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Laplace ve adomian ayrışma metodları ile Benney-Luke denkleminin çözümünü elde etme

ÖZ

Bu çalışmada, başlangıç değerleri verilen homojen olmayan Benney-Luke denklemini ele aldık. Laplace ve Adomian ayrışma metotları bu denkleme uygulanmıştır. Daha sonra, bu denklemin verilen başlangıç değerini sağlayan çözümü elde edilmiştir.

Anahtar Kelimeler: laplace ayrışma Metodu, Adomian ayrışma Metodu, Benney-Luke denklemi, parazit terimler.

Obtaining the solution of Benney-Luke Equation by Laplace and adomian decomposition methods

ABSTRACT

In this study, we consider the inhomogeneous Benney-Luke equation with its initial conditions. Laplace Decomposition Method and Adomian Decomposition Method are applied to this equation. Then, the solution yielding the given initial conditions is gained.

Keywords: Laplace Decomposition Method, Adomian Decomposition Method, Benney-Luke Equation, The Noise terms.

1. INTRODUCTION

In most areas of science and engineering, a great deal of issues can be expressed as nonlinear partial differential equations (shortly, N/LPDEs). To get more information in these areas, solutions of these kind of equations should be attained. Therefore, gaining solutions to these type of equations is getting more and more important in modern science. For this purpose, countless search have been made. At last, a lots of analytical and numerical methods have been built. Some of them related to this study are the perturbation method [1-2], the homotopy perturbation method [3-5], the delta perturbative method [6], the Adomian Decomposition method [7-12], the Modified Decomposition method [13-15] and the Laplace Decomposition method [16-20].

The purpose of this study is solving the nonlinear Benney-Luke equation by LDM and ADM. Then, it is shown that LDM and ADM gives the same solution. In addition, application of these methods demonstrates the efficiency, effectiveness, usefulness and simplicity of the methods in solving NLPDEs.

The nonlinear Benney-Luke equation [21] is a Sobolev type equation given by

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0 \quad (1)$$

where a and b are positive numbers, suct that $a - b = \sigma - \frac{1}{3}$. Here, σ is named the Bond number, which captures the effects of surface tension and gravity force. It is formally valid approximation for describing two-way water wave propagation in the presence of surface tension.

In this study, we first consider the following inhomogeneous form of the equation (1)

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 2t \quad (2)$$

with the initial conditions

$$u(x,0) = 1 \text{ and } u_t(x,0) = x. \quad (3)$$

In this application, we take the advantage of the noise terms [22]. It gives us the opportunity to see a fast convergence of the solution. The noise terms phenomenon may show up in only inhomegeneous PDEs. And also, this is applicable to all kind of inhomegeneous PDEs with any order. The noise terms usually provide us with the solution after two consecutive iterations, if the noise terms exist in the components u_0 and u_1.

The paper is organized as follows. In chapter 2, the decomposition methods, mentioned above, are described for the general form of equation (2) and its initial value (3). Then, these methods are applied to the equation (4) in chapter 3. In final chapter, a short conclusion is given.

2. OUTLINES OF THE METHODS

In this part, it is given the main lines of the Adomian and Laplace decomposition methods.

2.1. Laplace Decomposition Method (LDM)

This method is given as in the following manner. Let us consider the nonlinear partial differential equation in operator form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad (4)$$

with initial conditions $u(x,t) = f(x)$ and $u_t(x,0) = g(x)$.

In equation (4), D is a second order partial differantial operator w.r.t t, R is a remaining linear operator, N represents a general nonlinear differantial operator, $h(x,t)$ is called the source term.

Firstly, the laplace transform is applied to both sides of equation (4). Then, we have

$$L[Du(x,t) + Ru(x,t) + Nu(x,t)] = L[h(x,t)] \quad (5)$$

By the differentiation property of the Laplace transform, we get

$$L[u(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)] - \frac{1}{s^2}(L[Nu(x,t)] + L[Ru(x,t)]) \quad (6)$$

where s is Laplace domain function.

In this method, the linear terms $u(x,t)$ is defined as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (7)$$

The nonlinear terms $Nu(x,t)$ is given by infinite series

$$N\big(u(x,t)\big) = \sum_{n=0}^{\infty} A_n(x,t) \quad (8)$$

where $A_n(x,t)$ are the Adomian polynımials.

Adomian polynomials are defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\alpha^n}[N(\textstyle\sum_{n=0}^{\infty} \alpha^i u_i)]_{\alpha=0}, \quad n = 0,1,2,3,\ldots \quad (9)$$

By using this definition, the first few terms can be given as follows:

$$A_0 = N(u_0)$$

$$A_1 = N'(u_0)u_1 \quad (10)$$

$A_2 = N'(u_0)u_2 + N''(u_0)\frac{{u_1}^2}{2!}$

and other terms can be derived in a similar way.

Now, substituting (7) and (8) into equation (6) gives us

$$L[\sum_{n=0}^{\infty} u_n(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)] - \frac{1}{s^2}L[\sum_{n=0}^{\infty} A_n(x,t)] - \frac{1}{s^2}L[Ru(x,t)] \,. \quad (11)$$

Linearity of the Laplace transform converts it into the following form:

$$\sum_{n=0}^{\infty} L[u_n] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)] - \frac{1}{s^2}\sum_{n=0}^{\infty} L[A_n(u)] - \frac{1}{s^2}L[Ru(x,t)] \,. \quad (12)$$

Comparing both sides of (12) yields the following equalities:

$$L(u_0) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)],$$

$$L(u_1) = -\frac{1}{s^2}L[Ru_0] - \frac{1}{s^2}\sum_{n=0}^{\infty} L[A_0(u)], \quad (13)$$

$$L(u_2) = -\frac{1}{s^2}L[Ru_1] - \frac{1}{s^2}\sum_{n=0}^{\infty} L[A_1(u)],$$

and so on.

In general, the recursive relation is given by

$$L(u_{k+1}) = -\frac{1}{s^2}L[Ru_k] - \frac{1}{s^2}\sum_{n=0}^{\infty} L[A_k(u)],\ k \geq 0 \,. \quad (14)$$

Suppose that the inverse Laplace transform exists. We take the inverse Laplace transform of (13) and (14). Then, we gain the components, $u_0, u_1, u_2, \ldots, u_{n+1}$. Putting these components in the expansion (7) gives the solution rapidly.

2.2. Adomian Decomposition Method (ADM)

Let us consider the nonlinear partial differential equation (4). It can be written as

$$L_t u(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad (15)$$

where $L_t = \frac{\partial}{\partial t^2}$.

Firstly, let us rewrite the equation (15) as

$$L_t u(x,t) = h(x,t) - Ru(x,t) - Nu(x,t). \quad (16)$$

Then, applying ${L_t}^{-1}$ to both sides of this equation and using the initial conditions gives us

$$u(x,t) = f(x) + tg(x) + {L_t}^{-1}h(x,t) - {L_t}^{-1}Ru(x,t) - {L_t}^{-1}Nu(x,t) \quad (17)$$

where ${L_t}^{-1} = \int_0^t \int_0^t (.)dtdt$.

In this method, the same expansions (7) and (8) are used as well.

That is, the linear terms $u(x,t)$ is defined as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

And also, the nonlinear terms $Nu(x,t)$ is given by infinite series

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(x,t)$$

where $A_n(x,t)$ are the Adomian polynımials.

We obtain the recursive relation:

$$u_0(x,t) = f(x) + tg(x) + {L_t}^{-1}h(x,t), \quad (18)$$

$$u_{k+1}(x,t) = -{L_t}^{-1}Ru_k - {L_t}^{-1}A_k,\ k \geq 0. \quad (19)$$

This relations provide us with the components of the solutions.

The first few terms of solution are given as

$$u_1(x,t) = -{L_t}^{-1}Ru_0 - {L_t}^{-1}A_0, \quad (20)$$

$$u_2(x,t) = -{L_t}^{-1}Ru_1 - {L_t}^{-1}A_1. \quad (21)$$

After getting the components, we put them in the expansion (7). Then, the solution $u(x,t)$ is obtained.

3. APPLICATION OF THE METHODS

3.1. Solving B-L equation by LDM

Applying the Laplace transform both sides of the equation (2) exposed to the condition (3), we get

$$L[u_{tt}] = L[2t] + L[u_{xx}] - aL[u_{xxxx}] + bL[u_{xxtt}] - L[N(u)] \quad (22)$$

where $N(u) = u_t u_{xx} + 2u_x u_{xt}$.

Then, using the differentiation property of Laplace transformation and initial conditions gives

$$L[u] = \frac{1}{s} + \frac{x}{s^2} + \frac{1}{s^2}L[2t] + \frac{1}{s^2}(L[u_{xx}] - aL[u_{xxxx}] + bL[u_{xxtt}]) - \frac{1}{s^2}L[N(u)]. \quad (23)$$

Now, let us use the expansion (7) for the linear terms and the expansion (8) for the nonlinear ones. Then, we have the following recursive relation:

$$L(u_0) = \frac{1}{s} + \frac{x}{s^2} + \frac{1}{s^2}L[2t], \quad (24)$$

$$L(u_{k+1}) = \frac{1}{s^2}(L[u_{xx}] - aL[u_{xxxx}] + bL[u_{xxtt}]) - \frac{1}{s^2}L[A_k], k \geq 0. \quad (25)$$

Taking the inverse Laplace operator of both sides of (24) and (25) gives,

$$u_0 = 1 + xt + L^{-1}[\frac{1}{s^2}L[2t]] \quad (26)$$

$$u_{k+1} = L^{-1}\left[\frac{1}{s^2}(L[u_{xx}] - aL[u_{xxxx}] + bL[u_{xxtt}])\right] - L^{-1}\left[\frac{1}{s^2}L[A_k]\right],\ k \geq 0, \quad (27)$$

where $A_k(u) = u_{k_t}u_{k_{xx}} + 2u_{k_x}u_{k_{xt}}$.

Then, the components are gained as

$$u_0 = 1 + xt + \frac{t}{3}, \quad (28)$$

$$u_1 = -\frac{t}{3}, \quad (29)$$

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If we compare the first two components, it is seen that $\frac{t}{3}$ is the common term. This common term is called the noise term. Deleting the noise term from the first components gives the solution

$$u = 1 + xt. \quad (30)$$

3.2. Solving B-L equation by ADM

First, let us write the equation (2) as

$$L_t u(x,t) = 2t + u_{xx} - au_{xxxx} + bu_{xxtt} - N(u) \quad (31)$$

where L_t is a second order partial differential operator w.r.t t and $N(u) = u_t u_{xx} + 2u_x u_{xt}$.

Here, let us suppose that ${L_t}^{-1}$ exists. Then, applying it to both sides of the equation (31) and considering the conditions (3) gives

$$u(x,t) = 1 + xt + {L_t}^{-1}(2t) + {L_t}^{-1}(u_{xx} - au_{xxxx} + bu_{xxtt}) - {L_t}^{-1}(N(u)) \quad (32)$$

where ${L_t}^{-1} = \int_0^t \int_0^t (.)dtdt$.

Using the decomposition series for the linear terms, Adomian polynomial for the nonlinear terms and the realations (18)-(19) gives

$$u_0(x,t) = 1 + xt + {L_t}^{-1}(2t), \quad (33)$$

$$u_1(x,t) = {L_t}^{-1}(u_{0_{xx}} - au_{0_{xxxx}} + bu_{0_{xxtt}}) - {L_t}^{-1}(A_0) \quad (34)$$

and so on for the other components.

Then, we obtain

$$u_0(x,t) = 1 + xt + \frac{t}{3} \quad (35)$$

$$u_1 = -{L_t}^{-1}(2t) = -\frac{t}{3}, \quad (36)$$

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It is easily seen that $\frac{t}{3}$ is the noise term. Deleting this term from $u_0(x,t)$ provides us with the solution

$$u = 1 + xt. \quad (37)$$

4. CONCLUSION

In this study, we consider the B-L equation with its initial conditions. To obtain the solution of this equation, ADM and LDM are applied. Then, it is seen that the same exact solution is gained by both methods. In addition to their effectiveness and usefulness in solving linear partial differential equations, it is shown that these decomposition methods are powerful in solving nonlinear ones with initial and boundary conditions.

Compared to other methods for solving nonlinear partial differential equations with initial conditions, there is no need for linearization of nonlinear terms thanks to the Adomian polynomials. Moreover, we can easily and rapidly attain the solution with the help of the noise terms.

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