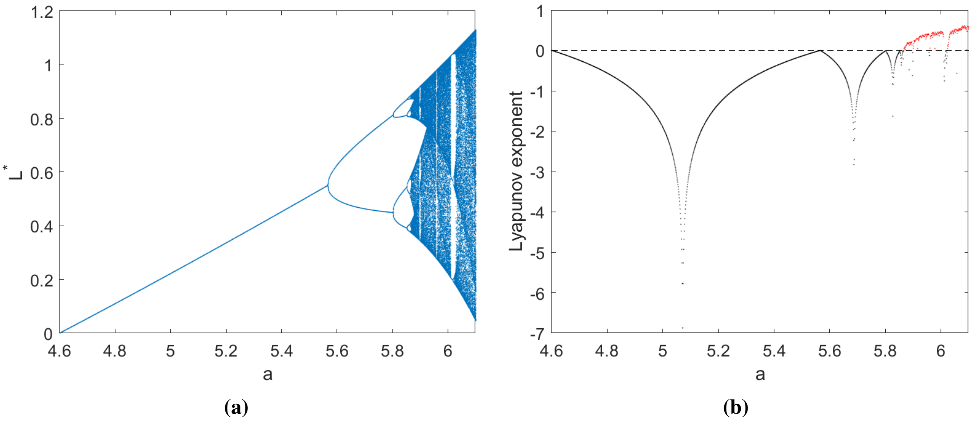


Figure 4.2: (a) The parameter a bifurcation diagram, and (b) the associated Lyapunov exponent.

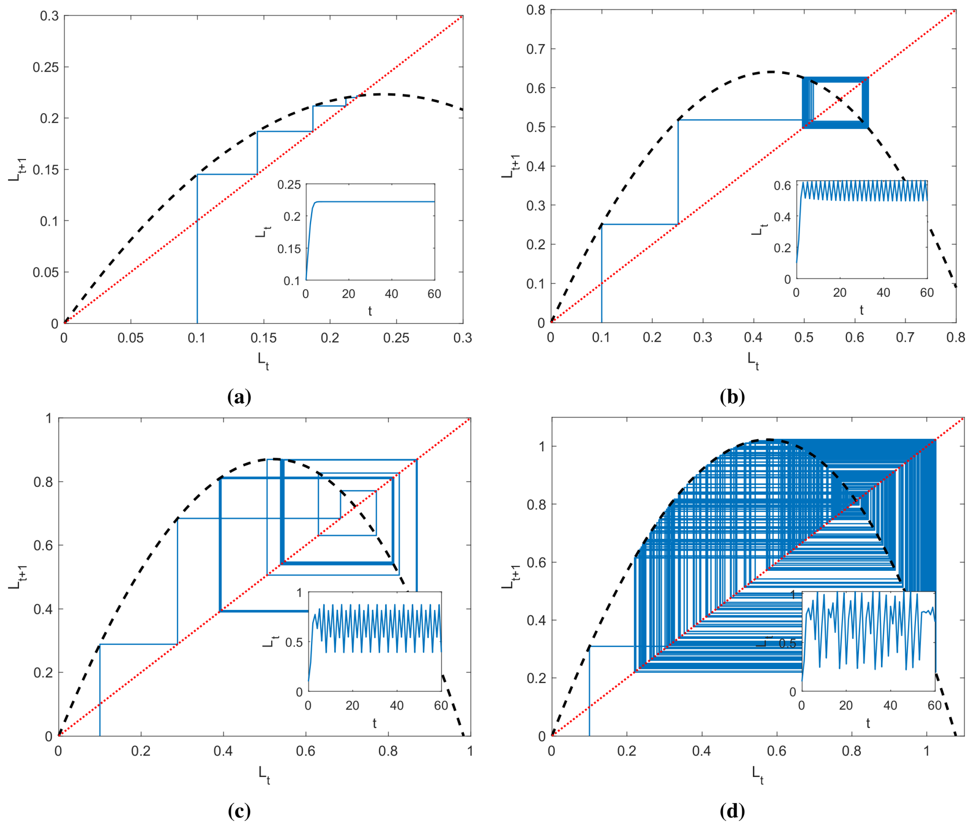
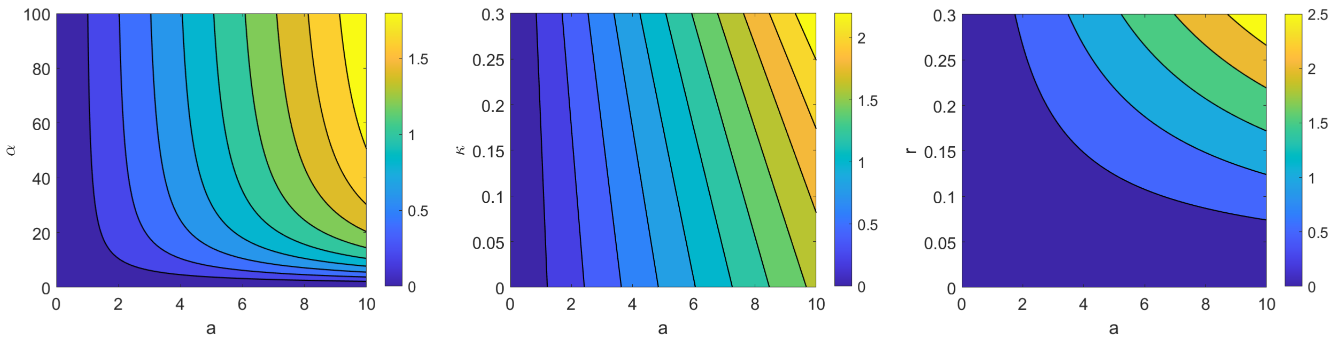


Figure 4.3: Cobweb diagram of map (2.3) and time series of L_t when (a) $a = 5$, (b) $a = 5.6$, (c) $a = 5.85$, and (d) $a = 6$.



(a) **(b)** **(c)**

Figure 4.4: Contour plot of S for different parameter spaces.

5. Discussion and Conclusion

The introduction of nonlinear deposit interest rate in the banking system can produce rich dynamics. The result shows that the bifurcation parameter is crucial to affect the banking loan dynamics. A higher bifurcation parameter means that the deposit interest rate becomes lower. When this happens, it can cause a period-doubling banking loan and even lead to chaos. To control this phenomenon, we can use the other parameters to reduce the risk of unstable banking loans by observing the sensitivity analysis. In our analysis, we can draw a comparison to the linear deposit interest rate that has been examined in previous research [22]. The authors of these studies analyze a deposit benchmark interest rate that is incorporated into the linear interest rate. They demonstrate that a decrease in the linear deposit interest rate can lead to periodic or even chaotic dynamics in banking loans. Our findings align with the fact that a greater value of the nonlinear interest rate parameter (equal to a lower interest rate) may indicate the presence of periodic or chaotic dynamics in banking loans. The model described in [22] can be reformulated as a logistic map. However, our present model cannot be converted into a logistic map, while it still yields comparable findings. This study only uses a very simple model that consists of deposit, equity, and loan, and of course, it is a monopoly model. Thus, for upcoming research, the addition of other bank balance sheet variables such as reserve requirement policy and/or the case of the duopoly model can be considered.

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