Sinopfbd, 2024; 9(1): 167-183 https://doi.org/10.33484/sinopfbd.1401042

Sinop Üniversitesi Fen Bilimleri Dergisi
Sinop Uni J Nat Sci
E-ISSN: 2564-7873

https://dergipark.org.tr/tr/pub/sinopfbd

Pell Collocation Approach For The Nonlinear Pantograph Differential Equations

Pınar ALBAYRAK

How to cite: Albayrak, P. (2024). Pell collocation approach for the nonlinear pantograph differential equations. *Sinop Üniversitesi Fen Bilimleri Dergisi*, *9*(1), 167-183. https://doi.org/10.33484/sinopfbd.1401042

Research Article

Corresponding Author
Pınar ALBAYRAK
pkanar@yildiz.edu.tr

ORCID of the Author
P.A: 0000-0002-7973-3500

Received: 07.12.2023
Accepted: 27.05.2024

Abstract
Pantograph equations, which we encounter in the branches of pure and applied mathematics such as electrodynamics, control systems and quantum mechanics, are essentially a particular form of the functional differential equations characterized with proportional delays. This study focuses on exploring the approximate solution to the Pantograph differential equation. Since there is no analytic solutions for this equation class, only the approximate solutions can be obtain. For this purpose, Pell Collocation Method which is one of the numerical solution methods is chosen. As the result of applying the method to the equation, an algebraic equation system has been gained and then the approximate solution has been found by using MATHEMATICA via the given initial conditions. The method is applied to the some test examples and then the results are presented by both graphically and by table. The error estimations show that the method works accurately and efficiently.

Keywords: Approximate solution, pantograph differential equation, collocation method

Lineer Olmayan Pantograf Diferansiyel Denklemleri İçin Pell Sıralama Yaklaşımı

Yıldız Technical University,
Department of Mathematics,
Istanbul, Türkiye

Öz
Elektrodinamik, kontrol sistemleri ve kuantum mekaniği gibi teorik ve uygulamalı matematiğin dallarında karşılaştığımız Pantograf denklemleri, orantısal gecikmeli fonksiyonel diferansiyel denklemlerin özel bir türüdür. Bu çalışmada, Pantograf diferansiyel denklemin yaklaşık çözümleri üzerine çalışılmıştır. Bu denklem sınıfı için analitik çözüm olmadığından sadece yaklaşık çözümleri bulunabilir. Bu amaçla sayısal çözüm yöntemlerinden biri olan Pell sıralama yöntemi seçilmiştir. Yöntemin denkleme uygulanması sonucunda bir cebirsel denklem sistemi elde edilmiş ve MATHEMATICA programı kullanılarak verilen başlangıç koşulları ile yaklaşık çözüm bulunmuştur. Bu yöntem bazı test örneklerine uygulanmış ve sonuçlar hem grafiksel olarak hem de tablo olarak ifade edilmiştir. Hata analizleri bu yöntemin doğru ve etkili çalıştığını göstermiştir.

Anahtar Kelimeler: Yaklaşık çözüm, pantograf diferansiyel denklemi, sıralama yöntemi

Introduction

Many problems that we encounter in every aspect of our lives are modeled via mathematics and solutions of the models are one of the main topic for the researchers. Differential equations take an important place in these modeling. One of these differential equations is the Pantograph differential equation. The Pantograph differential equation (PDE) is a special class of the functional differential equations. Actually, Pantograph is a device that collects electric current from electric poles in vehicles such as trains and trams. For the modelling of the problem, PDE was firstly mentioned in the study [1] by J.R. Ockendon and A.B.Tayler in 1971. In this study, the Pantograph of the electric locomotive was modeled. Later, many studies were conducted on this subject. Since the PDE equation does not have an exact solution, only approximate solution can be found. To obtain the approximate solution, various numerical methods such as homotopy methods, Haar wavelets, Legendre approximations, Sinc collocation method can be considered. To get more information about these numerical methods, readers can be look into [2-11]. In recent years different numerical approaches are applied to the PDE. For instance, Sedaghat et. al. applied a numerical approach to find the approximate solution of the PDE with the help of Chebyshev polynomials in [12]. In the work [13] Jafari et. al. provide an efficient transferred Legendre pseudospectral method for solving PDE. M. M. Bahşi and M. Çevik resorted to the Perturbation approach for Pantograph delay differential equation (PDDE) in [14]. R. Alrebdi and H. K. Al-jeaid in [15] examined the PDDE with the help of the Laplace transformation, which is one of the integral transformations. For more work related to PDE see [16-20].

In recent years, some collocation methods to solve the linear and nonlinear differential, integral and pantograph equations have been presented in many articles. Such as Legendre-Gauss collocation method [21], Chebyshev collocation method [22], Fibonacci wavelet collocation method [23], Hermite collocation approach [24], Legendre spectral collocation method [25] and Lagrange-collocation method [26]. In addition to these methods, there are also Pell collocation method and Pell-Lucas collocation methods. These two collocation methods based on Pell and Pell-Lucas polynomials respectively. These polynomials belong to families of orthogonal polynomials and are characterized by recursive expressions and satisfy the following properties [27]: Pell polynomials, $P_n(x)$, are defined as follows

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \ , \ \ n \geqslant 0$$

where $P_0(x) = 0, \ P_1(x) = 1$. Pell-Lucas polynomials, $Q_n(x)$, are defined by

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \ , \ \ n \geqslant 0$$

where $Q_0(x) = 2, \ Q_1(x) = 2x$. There are many studies using methods based on Pell and Pell-Lucas polynomials [28-32]. In our paper, we tried to obtain the approximate solution by applying the Pell collocation method (PCM) to a class of nonlinear PDE (NPDE).

Let us consider a class of NPDE defined as

$$\begin{aligned}&\sum_{s=0}^{n}\sum_{p=0}^{m} R_{sp}(x)v^{p}\left(\alpha_{sp}x+\beta_{sp}\left(x\right)\right)v^{(s)}(\lambda_{sp}x+\gamma_{sp}\left(x\right))\\&+\sum_{s=1}^{n}\sum_{p=1}^{m} Q_{sp}(x)v^{(p)}\left(\alpha_{sp}x+\beta_{sp}\left(x\right)\right)v^{(s)}(\lambda_{sp}x+\gamma_{sp}\left(x\right))=g(x),\ \text{for } a\leq x\leq b\end{aligned} \tag{1}$$

with the initial conditions

$$\sum_{s=0}^{n}\left[a_{js}v^{(s)}(0)+b_{js}v^{(s)}(0)\right]=\delta_{j}, \qquad j=0,1. \tag{2}$$

Here, $v^{(0)}(x)=v(x),\ v^{0}(x)=1$. Also, $v(x)$ is an unknown function; the functions $R_{sp}(x)$, $Q_{sp}(x)$ and $g(x)$ are continuous on $[0,1]$; $a_{js},\ b_{js},\alpha_{sp},\ \lambda_{sp}$ and δ_j are constants. Additionally, $\beta_{sp}\left(x\right)$ and $\gamma_{sp}\left(x\right)$ are either appropriate constants or random variables. In our approach, we will consider the approximate solution to be the truncated Pell series, represented as:

$$v\left(x\right)=\sum_{m=1}^{M+1} c_{m}P_{m}(x). \tag{3}$$

In this series, $P_m(x)$ $(1\leq m\leq M+1)$ are the Pell polynomials; c_m are the coefficients to be found and $M\in\mathbb{Z}^{+}(M\geq n)$. The first few Pell polynomials are as follows:

$$\begin{aligned}P_0(x)=&0,\\P_1(x)=&\mathbf{1},\\P_2(x)=&2x,\\P_3(x)=&\mathbf{4}x^2+\mathbf{1},\\P_4(x)=&8x^3+4x,\\P_5(x)=&16x^4+12x^2+1,\\&\vdots\end{aligned} \tag{4}$$

Fundamental Relations

Let Eq. (3) be our approximate solution of Eq. (1). In order to obtain the approximate solution, we will try to write Eq. (1) in matrix form according to the solution of Eq. (3).
$i)$ If $\alpha_{sp}=\ \lambda_{sp}=1,\beta_{sp}\left(x\right)\ =\gamma_{sp}\left(x\right)=0,$ Pell polynomials in Eq. (4) can be represented in matrix format as

$$\mathbf{P}\left(x\right)=\mathbf{\Gamma}\left(x\right)\mathbf{N}. \tag{5}$$

Here, $P(x) = [P_1(x)\ \ P_2(x) \cdots P_{M+1}(x)]$, $\mathbf{\Gamma}(x) = \left(1\ x\ x^2\ x^3 ... x^M\right)$, $\mathbf{C} = (c_1\ c_2\ \cdots c_{M+1})^T$ and

$$\mathbf{N} = \begin{bmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \cdots \\ 0 & 2 & 0 & 4 & 0 & 6 & 0 & 8 & 0 & \cdots \\ 0 & 0 & \mathbf{4} & 0 & \mathbf{12} & 0 & \mathbf{24} & 0 & \mathbf{40} & \cdots \\ 0 & 0 & 0 & 8 & 0 & 32 & 0 & 80 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mathbf{16} & 0 & \mathbf{80} & 0 & \mathbf{240} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 32 & 0 & 192 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{64} & 0 & \mathbf{448} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{256} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So, we can express Eq. (3) as

$$v(x) = \mathbf{P}(x)\,\mathbf{C}. \tag{6}$$

If we put Eq. (5) in Eq. (6), then $v(x)$ will be

$$\begin{aligned} v(x) &\cong v_M(x) = \mathbf{\Gamma}(x)\,\mathbf{NC}, \\ v'(x) &\cong v'_M(x) = \mathbf{\Gamma}(x)\,\mathbf{BNC}, \\ v''(x) &\cong v''_M(x) = \mathbf{\Gamma}(x)\,\mathbf{B}^2\mathbf{NC}, \\ &\vdots \\ v^{(s)}(x) &\cong v^{(s)}_M(x) = \mathbf{\Gamma}(x)\,\mathbf{B}^s\mathbf{NC}. \end{aligned} \tag{7}$$

Also,

$$\begin{aligned} \mathbf{\Gamma}'(x) &= \mathbf{\Gamma}(x)\,\mathbf{B}, \\ \mathbf{\Gamma}''(x) &= \mathbf{\Gamma}(x)\,\mathbf{B}^2, \\ \mathbf{\Gamma}'''(x) &= \mathbf{\Gamma}(x)\,\mathbf{B}^3, \\ &\vdots \\ \mathbf{\Gamma}^{(s)}(x) &= \mathbf{\Gamma}(x)\,\mathbf{B}^s. \end{aligned} \tag{8}$$

In Eq. (8)

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & M \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \mathbf{B}^0=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\mathbf{\Gamma}_{1,0}=\begin{bmatrix} 1 & x_0 & ... & x_0^M \\ 1 & x_1 & ... & x_1^M \\ 1 & \vdots & ... & \vdots \\ 1 & x_M & ... & x_M^M \end{bmatrix}.$$

$ii)$ If α_{sp} , λ_{sp} , $\beta_{sp}(x)$ and $\gamma_{sp}(x)$ are either random constants or variables, then the approximate solution using the Pell polynomials in Eq. (3) can be represented in the matrix format

$$v\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\cong v_M\left(\lambda_{sp}x+\gamma_{sp}(x)\right)=\mathbf{P}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{C}. \tag{9}$$

By considering Eq. (5) and Eq. (6) in Eq. (9), we get the matrix forms

$$\begin{aligned} v\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &\cong v_M\left(\lambda_{sp}x+\gamma_{sp}(x)\right)=\mathbf{P}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{C}=\mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{NC}, \\ v'\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &\cong v'_M\left(\lambda_{sp}x+\gamma_{sp}(x)\right)=\mathbf{\Gamma}_{(\lambda,\ \gamma)}(x)\,\mathbf{BNC}, \\ v''\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &\cong v''_M\left(\lambda_{sp}x+\gamma_{sp}(x)\right)=\mathbf{\Gamma}_{(\lambda,\ \gamma)}(x)\,\mathbf{B}^2\mathbf{NC}, \\ &\vdots \\ v^{(s)}\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &\cong v^{(s)}_M\left(\lambda_{sp}x+\gamma_{sp}(x)\right)=\mathbf{\Gamma}_{(\lambda,\ \gamma)}(x)\,\mathbf{B}^s\mathbf{NC}. \end{aligned} \tag{10}$$

Furthermore, the relationships between $\mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)$ and its derivatives
$\mathbf{\Gamma}'\left(\lambda_{sp}x+\gamma_{sp}(x)\right)$, $\mathbf{\Gamma}''\left(\lambda_{sp}x+\gamma_{sp}(x)\right)$,..., $\mathbf{\Gamma}^{(s)}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)$ are described as

$$\begin{aligned} \mathbf{\Gamma}'\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &= \mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{B}, \\ \mathbf{\Gamma}''\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &= \mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{B}^2, \\ \mathbf{\Gamma}'''\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &= \mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{B}^3, \\ &\vdots \\ \mathbf{\Gamma}^{(s)}\left(\lambda_{sp}x+\gamma_{sp}(x)\right) &= \mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}(x)\right)\mathbf{B}^s, \end{aligned} \tag{11}$$

E-ISSN: 2564-7873

where

$$\mathbf{\Gamma}_{\lambda,\,\gamma}=\begin{bmatrix}\mathbf{\Gamma}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)\\ \mathbf{\Gamma}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)\\ \vdots\\ \mathbf{\Gamma}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix}=\begin{bmatrix}1 & \lambda_{sp}x_0+\gamma_{sp}\left(x_0\right) & ... & \left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)^M\\ 1 & \lambda_{sp}x_1+\gamma_{sp}\left(x_1\right) & ... & \left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)^M\\ 1 & \vdots & ... & \vdots\\ 1 & \lambda_{sp}x_M+\gamma_{sp}\left(x_M\right) & ... & \left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)^M\end{bmatrix}.$$

With the help of Eq. (10) and Eq. (11), we find

$$v^{(s)}\left(\lambda_{sp}x+\gamma_{sp}\left(x\right)\right)=\mathbf{\Gamma}\left(\lambda_{sp}x+\gamma_{sp}\left(x\right)\right)\mathbf{B}^s\mathbf{NC}. \tag{12}$$

The Pell collocation points are defined by

$$x_i=a+\frac{(b-a)\,i}{M},\, i=0,1,...,M. \tag{13}$$

If we substitute these grid points into Eq. (12), we get

$$v^{(s)}\left(\lambda_{sp}x_i+\gamma_{sp}\left(x_i\right)\right)=\mathbf{\Gamma}_{\lambda,\,\gamma}\left(x_i\right)\mathbf{B}^s\mathbf{NC},\ s=0,1,...,n \tag{14}$$

and the closed form of the Eq. (14) can be stated as:

$$\mathbf{V}^{(s)}=\mathbf{\Gamma}_{\lambda,\,\gamma}\mathbf{B}^s\mathbf{NC},\ s=0,1,...,n. \tag{15}$$

Here

$$\mathbf{V}^{(s)}=\begin{bmatrix}v^{(s)}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)\\ v^{(s)}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)\\ \vdots\\ v^{(s)}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix}.$$

In addition, the matrix forms of $\left(\hat{\mathbf{V}}\right)^{p}\ \mathbf{V}^{(s)}$ and $\left(\hat{\mathbf{V}}\right)^{(p)}\ \mathbf{V}^{(s)}$ which emerges in Eq. (1) are

$$\left(\hat{\mathbf{V}}\right)^{p}\ \mathbf{V}^{(s)}=\begin{bmatrix} v^{p}\left(\alpha_{sp}x_0+\beta_{sp}\left(x_0\right)\right)v^{(s)}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)\\ v^{p}\left(\alpha_{sp}x_1+\beta_{sp}\left(x_1\right)\right)v^{(s)}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)\\ \vdots\\ v^{p}\left(\alpha_{sp}x_M+\beta_{sp}\left(x_M\right)\right)v^{(s)}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix}$$

$$=\begin{bmatrix} v^{p}\left(\alpha_{sp}x_0+\beta_{sp}\left(x_0\right)\right) & 0 & \ldots & 0\\ 0 & v^{p}\left(\alpha_{sp}x_1+\beta_{sp}\left(x_1\right)\right) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & v^{p}\left(\alpha_{sp}x_M+\beta_{sp}\left(x_M\right)\right)\end{bmatrix}$$

$$\times\begin{bmatrix} v^{(s)}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)\\ v^{(s)}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)\\ \vdots\\ v^{(s)}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix} \tag{16}$$

$$\left(\hat{\mathbf{V}}\right)^{(p)}\ \mathbf{V}^{(s)}=\begin{bmatrix} v^{(p)}\left(\alpha_{sp}x_0+\beta_{sp}\left(x_0\right)\right)v^{(s)}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)\\ v^{(p)}\left(\alpha_{sp}x_1+\beta_{sp}\left(x_1\right)\right)v^{(s)}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)\\ \vdots\\ v^{(p)}\left(\alpha_{sp}x_M+\beta_{sp}\left(x_M\right)\right)v^{(s)}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix}$$

$$=\begin{bmatrix} v^{(p)}\left(\alpha_{sp}x_0+\beta_{sp}\left(x_0\right)\right) & 0 & \ldots & 0\\ 0 & v^{(p)}\left(\alpha_{sp}x_1+\beta_{sp}\left(x_1\right)\right) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & v^{(p)}\left(\alpha_{sp}x_M+\beta_{sp}\left(x_M\right)\right)\end{bmatrix}$$

$$\times\begin{bmatrix} v^{(s)}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right)\\ v^{(s)}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right)\\ \vdots\\ v^{(s)}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix}$$

where

$$\hat{\mathbf{V}}=\hat{\mathbf{\Gamma}}\ \hat{\mathbf{N}}\ \hat{\mathbf{C}}\text{ and }\left(\hat{\mathbf{V}}\right)^{(p)}=\hat{\mathbf{\Gamma}}\left(\hat{\mathbf{B}}\right)^{p}\ \hat{\mathbf{N}}\ \hat{\mathbf{C}}, \tag{17}$$

$$\hat{\mathbf{\Gamma}}_{\lambda,\gamma}=\begin{bmatrix} \mathbf{\Gamma}\left(\lambda_{sp}x_0+\gamma_{sp}\left(x_0\right)\right) & 0 & \ldots & 0\\ 0 & \mathbf{\Gamma}\left(\lambda_{sp}x_1+\gamma_{sp}\left(x_1\right)\right) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \mathbf{\Gamma}\left(\lambda_{sp}x_M+\gamma_{sp}\left(x_M\right)\right)\end{bmatrix},$$

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & ... & 0 \\ 0 & \mathbf{B} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \mathbf{B} \end{bmatrix}, \ \hat{\mathbf{N}} = \begin{bmatrix} \mathbf{N} & 0 & ... & 0 \\ 0 & \mathbf{N} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \mathbf{N} \end{bmatrix}, \ \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & 0 & ... & 0 \\ 0 & \mathbf{C} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \mathbf{C} \end{bmatrix}.$$

By alternating the grid points given in Eq. (13) into Eq. (16), the following equation system is obtained

$$\begin{aligned} &\sum_{s=0}^{n}\sum_{p=0}^{m} R_{sp}(x_i) v^p\left(\alpha_{sp}x_i + \beta_{sp}(x_i)\right) v^{(s)}(\lambda_{sp}x_i + \gamma_{sp}(x_i)) \\ &+ \sum_{s=1}^{n}\sum_{p=1}^{m} Q_{sp}(x_i) v^{(p)}\left(\alpha_{sp}x_i + \beta_{sp}(x_i)\right) v^{(s)}(\lambda_{sp}x_i + \gamma_{sp}(x_i)) \\ &= g(x_i), \end{aligned} \tag{18}$$

which can be represented using Eqs. (14) and (16) as

$$\sum_{s=0}^{n}\sum_{p=0}^{m} \mathbf{R}_{sp}\left(\hat{\mathbf{V}}\right)^p \ \mathbf{V}^{(s)} + \sum_{s=1}^{n}\sum_{p=1}^{m} \mathbf{Q}_{sp}\left(\hat{\mathbf{V}}\right)^{(p)} \ \mathbf{V}^{(s)} = \mathbf{G}, \tag{19}$$

where

$$\begin{aligned} \mathbf{R}_{sp} &= diag\left[R_{sp}(x_0) \ \ R_{sp}(x_1) \ \ ... \ \ R_{sp}(x_M)\right], \\ \mathbf{Q}_{sp} &= diag\left[Q_{sp}(x_0) \ \ Q_{sp}(x_1) \ \ ... \ \ Q_{sp}(x_M)\right], \\ \mathbf{G} &= \begin{bmatrix} g(x_0) & g(x_1) & ... & g(x_M) \end{bmatrix}^T. \end{aligned}$$

By substituting the relations (15) and (17) into Eq. (19), the fundamental matrix equation is achieved as

$$\left\{\sum_{s=0}^{n}\sum_{p=0}^{m} \mathbf{R}_{sp}\left(\hat{\mathbf{\Gamma}}_{\alpha,\beta}\,\hat{\mathbf{N}}\,\hat{\mathbf{C}}\right)^p \mathbf{\Gamma}_{\lambda,\gamma}\mathbf{B}^s\mathbf{N} + \sum_{s=1}^{n}\sum_{p=1}^{m} \mathbf{Q}_{sp}\hat{\mathbf{\Gamma}}_{\alpha,\beta}\left(\hat{\mathbf{B}}\right)^p \hat{\mathbf{N}}\,\hat{\mathbf{C}}\ \mathbf{\Gamma}_{\lambda,\gamma}\mathbf{B}^s\mathbf{N}\right\}\mathbf{C} = \mathbf{G}. \tag{20}$$

Briefly, Eq. (20) can also be shown as,

$$\mathbf{WC} = \mathbf{G} \ \text{ or } \ [\mathbf{W};\mathbf{G}], \tag{21}$$

where

$$\mathbf{W} = \sum_{s=0}^{n}\sum_{p=0}^{m} \mathbf{R}_{sp}\left(\hat{\mathbf{\Gamma}}_{\alpha,\beta}\,\hat{\mathbf{N}}\,\hat{\mathbf{C}}\right)^p \mathbf{\Gamma}_{\lambda,\gamma}\mathbf{B}^s\mathbf{N} + \sum_{s=1}^{n}\sum_{p=1}^{m} \mathbf{Q}_{sp}\hat{\mathbf{\Gamma}}_{\alpha,\beta}\left(\hat{\mathbf{B}}\right)^p \hat{\mathbf{N}}\,\hat{\mathbf{C}}\ \mathbf{\Gamma}_{\lambda,\gamma}\mathbf{B}^s\mathbf{N}.$$

Here, Eq. (21) represents a system comprising $(M+1)$ nonlinear algebraic equations involving $(M+1)$ unknown Pell coefficients. Utilizing Eq. (15) for the values a and b, we formulate the matrix representation of the conditions stated in Eq. (2) as

$$\left\{\sum_{s=0}^{n-1}\left[a_{js}\mathbf{\Gamma}\left(0\right)+b_{js}\mathbf{\Gamma}\left(0\right)\right]\left(\mathbf{B}\right)^{(s)}\mathbf{N}\right\}\mathbf{C}=\delta_j,\ j=0,1,2,...,n-1$$

alternatively, this can be expressed as

$$\mathbf{U}_j\,\mathbf{C}=[\delta_j]\qquad\text{or}\qquad[\mathbf{U}_j;\delta_j]\,;\quad j=0,1,2,...,n-1 \tag{22}$$

Here

$$\mathbf{U}_j=\sum_{s=0}^{n-1}\left[a_{js}\mathbf{\Gamma}\left(0\right)+b_{js}\mathbf{\Gamma}\left(0\right)\right]\left(\mathbf{B}\right)^{(s)}\mathbf{N}=\left[u_{j0}\ \ u_{j1}\ \ u_{j2}\ ...\ u_{jM}\right].$$

Hence, through the substitution of the condition matrices in (22) with the n rows of the augmented matrix in (21), the new augmented matrix is achieved as

$$\left[\hat{\mathbf{W}};\hat{\mathbf{G}}\right]=\begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0M} & ; & g(x_0)\\ w_{10} & w_{11} & w_{12} & \cdots & w_{1M} & ; & g(x_1)\\ w_{20} & w_{21} & w_{22} & \cdots & w_{2M} & ; & g(x_2)\\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ w_{(M-n)0} & w_{(M-n)1} & w_{(M-n)2} & \cdots & w_{(M-n)M} & ; & g(x_{M-n})\\ u_{00} & u_{01} & u_{02} & \cdots & u_{0M} & ; & \delta_0\\ u_{10} & u_{11} & u_{12} & \cdots & u_{1M} & ; & \delta_1\\ u_{20} & u_{21} & u_{22} & \cdots & u_{2M} & ; & \delta_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ u_{(n-1)0} & u_{(n-1)1} & u_{(n-1)2} & \cdots & u_{(n-1)M} & ; & \delta_{n-1} \end{bmatrix} \tag{23}$$

Thus, the determination of the unknown Pell coefficients $c_m,\ m=1,2,...,M+1$ is achieved through the solution of the system outlined in Eq. (23). Next, the coefficients are inserted into Eq. (3) to derive the approximate solution.

Error Estimation

To determine the accuracy of the proposed method, we define the error function $E_M(x)$ as following

$$E_M(x)=|v_M(x)-v(x)|. \tag{24}$$

Here, $v_M(x)$ is the approximate solution and $v(x)$ is the exact solution of Eq.(1).

Illustrative Example

This section provides four numerical examples to showcase the effectiveness of the proposed method. By using the error function $E_M(x)$, the method has been tested on these problems. The numerical results obtained have been displayed through tables and graphics.

Example 1. Let us examine the differential equation:

$$v''(x) + v'(\frac{x}{4})v\left(\frac{x}{5}\right) - v^2\left(\frac{x}{2}\right) = -\frac{x^4}{16} + \frac{x^3}{50} - \frac{x^2}{2} + \frac{x}{2} + 1, \tag{25}$$

with the initial conditions

$$v(0) = 1, \;\; v'(0) = 0.$$

Eq. (25) is the second-order nonlinear pantograph differential equation and the function $v(x) = x^2 + 1$ is the exact solution of this equation. The solution $v(x)$ approximated by the Pell polynomials is obtained as

$$v\,(x) = \sum_{m=1}^{M+1} c_m P_m(x)$$

where $M = 2,\; R_{20}\,(x) = 1,\; \alpha_{20} = 1,\; \beta_{20} = 0,\; R_{11}\,(x) = 1,\; \alpha_{11} = \frac{1}{4},\; \beta_{11} = 0,\; \lambda_{11} = \frac{1}{5},\; \gamma_{11} = 0,\; R_{01}\,(x) = -1,\; \alpha_{01} = \frac{1}{2},\; \beta_{01} = 0,\; \lambda_{01} = \frac{1}{2},\; \gamma_{01} = 0$, and $g\,(x) = -\frac{x^4}{16} + \frac{x^3}{50} - \frac{x^2}{2} + \frac{x}{2} + 1$. Thus, for $M = 2$ the set of obtained collocation points by Eq. (13) are computed as

$$x_0 = 0,\;\; x_1 = \frac{1}{2},\; x_2 = 1$$

From Eq. (20), we obtain

$$\left\{\mathbf{R}_{20}\boldsymbol{\Gamma}_{1,0}\mathbf{B}^2\mathbf{N} + \mathbf{R}_{11}\hat{\boldsymbol{\Gamma}}_{\frac{1}{5},0}\hat{\mathbf{N}}\hat{\mathbf{C}}\boldsymbol{\Gamma}_{\frac{1}{4},0}\mathbf{B}\mathbf{N} + \mathbf{R}_{01}\hat{\boldsymbol{\Gamma}}_{\frac{1}{2},0}\,\hat{\mathbf{N}}\,\hat{\mathbf{C}}\,\boldsymbol{\Gamma}_{\frac{1}{2},0}\mathbf{N}\right\}\mathbf{C} = \mathbf{G}$$

where

$$\mathbf{W} = \mathbf{R}_{20}\boldsymbol{\Gamma}_{1,0}\mathbf{B}^2\mathbf{N} + \mathbf{R}_{11}\hat{\boldsymbol{\Gamma}}_{\frac{1}{5},0}\hat{\mathbf{N}}\hat{\mathbf{C}}\boldsymbol{\Gamma}_{\frac{1}{4},0}\mathbf{B}\mathbf{N} + \mathbf{R}_{01}\hat{\boldsymbol{\Gamma}}_{\frac{1}{2},0}\,\hat{\mathbf{N}}\,\hat{\mathbf{C}}\,\boldsymbol{\Gamma}_{\frac{1}{2},0}\mathbf{N}$$

$$\begin{aligned}
\mathbf{R}_{20} &= \mathbf{R}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; \mathbf{R}_{01} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\
\boldsymbol{\Gamma}_{1,0} &= \boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}\,(0) \\ \boldsymbol{\Gamma}\left(\frac{1}{2}\right) \\ \boldsymbol{\Gamma}\,(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix},\; \hat{\boldsymbol{\Gamma}}_{1,0} = \begin{bmatrix} \boldsymbol{\Gamma}\,(0) & 0 & 0 \\ 0 & \boldsymbol{\Gamma}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \boldsymbol{\Gamma}\,(1) \end{bmatrix}, \\
\boldsymbol{\Gamma}_{\frac{1}{2},0} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix},\; \hat{\boldsymbol{\Gamma}}_{\frac{1}{2},0} = \begin{bmatrix} \boldsymbol{\Gamma}_{\frac{1}{2},0}\,(0) & 0 & 0 \\ 0 & \boldsymbol{\Gamma}_{\frac{1}{2},0}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \boldsymbol{\Gamma}_{\frac{1}{2},0}\,(1) \end{bmatrix}
\end{aligned}$$

$$\mathbf{\Gamma}_{\frac{1}{4},0} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{8} & \frac{1}{64} \\ 1 & \frac{1}{4} & \frac{1}{16} \end{bmatrix}, \ \hat{\mathbf{\Gamma}}_{\frac{1}{4},0} = \begin{bmatrix} \mathbf{\Gamma}_{\frac{1}{4},0}(0) & 0 & 0 \\ 0 & \mathbf{\Gamma}_{\frac{1}{4},0}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \mathbf{\Gamma}_{\frac{1}{4},0}(1) \end{bmatrix}$$

$$\mathbf{\Gamma}_{\frac{1}{5},0} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{10} & \frac{1}{100} \\ 1 & \frac{1}{5} & \frac{1}{25} \end{bmatrix}, \ \hat{\mathbf{\Gamma}}_{\frac{1}{5},0} = \begin{bmatrix} \mathbf{\Gamma}_{\frac{1}{5},0}(0) & 0 & 0 \\ 0 & \mathbf{\Gamma}_{\frac{1}{5},0}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \mathbf{\Gamma}_{\frac{1}{5},0}(1) \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \ \hat{\mathbf{N}} = \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \ \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & 0 \\ 0 & \mathbf{B} & 0 \\ 0 & 0 & \mathbf{B} \end{bmatrix} \ \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & 0 & 0 \\ 0 & \mathbf{C} & 0 \\ 0 & 0 & \mathbf{C} \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 1 \\ \frac{7191}{6400} \\ \frac{383}{400} \end{bmatrix}.$$

From Eq. (22), the matrix representation of the initial condition is

$[\mathbf{U}_0;\ \delta_0] = [1\ \ 0\ \ 1\ \ ;\ \mathbf{1}],\ [\mathbf{U}_1;\ \delta_1] = [0\ \ 2\ \ 0\ \ ;\ \mathbf{0}]$.

Hence, the resulting augmented matrix $[\hat{\mathbf{W}}; \hat{\mathbf{G}}]$ is obtained as following

$$[\hat{\mathbf{W}}; \hat{\mathbf{G}}] = \begin{bmatrix} -a-c & 2a+2c & 8-c-a & ; & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & ; & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} & ; & \mathbf{0} \end{bmatrix}.$$

The matrix of Pell coefficients C is established by solving this system:

$$\mathbf{C} = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}^T$$

Then, for $M = 2$, the solution approximated using the Pell polynomials is

$v_2(x) = x^2 + 1.$

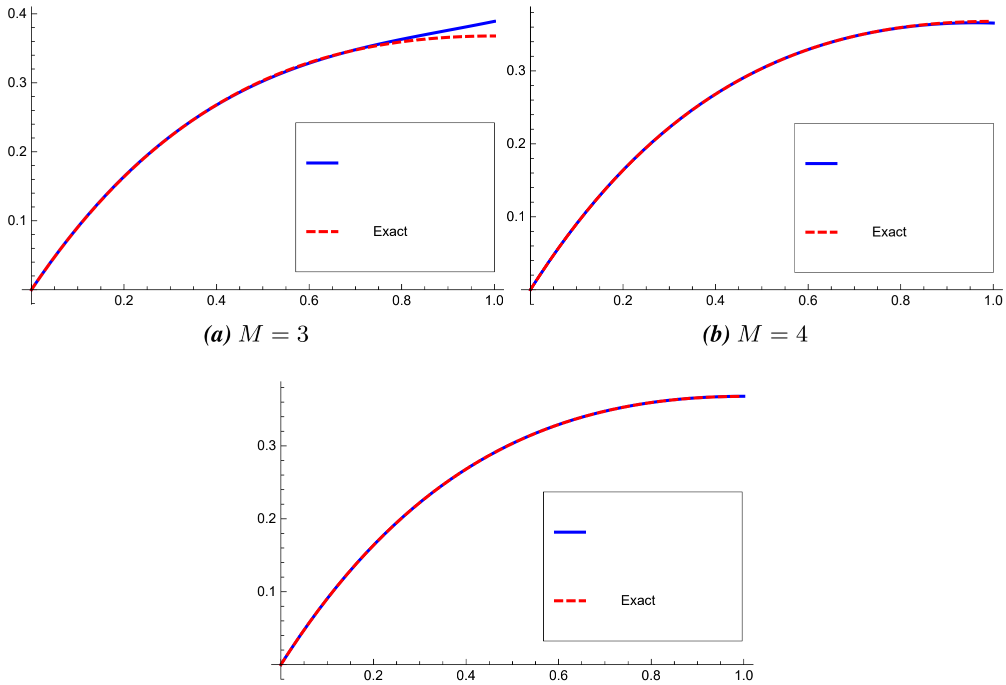
Example 2. Consider the following differential equation

$$v''(x) - v(x) + \frac{8}{x^2} v^2\left(\frac{x}{2}\right) = 0; \quad v(0) = 0, v'(0) = 1. \tag{26}$$

We know that the exact solution of Eq. (26) is found by $v(x) = xe^{-x}$. Table 1 provides the error function values and offers a numerical comparison between the proposed method and the modified differential transform method (MDTM) [33] for $M = 8$ and $M = 11$. Figure 1 depicts a visual comparison between the approximate and exact solutions derived using the proposed method for $M = 3, 4, 5$.

Table 1. *The numerical results of the error function E_M are provided for different values of M for Example 2*

x	$E_8(MDTM)$	$E_{11}(MDTM)$	E_8	E_{11}
0.1	3.56812×10^{-12}	3.59265×10^{-12}	2.21233×10^{-9}	3.60961×10^{-13}
0.3	4.67857×10^{-10}	4.52835×10^{-12}	6.10001×10^{-9}	9.75359×10^{-13}
0.5	4.58254×10^{-8}	5.85598×10^{-11}	7.06047×10^{-9}	1.14569×10^{-12}
0.7	9.28161×10^{-7}	2.78563×10^{-10}	5.60762×10^{-9}	1.11228×10^{-12}
0.9	8.72761×10^{-6}	6.64594×10^{-9}	8.19878×10^{-10}	1.40254×10^{-12}



(a) $M = 3$ ***(b)*** $M = 4$ ***(c)*** $M = 5$

Figure 1. *Graphical comparison illustrating exact and approximate solutions for Example 2 at* $M = 3, 4, 5$

Example 3. Let us consider following differential equation [34]

$$v''(x) + 2v(x) - v^2(x) + v\left(\frac{x^3}{8}\right) = g(x); \quad v(0) = 0, v'(0) = 1 \tag{27}$$

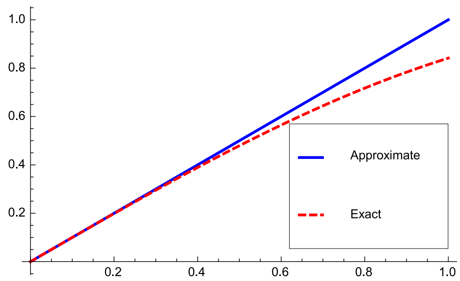
where

$$g(x) = \sin x - \sin^2 x + \sin\left(\frac{x^3}{8}\right)$$

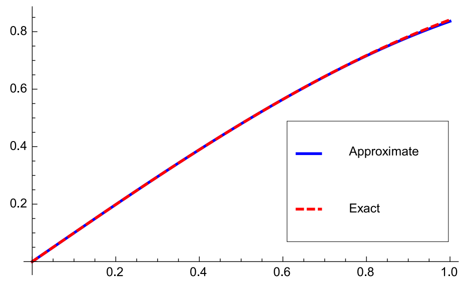
$v(x) = \sin x$ is the exact solution of Eq.(27). Values of the error function specified in Eq.(24) for Eq.(27) are displayed in Table 2, for $M = 9, 10, 11$. Figure 2 depicts the graphical representation of the estimated error function for $M = 2, 3$ and 4.

Table 2. *The numerical results of the error function E_M are provided for $M = 9, 10, 11$ for Example 3*

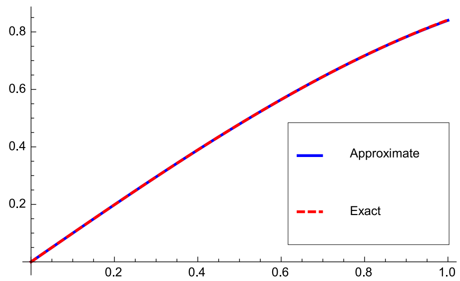
x	E_9	E_{10}	E_{11}
0.1	4.49765×10^{-13}	4.39510×10^{-14}	5.55112×10^{-16}
0.3	1.83259×10^{-12}	1.69309×10^{-13}	2.05391×10^{-15}
0.5	3.08087×10^{-12}	2.83495×10^{-13}	3.38618×10^{-15}
0.7	4.14413×10^{-12}	3.87912×10^{-13}	4.55191×10^{-15}
0.9	6.41098×10^{-12}	9.74887×10^{-13}	8.88178×10^{-16}



(a) $M = 2$



(b) $M = 3$



(c) $M = 4$

Figure 2. *The graphical comparison illustrating exact and approximate solutions for Example 3 at M = 2, 3, 4*

Example 4. Let us consider the following differential equation

$$xv'(\frac{x}{2})v^2(x) + v''(\frac{x}{4}) + v\left(\frac{x}{2}\right) + v'(x-0.5) = g(x); \quad 0 \leq x \leq 1; \quad v(0) = 1, \; v'(0) = 0 \tag{28}$$

where

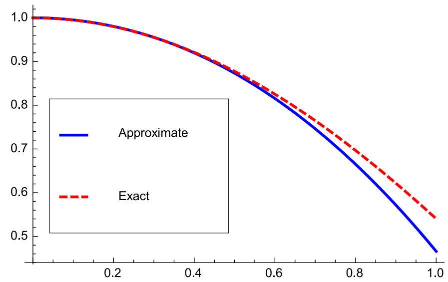
$$g(x) = -\sin(x-0.5) + \cos(x/2) - \cos(x/4) - x\sin(x/2)\cos^2(x).$$

$v(x) = \cos(x)$ is the exact solution of Eq.28. For $M = 5, 7, 9$; values of the error function specified in Eq.(24) for Eq.(28) are showed in Table 3. The graphical representation of the estimated error function for $M = 3, 4, 5$ is depicted in Figure 3.

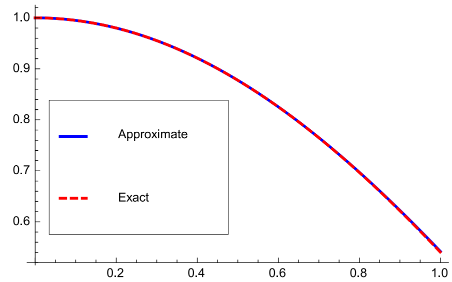
E-ISSN: 2564-7873

Table 3. *The numerical results of the error function E_M are provided for different values of M for Example 4*

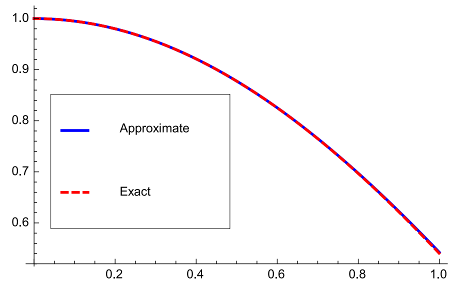
x	E_5	E_7	E_9
0.1	3.91136×10^{-7}	3.34594×10^{-9}	1.23668×10^{-11}
0.3	2.81310×10^{-6}	2.21417×10^{-8}	6.88932×10^{-11}
0.5	3.83995×10^{-5}	2.74960×10^{-7}	6.02225×10^{-10}
0.7	2.73856×10^{-4}	1.41557×10^{-6}	2.76425×10^{-9}
0.9	1.17411×10^{-3}	2.09302×10^{-6}	2.20612×10^{-8}



(a) $M = 3$



(b) $M = 4$



(c) $M = 5$

Figure 3. *Graphical comparison illustrating exact and approximate solutions for Example 4 at M = 3, 4, 5*

Conclusion

This study utilized the Pell collocation method to solve a class of nonlinear Pantograph differential equations. The method's efficiency and accuracy are demonstrated through four distinct examples. The approximate and error results obtained are compared with those obtained using the modified differential transform method. From these comparisons, it can be inferred that the method is notably effective in acquiring approximate solutions for nonlinear Pantograph differential equations.

Acknowledgments -

Funding/Financial Disclosure The author has no received any financial support for the research, or publication of this study.

Ethics Committee Approval and Permissions The work does not require ethics committee approval and any private permission.

Conflict of Interests The author stated that there are no conflict of interest in this article.

Authors Contribution The author read and approved the final manuscript.

References

[1] Ockendon, J. R., & Tayler, A. B. (1971). The dynamics of a current collection system for an electric locomotive. *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences*, 322 (1551), 447-468.

[2] Alkan, S., Aydin, M. N., & Coban, R. (2019). A numerical approach to solve the model of an electromechanical system. *Mathematical Methods in the Applied Sciences, 42*(16), 5266-5273.

[3] Alkan, S., & Secer, A. (2018). A collocation method for solving boundary value problems of fractional order. *Sakarya University Journal of Science, 22*(6), 1601-1608.

[4] Hesameddini, E., & Asadollahifard, E. (2015). Numerical solution of multi-order fractional differential equations via the sinc-collocation method. *Iranian Joıurnal of Numerical Analysis and Optimization, 5*(1), 37-48.

[5] Nagy, A. M. (2017). Numerical solution of time fractional nonlinear Klein–Gordon equation using Sinc–Chebyshev collocation method. *Applied Mathematics and Computation*, 310, 139-148.

[6] Zhi, M., Aiguo, X., Zuguo, Y., & Long, S. (2014). Finite difference and Sinc-collocation approximations to a class of fractional diffusion-wave equations. *Journal of Applied Mathematics*, 536030.

[7] Moshtaghi, N., & Saadatmandi, A. (2021). Numerical solution of time fractional cable equation via the Sinc-Bernoulli collocation method. *Journal of Applied and Computational Mechanics, 7*(4), 1916-1924.

[8] Jalili, P., Jalili, B., Ahmad, I., Hendy, A., Ali, M., & Ganji, D. D. (2024). Python approach for Using homotopy perturbation method to investigate heat transfer problems, *Case Studies in Thermal Engineering, 54,* 104049.

[9] Hatipoglu, V. F. (2021). A novel model for the contamination of a system of three artificial lakes. *Discrete and Continuous Dynamical Systems-S, 14*(7), 2261-2272.

[10] Hatipoglu, V. F. (2019). A numerical algorithm for the solution of nonlinear fractional differential equations via beta-derivatives. *Mathematical Methods in the Applied Sciences, 42*(16), 5258-5265.

[11] Bayram, M., Hatipoglu, V. F., Alkan, S., & Das, S. E. (2018). A solution method for integro-differential equations of conformable fractional derivative. *Thermal Science, 22*(1), S7-S14.

[12] Sedaghat, S., Ordokhani, Y., & Dehghan, M. (2012). Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials. *Communications in Nonlinear Science and Numerical Simulation, 17*(12), 4815-4830.

[13] Jafari, H., Mahmoudi, M., & Skandari, M. H. N. (2021). A new numerical method to solve pantograph delay differential equations with convergence analysis. *Advances in Difference Equations, 2021*(1), 129.

[14] Bahşi, M. M., & Çevik, M. (2015). Numerical solution of pantograph-type delay differential equations using perturbation-iteration algorithms. *Journal of Applied Mathematics*, 139821.

[15] Alrebdi, R., & Al-Jeaid, H. K. (2023). Accurate solution for the pantograph delay differential equation via Laplace transform. *Mathematics, 11*(9), 2031.

[16] Izadi, M., & Srivastava, H. M. (2021). A novel matrix technique for multi-order pantograph differential equations of fractional order. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 477*(2253), 20210321.

[17] Abdo, M. S., Abdeljawad, T., Kucche, K. D., Ali, S. M., & Jeelani, M.B. (2021). On nonlinear pantograph fractional differential equations with Atangana–Baleanu–Caputo derivative. *Advances in Difference Equations, 2021*(1), 65.

[18] Rahimkhani, P., Ordokhani, Y., & Babolian, E. (2017). Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet. *Journal of Computational and Applied Mathematics*, 309, 493–510.

[19] Ali, I., Brunner, H., & Tang, T. (2009). Spectral methods for pantograph-type differential and integral equations with multiple delays. *Frontiers of Mathematics in China, 4*(1), 49–61.

[20] Rabiei, K., & Ordokhani, Y. (2019). Solving fractional pantograph delay differential equations via fractional-order Boubaker polynomials. *Engineering with Computers, 35*(4), 1431–1441.

[21] Xu, Y., Zhang, Y., & Zhao, J. (2019). Error analysis of the Legendre-Gauss collocation methods for the nonlinear distributed-order fractional differential equation. *Applied Numerical Mathematics*, 142, 122-138.

[22] Gebril, E., El-Azab, M. S., & Sameeh, M. (2024). Chebyshev collocation method for fractional Newell-Whitehead-Segel equation. *Alexandria Engineering Journal*, 87, 39-46.

[23] Manohara, G., & Kumbinarasaiah, S. (2024). An innovative Fibonacci wavelet collocation method for the numerical approximation of Emden-Fowler equations. *Applied Numerical Mathematics*, 201, 347-369.

[24] Kumari, A., & Kukreja, V. K. (2023). Study of 4th order Kuramoto-Sivashinsky equation by septic Hermite collocation method. *Applied Numerical Mathematics*, 188, 88-105.

[25] Saad, K. M. (2020). New fractional derivative with non-singular kernel for deriving Legendre spectral collocation method. *Alexandria Engineering Journal, 59*(4), 1909-1917.

[26] Wang, K., & Wang, Q. (2013). Lagrange collocation method for solving Volterra–Fredholm integral equations. *Applied Mathematics and Computation, 219*(21), 10434-10440.

[27] Horadam, A. F., & Mahon, J. M. (1985). Pell and pell-lucas polynomials. *Fibonacci Quart, 23*(1), 7-20.

[28] Taghipour, M., & Aminikhah, H. (2022). A fast collocation method for solving the weakly singular fractional integro-differential equation. *Computational and Applied Mathematics, 41*(4), 142.

[29] Sabermahani, S., Ordokhani, Y., & Razzaghi, M. (2023). Ritz-generalized Pell wavelet method: Application for two classes of fractional pantograph problems.*Communications in Nonlinear Science and Numerical Simulation* , 119, 107138.

[30] Sahin, M., & Sezer, M. (2018). Pell-Lucas collocation method for solving high-order functional differential equations with hybrid delays. *Celal Bayar University Journal of Science, 14*(2), 141-149.

[31] Cayan, S., & Sezer, M. (2019). Pell polynomial approach for Dirichlet problem related to partial differential equations, *Journal of Science and Arts, 19*(3), 613-628.

[32] Yüzbasi, Ş., &Yildirim, G. (2020). Pell-Lucas collocation method to solve high-order linear Fredholm-Volterra integro-differential equations and residual correction, *Turkish Journal of Mathematics, 44*(4), 1065-1091.

[33] Noori, S. R. M., & Taghizadeh, N. (2020). Modified differential transform method for solving linear and nonlinear pantograph type of differential and Volterra integro-differential equations with proportional delays. *Advances in Difference Equations, 2020*(1), 1-25.

[34] Gümgüm, S., Savaşaneril, N. B., Kürkçü, Ö. K.,& Sezer, M. (2020). Lucas polynomial solution of nonlinear differential equations with variable delays. *Hacettepe Journal of Mathematics and Statistics, 49*(2), 553-564.