Hagia Sophia Journal of Geometry
e-ISSN 2687-4261
Vol. 6, No. 1, 45-68, 2024
Research Article

Ruled Surfaces with T_1N_1-Smarandache Base Curve Obtained from the Successor Frame

Gülşah UZUN [1], Süleyman ŞENYURT [2], Kübra AKDAĞ [3]

Abstract
In this study, ruled surfaces formed by the movement of the Frenet vectors of the successor curve along the Smarandache curve obtained from the tangent and principal normal vectors of the successor curve of a curve were defined. Then, the Gaussian and mean curvatures of each ruled surface were calculated. It has been shown that the ruled surface formed by the tangent vector of the successor curve moving along the Smarandache curve is a developable ruled surface. In addition, it was found that the surface formed by the principal normal vector of the successor curve along the Smarandache curve is a minimal developable ruled surface if the principal curve is planar. Conditions are given for other surfaces to be developable or minimal surfaces. Finally, the examples of these surfaces were provided and their shapes were drawn.

Keywords and 2020 Mathematics Subject Classification
Keywords: Smarandache ruled surfaces — mean curvature — Gaussian curvature — successor curve.
MSC: 53A04, 53A05

[1,2,3] *Department of Mathematics, Faculty of Science and Arts, Ordu University, 52200, Ordu, Türkiye*
[1] ✉qulsahqaya@hotmail.com, [2] ✉ssenyurt@odu.edu.tr, [3] ✉kubra28grsn@gmail.com
Corresponding author: Süleyman ŞENYURT
Article History: Received 6 December 2023; Accepted 24 May 2024

1. Introduction

The image of a function with two real variables in three-dimensional space is a surface. Surfaces are used in many fields, such as architecture and engineering [1]. In 1795, Monge defined the ruled surface as the surface formed by the movement of the line along a curve. Any ruled surface is formed as a result of the continuous movement of a line along any curve. These curves are called the base curve and the director curve, respectively. The curvature of surfaces was defined by Gauss in the 19th century, and therefore it was named Gaussian curvature [2]. Gaussian curvatures are related to the dimensions of the surface [3]. Since the average curvature of the surface is a ratio, it is independent of the size of the surface. Thus far, many studies [4–14] on the Gaussian curvatures of surfaces have been conducted.

There are many special curves in differential geometry. One of them is the successor curve. This curve is defined as a new curve; such that the tangent of one curve is the principal normal of the other curve, by Menninger [15] in 2014. Later, Masal [16] investigated the relationships between the position vectors of this curve and defined successor planes. Thus far, many studies have been conducted on this concept [17, 18]. Another special curve is the Smarandache curve defined in Minkowski space [19–22].

In recent years, many studies have been conducted on ruled surfaces whose base curve is the Smarandache curve. Some of these studies can be accessed from [23–30].

In this paper, we present some special ruled surfaces with T_1N_1-Smarandache curves obtained from their successor frames. We investigate the properties of these ruled surfaces by means of Gaussian and mean curvatures. We then obtain the conditions that which of these surfaces are developable and which of these are minimal. We visualize the main idea by providing examples.

2. Preliminaries

This section provides some basic notions needed for the following sections. Throughout this paper, let $\alpha = \alpha(s)$ and $\beta = \beta(s)$ be two differentiable unit speed curves in E^3 and their Frenet apparatus be $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$, respectively. Then,

$$T = \alpha',\ N = \frac{\alpha''}{\|\alpha''\|},\ B = T \wedge N,\ \kappa = \left\|\alpha''\right\|,\ \tau = \left\langle N', B\right\rangle,\ \ T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N.$$

The surface formed by a line moving depending on the parameter of a curve is called a ruled surface, and its parametric expression is $X(s,v) = \alpha(s) + v r(s)$. Here, v is a constant. Besides, α and r are referred to as the base curve and the director curve of X, respectively. The normal vector field N_X, the Gaussian curvature K_X, and the mean curvature H_X of $X(s,v)$ are as follows:

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{1}$$

$$K_X = \frac{eg - f^2}{EG - F^2},\quad H_X = \frac{Eg - 2fF + eG}{2(EG - F^2)}. \tag{2}$$

Here,

$$E = \langle X_s, X_s\rangle,\quad F = \langle X_s, X_v\rangle,\quad G = \langle X_v, X_v\rangle, \tag{3}$$

$$e = \langle X_{ss}, N_X\rangle,\quad f = \langle X_{sv}, N_X\rangle,\quad g = \langle X_{vv}, N_X\rangle. \tag{4}$$

Definition 1. *(see [15, 16]) If the unit tangent vector of α is the principal normal vector of β, then β is called the successor curve of α.*

Theorem 2. *(see [16]) Let β be the successor curve of α. The Frenet apparatus of β are as follows:*

$$T_1 = -\cos\theta N + \sin\theta B,\quad N_1 = T,\ B_1 = \sin\theta N + \cos\theta B,\quad \kappa_1 = \kappa\cos\theta,\quad \tau_1 = \kappa\sin\theta.$$

Moreover,

$$\theta(s) = \theta_0 + \int \tau(s)ds.$$

Here, θ is the angle between binormal vectors B and B_1.

Definition 3. *(see [22]) A regular curve in Minkowski space, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Let β be the Successor curve of α. It can be observed that the unit curve γ, inspired in [22], produces Smarandache curves, for all $s \in I \subseteq \mathbb{R}$, such that

$$\gamma(s) = \frac{aT + bN + cB}{\sqrt{a^2 + b^2 + c^2}},\ a, b, c \in \mathbb{R}.$$

Here, if a, b, and c are nonzero, the Smarandache curves produced by $\gamma(s)$ are denoted by $\{T\,N\,B\}$-Smarandache Curves. This paper consider $\{T\,N\}$-Smarandache curves.

3. Ruled surfaces with T_1N_1-Smarandache base curve obtained from the successor frame

Definition 4. *Let the successor curve of the curve α be β . The ruled surface formed by tangent vector T_1 along the Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve β as follows:*

$$\Phi(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + vT_1 = \frac{1}{\sqrt{2}}(T - \cos\theta N + \sin\theta B) + v(-\cos\theta N + \sin\theta B). \tag{5}$$

Theorem 5. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $\Phi(s,v)$ are as follows:*

$$K_\Phi = 0, \quad H_\Phi = \frac{2\big(\sin\theta + (1+v\sqrt{2})^2\cos^2\theta\big) - \kappa\tau(1+v\sqrt{2})}{\cos^2\theta\sin^2\theta(1+v\sqrt{2})^2}.$$

Proof. Partial derivatives of equation (5) are

$$\Phi_s = \frac{\kappa\big((1+v\sqrt{2})\cos\theta T + N\big)}{\sqrt{2}}, \quad \Phi_v = (-\cos\theta N + \sin\theta B), \quad \Phi_{sv} = \kappa\cos\theta T, \quad \Phi_{vv} = 0,$$

$$\Phi_{ss} = -\frac{\big(\kappa'((1+v\sqrt{2})\cos\theta) - \kappa\tau((1+v\sqrt{2})\sin\theta) - \kappa^2\big)T + (\kappa^2(1+v\sqrt{2})\cos\theta + \kappa')N - \kappa\tau B}{\sqrt{2}}.$$

Thus, from equation (1) the normal N_Φ of the surface is given as

$$N_\Phi = \frac{\sin\theta T - ((1+v\sqrt{2})\sin\theta\cos\theta)N - ((1+v\sqrt{2})\cos^2\theta)B}{\sqrt{\sin^2\theta + \cos^2(1+v\sqrt{2})^2\theta(1+\sin^2\theta)}}.$$

Moreover, in equations (3) and (4) the coefficients of fundamental forms are

$$E_\Phi = \frac{\kappa^2\big((1+v\sqrt{2})^2\cos^2\theta + 1\big)}{2}, \quad F_\Phi = -\frac{\kappa\cos\theta}{\sqrt{2}}, \quad G_\Phi = 1,$$

$$e_\Phi = \frac{\kappa^2\big(\sin\theta + (1+v\sqrt{2})^2\cos^2\theta\big) - \kappa\tau(1+v\sqrt{2})}{\sqrt{2}\sqrt{\sin^2\theta + \cos^2\theta(1+v\sqrt{2})^2(1+\sin^2\theta)}}, \quad f_\Phi = g_\Phi = 0$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Corollary 6. *The ruled surface $\Phi(s,v)$ is a developable surface.*

Definition 7. *Let the successor curve of the curve α be β. The ruled surface formed by principal normal vector N_1 along the Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve β as follows:*

$$Q(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + vN_1 = \frac{1}{\sqrt{2}}(T - \cos\theta N + \sin\theta B) + vT. \tag{6}$$

Theorem 8. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $Q(s,v)$ are as follows:*

$$K_Q = 0, \quad H_Q = \frac{\kappa\tau}{1+v\sqrt{2}}.$$

Proof. Partial derivatives of equation (6) are

$$Q_s = \frac{\kappa\big(-\cos\theta T + (1+v\sqrt{2})N\big)}{\sqrt{2}}, \quad Q_v = T, \quad Q_{sv} = \kappa N, \quad Q_{vv} = 0,$$

$$Q_{ss} = -\frac{\big(\kappa' + \kappa\tau\sin\theta - \kappa^2(1+v\sqrt{2})\big)T + \big(\kappa^2\cos\theta - \kappa'(1+v\sqrt{2})\big)N - \kappa\tau(1+v\sqrt{2})B}{\sqrt{2}}.$$

Thus, from equation (1) the normal N_Q of the surface is given as $N_Q = -B$. Moreover, in equations (3) and (4) the coefficients of fundamental forms are

$$E_Q = \frac{\kappa^2\big(\cos^2\theta + (1+v\sqrt{2})^2\big)}{2}, \quad F_Q = \frac{\kappa\cos\theta}{\sqrt{2}}, \quad G_Q = 1, \quad e_Q = \kappa\tau(1+v\sqrt{2}), \quad f_Q = g_Q = 0$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Corollary 9. *Let the successor curve of the β curve be α. If the curve α is planar, the ruled surface $Q(s,v)$ is the minimal developable surface.*

Definition 10. *Let the successor curve of the curve α be β. The ruled surface formed by binormal vector B_1 along the Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve β as follows:*

$$M(s,v) = \frac{1}{\sqrt{2}}(T_1+N_1)+vB_1 = \frac{1}{\sqrt{2}}(T-\cos\theta N+\sin\theta B)+v(\sin\theta N+\cos\theta B). \tag{7}$$

Theorem 11. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $M(s,v)$ are as follows:*

$$K_M = -\frac{2\sin^2\theta\cos^2}{\left(\cos^2\theta+(\cos\theta-v\sqrt{2}\sin\theta)^2\right)^2},\quad H_M = \frac{\tau(\sin 2\theta - v\sqrt{2})-\kappa\cos\theta\left((\cos\theta-v\sqrt{2}\sin\theta)^2-\cos^2\theta+\sin^2\theta\right)^{\frac{3}{2}}}{\sqrt{2}\kappa\left(\cos^2\theta+(\cos\theta-v\sqrt{2}\sin\theta)^2\right)}.$$

Proof. Partial derivatives of equation (7) are

$$M_s = \frac{\kappa\left((\cos\theta-v\sqrt{2}\sin\theta)T+N\right)}{\sqrt{2}},\quad M_v=\sin\theta N+\cos\theta B,\quad M_{sv}=-\kappa\sin\theta T,\quad M_{vv}=0,$$

$$M_{ss} = -\frac{\left(\kappa^2+\kappa'(\cos\theta-v\sqrt{2}\sin\theta)+\kappa\tau(\sin\theta-v\sqrt{2}\cos\theta)\right)T+\left(\kappa'+\kappa^2(\cos\theta-v\sqrt{2}\sin\theta)\right)N+\kappa\tau B}{\sqrt{2}}.$$

Thus, from equation (1) the normal N_M of the surface is given as

$$N_M = \frac{\cos\theta T-\cos\theta(\cos\theta-v\sqrt{2}\sin\theta)N+\sin\theta(\cos\theta-v\sqrt{2}\sin\theta)B}{\sqrt{\cos^2\theta+(\cos\theta-v\sqrt{2}\sin\theta)^2}}.$$

Moreover, equations (3) and (4) the coefficients of fundamental forms are

$$E_M = \frac{\kappa^2\left((\cos\theta-v\sqrt{2}\sin\theta)^2+1\right)}{2},\quad F_M=\frac{\kappa\sin\theta}{\sqrt{2}},\quad G_M=1,\quad g_M=0,$$

$$e_M = \frac{\kappa\tau(\sin 2\theta-v\sqrt{2})-\left(\kappa^2\cos\theta(\cos\theta-v\sqrt{2}\sin\theta)^2-1\right)}{\sqrt{2}\sqrt{\cos^2\theta+\cos\theta(\cos\theta-v\sqrt{2}\sin\theta)^2}},\quad f_M=\frac{-\kappa\sin\theta\cos\theta}{\sqrt{2}\sqrt{\cos^2\theta+\cos\theta(\cos\theta-v\sqrt{2}\sin\theta)^2}},$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Corollary 12. *If the $\theta=\pi+k\pi$ $(k\in\mathbb{N})$ or or $\theta=\dfrac{\pi}{2}+k\pi$ $(k\in\mathbb{N})$, the ruled surface $M(s,v)$ is a developable surface.*

Definition 13. *Let the successor curve of the curve α be β. The ruled surface formed by the vector T_1N_1 along the Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve as β follows:*

$$\mu(s,v) = \frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+N_1) = \frac{1+v}{\sqrt{2}}(T-\cos\theta N+\sin\theta B). \tag{8}$$

Theorem 14. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $\mu(s,v)$ are as follows:*

$$K_\mu=0,\quad H_\mu=\frac{\sqrt{2}\sin\theta\left(-2\tau\sin\theta-\kappa(1+\cos^2)\right)}{2\kappa(1+v\sqrt{2})(1+\cos^2)}.$$

Proof. Partial derivatives of equation (8) are

$$\mu_s=\frac{(1+v)\kappa(\cos\theta T+N)}{\sqrt{2}},\quad \mu_v=\frac{T-\cos\theta N+\sin\theta B}{\sqrt{2}},\quad \mu_{sv}=\frac{\kappa(\cos\theta T+N)}{\sqrt{2}},\quad \mu_{vv}=0,$$

$$\mu_{ss}=-\frac{(1+v)\big(\kappa'\cos\theta-\kappa\tau\sin\theta-\kappa^2(1+v)\big)T+(\kappa'+\kappa^2\cos\theta(v-\sin\theta)+\kappa\tau B}{\sqrt{2}}.$$

Thus, from equation(1) the normal N_μ of the surface is given as

$$N_\mu=\sin\theta T-\cos\theta\sin\theta N-\sin\theta^2 B.$$

Moreover, in equations (3) and (4) the coefficients of fundamental forms are

$$E_\mu=\frac{(1+v)^2\kappa^2(\cos^2\theta+1)}{2},\quad F_\mu=0,\quad G_\mu=1,\quad e_\mu=\frac{(1+v)^2\kappa\sin\theta\big(2\tau\sin\theta+\kappa(1+\cos^2\theta)\big)}{\sqrt{2}},\quad f_\mu=g_\mu=0$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Corollary 15. *If $\theta=k\pi$ $(k\in\mathbb{N})$, the ruled surface $\mu(s,v)$ is a developable surface.*

Definition 16. *Let the successor curve of the curve α be β. The ruled surface formed by the vector T_1B_1 along the Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve β as follows:*

$$\psi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+B_1)=\frac{1}{\sqrt{2}}(T-\cos\theta N+\sin\theta B)+\frac{v}{\sqrt{2}}\big((\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B\big).\quad (9)$$

Theorem 17. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $\psi(s,v)$ are as follows:*

$$K_\psi=-\frac{(\sin^2\theta-\cos^2\theta)^2}{\big((1+\sin 2\theta)+2(\cos\theta-v(\sin\theta-\cos\theta))^2\big).\big((\cos\theta-v(\sin\theta-\cos\theta))^2+\sin 2\theta\big)},$$

$$H_\psi=-\frac{\tau(1+2v)+\kappa(\sin\theta+\cos\theta)\big((\cos\theta-v(\sin\theta-\cos\theta))^2+\sin 2\theta-2\big)}{\kappa\sqrt{2}\sqrt{(1+\sin 2\theta)+2(\cos\theta-v(\sin\theta-\cos\theta))^2}.\big((\cos\theta-v(\sin\theta-\cos\theta))^2+\sin 2\theta\big)}.$$

Proof. Partial derivatives of equation (9) are

$$\psi_s=\frac{\kappa\big((\cos\theta-v(\sin\theta-\cos\theta))T+N\big)}{\sqrt{2}},\quad \psi_v=\frac{(sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B}{\sqrt{2}},\quad \psi_{sv}=\frac{\kappa(\cos\theta-\sin\theta)T}{\sqrt{2}},\quad \psi_{vv}=0,$$

$$\psi_{ss}=-\frac{\big(\kappa'(\cos\theta-v(\sin\theta-\cos\theta))-\kappa\tau(\sin\theta+v(\sin\theta+\cos\theta))+\kappa^2\big)T+\big(\kappa'+\kappa^2(\cos\theta-v(\sin\theta-\cos\theta))\big)N+\kappa\tau B}{\sqrt{2}}.$$

Thus, from equation (1) the normal N_ψ of the surface is given as

$$N_\psi=-\frac{(\sin\theta+\cos\theta)T-\big(\cos\theta(\sin\theta+\cos\theta)-v(\sin^2\theta-\cos^2\theta)\big)N+\big(\cos\theta(\sin\theta-\cos\theta)-v(1+\sin 2\theta)\big)B}{\sqrt{(1+\sin 2\theta)+2\big(\cos\theta-v(\sin\theta-\cos\theta)\big)^2}}.$$

Moreover, in equations (3) and (4) the coefficients of fundamental forms are

$$E_\psi=\frac{\kappa^2\big((\cos\theta-v(\sin\theta-\cos\theta))^2+1\big)}{2},\quad F_\psi=\frac{\kappa(\cos\theta-\sin\theta)}{2},\quad G_\psi=1,\quad g_\psi=0,$$

$$e_\psi=\frac{\kappa\tau(1+2v)-\kappa^2(\sin\theta+\cos\theta)\big((\cos\theta-v(\sin\theta-\cos\theta))^2-1\big)}{\sqrt{2(1+2\sin\theta)+4\big(\cos\theta-v(\sin\theta-\cos\theta)\big)^2}},\quad f_\psi=\frac{\kappa(\sin^2\theta-\cos^2\theta)}{\sqrt{2(1+2\sin\theta)+4\big(\cos\theta-v(\sin\theta-\cos\theta)\big)^2}}$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Definition 18. *Let the successor curve of the curve α be β. The ruled surface formed by the vector N_ψ along the Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve β as follows:*

$$\eta(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(N_1+B_1)=\frac{1}{\sqrt{2}}(T-\cos\theta N+\sin\theta B)+\frac{v}{\sqrt{2}}(T+\sin\theta N+\cos\theta B).\quad (10)$$

Theorem 19. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $\eta(s,v)$ are as follows:*

$$K_\eta = -\frac{\cos^2\theta(1+\sin 2\theta)}{\Big(\big((1+v)^2+(v\sin\theta-\cos\theta)^2\big)\cos^2\theta+\big((1+v)+(v\sin^2\theta-\sin\theta\cos\theta)\big)\Big).\big((1+v)^2-(v\sin\theta+\cos\theta)^2-(1+\sin 2\theta)\big)},$$

$$H_\eta = -\frac{\kappa\cos\theta\big((1+v)^2-(v\sin\theta-\cos\theta)^2-(1+\sin 2\theta)\big)+\tau(1+v)\big((1+v)v(\sin^2\theta-\cos^2\theta)-\sin 2\theta\big)}{\sqrt{2}\kappa\big((-v\sin\theta+\cos\theta)^2+(1+v)^2-(1+\sin 2\theta)^2\big).\sqrt{\big((1+v)^2+(v\sin\theta-\cos\theta)^2\big)\cos^2\theta+\big(\sin\theta+(v\sin\theta-\cos\theta)+(1+v)^2\big)}}.$$

Proof. Partial derivatives of equation (10) are

$$\eta_s = \frac{\kappa\big((-v\sin\theta+\cos\theta)T+(1+v)N\big)}{\sqrt{2}},\quad \eta_v = \frac{T+\sin\theta N+\cos\theta B}{\sqrt{2}},\quad \eta_{sv} = \frac{\kappa(-\sin\theta T+N)}{\sqrt{2}},\quad \eta_{vv}=0,$$

$$\eta_{ss} = -\frac{\big(-\kappa\tau(v\cos\theta+\sin\theta)-\kappa'(v\sin\theta+\cos\theta)+\kappa^2(1+v)\big)T+\big(\kappa^2(-v\sin\theta+\cos\theta)+\kappa'(1+v)\big)N+\kappa\tau(1+v)B}{\sqrt{2}}.$$

Thus, from equation (1) the normal N_η of the surface is given as

$$N_\eta = \frac{(1+v)\cos\theta T+(v\sin\theta-\cos\theta)\cos\theta N+(\sin\theta(v\sin\theta-\cos\theta)+(1+v))B}{\sqrt{\big((1+v)^2+(v\sin\theta-\cos\theta)^2\big)\cos^2\theta+\big(\sin\theta(v\sin\theta-\cos\theta)+(1+v)\big)^2}}.$$

Moreover, in equations (3) and (4) the coefficients of fundamental forms are

$$E_\eta = \frac{\kappa^2\big((-v\sin\theta+\cos\theta)^2+(1+v)^2\big)}{2},\quad F_\eta = \frac{\kappa(\cos\theta+\sin\theta)}{2},\quad G_\eta=1,\quad g_\eta=0,$$

$$e_\eta = \frac{\kappa^2\cos\theta\big((1+v)^2-(v\sin\theta-\cos\theta)^2\big)+\kappa\tau(1+v)\big((1+v)v(\sin^2\theta-\cos^2\theta)-\sin 2\theta\big)}{\sqrt{2\cos^2\theta\big((1+v)^2+(v\sin\theta-\cos\theta)^2\big)+2\big(\sin\theta(v\sin\theta-\cos\theta)+(1+v)\big)^2}},$$

$$f_\eta = -\frac{\kappa\cos\theta(\sin\theta+\cos\theta)}{\sqrt{2\cos^2\theta\big((1+v)^2+(v\sin\theta-\cos\theta)^2\big)+2\big(\sin\theta(v\sin\theta-\cos\theta)+(1+v)\big)^2}}$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Corollary 20. *If $\theta = \dfrac{\pi}{2}+k\pi$ $(k\in\mathbb{N})$, the ruled surface $\eta(s,v)$ is a developable surface.*

Definition 21. *Let the successor curve of the curve α be β. The ruled surface formed by the vector $T_1N_1B_1$ along he Smarandache curve T_1N_1 obtained from the tangent vector T_1 and principal vector N_1 of the curve β as follows:*

$$\Gamma(s,v) = \frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1) = \frac{1}{\sqrt{2}}(T-\cos\theta N+\sin\theta B)+\frac{v}{\sqrt{3}}\big(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B\big). \quad (11)$$

Theorem 22. *Let the successor curve of the curve α be β. The Gaussian and mean curvatures of the ruled surface $\Gamma(s,v)$ are as follows:*

$$K_\Gamma = -\frac{2\big((\sqrt{3}+v\sqrt{2})(\sin\theta+\cos\theta)-\sqrt{3}\cos\theta(\sin\theta+\cos\theta)+v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)^2}{\begin{array}{l}\Big(\big(\sqrt{3}\cos\theta-v\sqrt{2}(\sin\theta-\cos\theta)\big)^2+(\sqrt{3}+v\sqrt{2})^2-\sin^2\theta\Big)\\ .\Big((\sqrt{3}+v\sqrt{2})^2(\sin\theta+\cos\theta)^2+(\sqrt{3}(\cos\theta\sin\theta-\cos^2\theta-1)+v\sqrt{2}\sin 2\theta)^2+\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)-v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)^2\Big)\end{array}},$$

$$H_\Gamma = -\frac{\begin{array}{l}\kappa(\sqrt{3}+v\sqrt{2})^2(\sin\theta+\cos\theta)+\sqrt{3}\cos\theta\sin\theta(\sqrt{3}\cos\theta-v\sqrt{2}\sin\theta)+\kappa(\sqrt{3}+v\sqrt{2})\cos^2\theta\big((\sqrt{3}+v\sqrt{2})\cos^2\theta-v\sqrt{2}\big)\\ +\tau(\sqrt{3}+v\sqrt{2})(2\sqrt{3}+v\sqrt{2})+2\sin\theta\big((\sqrt{3}+v\sqrt{2})(\cos\theta+\sin\theta)-\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)+v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)\end{array}}{\begin{array}{l}\sqrt{2}\Big((\sqrt{3}+v\sqrt{2})^2(\sin\theta+\cos\theta)^2+(\sqrt{3}(\cos\theta\sin\theta-\cos^2\theta-1)+v\sqrt{2}\sin 2\theta)^2+\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)-v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)^2\Big)^{\frac{1}{2}}\\ .\Big(\big(\sqrt{3}\cos\theta-v\sqrt{2}(\sin\theta-\cos\theta)\big)^2+(\sqrt{3}+v\sqrt{2})^2-\sin^2\theta\Big)\end{array}}.$$

Proof. Partial derivatives of equation (11) are

$$\Gamma_s=\frac{\kappa\Big(\big(\sqrt{2}\cos\theta-(\sqrt{3}+v\sqrt{2})\sin\theta)\big)T+(\sqrt{3}+v\sqrt{2})N\Big)}{\sqrt{6}},\quad \Gamma_{sv}=\frac{\kappa\big((\cos\theta-\sin\theta)T+N\big)}{\sqrt{3}},$$

$$\Gamma_v=\frac{T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B}{\sqrt{3}},\quad \Gamma_{vv}=0,$$

$$\Gamma_{ss}=-\frac{\begin{array}{c}\kappa\big(\kappa'\big(\sqrt{3}\cos\theta-v\sqrt{2}(\sin\theta-\cos\theta)\big)-\kappa\tau\big(\sqrt{3}\cos\theta-v\sqrt{2}(\sin\theta+\cos\theta)\big)-\kappa^2(\sqrt{3}+v\sqrt{2})\big)T\\+\kappa\big(\kappa'(\sqrt{3}+v\sqrt{2})+\kappa^2(\sqrt{3}\cos\theta-v\sqrt{2}(\sin\theta-\cos\theta))\big)N+\kappa^2\tau(\sqrt{3}+v\sqrt{2})B\end{array}}{\sqrt{6}}.$$

Thus, from equation (1) the normal N_Γ of the surface is given as

$$N_\Gamma=-\frac{(\sqrt{3}+v\sqrt{2})(\sin\theta+\cos\theta)T-\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)-v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)N+\big(\sqrt{3}(\cos\theta\sin\theta-\cos^2\theta-1)+v\sqrt{2}\sin 2\theta\big)B}{(\sqrt{3}+v\sqrt{2})^2(\sin\theta+\cos\theta)^2+\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta-1)+v\sqrt{2}\sin2\theta\big)^2+\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)-v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)^2}.$$

Moreover, in equations (3) and (4) the coefficients of fundamental forms are

$$E_\Gamma=\frac{\kappa^2\big(\big(\sqrt{3}\cos\theta-v\sqrt{2}(\sin\theta-\cos\theta)\big)^2+(\sqrt{3}+v\sqrt{2})^2\big)}{6},\quad F_\Gamma=\frac{\kappa\sin\theta}{6},\quad G_\Gamma=1,$$

$$e_\Gamma=\frac{\begin{array}{c}-\kappa^3\big((\sqrt{3}+v\sqrt{2})^2(\sin\theta+\cos\theta)+\sqrt{3}\cos\theta\sin\theta(\sqrt{3}\cos\theta-v\sqrt{2}\sin\theta)-\kappa\tau(\sqrt{3}+v\sqrt{2})(2\sqrt{3}+v\sqrt{2})\\-(\sqrt{3}+v\sqrt{2})\cos^2\theta((\sqrt{3}+v\sqrt{2})\cos^2\theta-v\sqrt{2}\sin\theta)\big)\end{array}}{\sqrt{6}\sqrt{\begin{array}{c}(\sqrt{3}+v\sqrt{2})^2(1\sin2\theta)+\big(\sqrt{3}(\cos\theta\sin\theta-\cos^2\theta-1)+v\sqrt{2}\sin2\theta\big)^2\\+\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)-v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)^2\end{array}}},$$

$$f_\Gamma=\frac{\kappa\big((\sqrt{3}+v\sqrt{2})(\sin\theta+\cos\theta)-\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)+v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)}{\sqrt{6}\sqrt{\begin{array}{c}(\sqrt{3}+v\sqrt{2})^2(1\sin2\theta)+\big(\sqrt{3}(\cos\theta\sin\theta-\cos^2\theta-1)+v\sqrt{2}\sin2\theta\big)^2\\+\big(\sqrt{3}(\cos\theta\sin\theta+\cos^2\theta)-v\sqrt{2}(\sin^2\theta-\cos^2\theta)\big)^2\end{array}}},\quad g_\Gamma=0$$

respectively. Thus, by using equation (2) the Gaussian and mean curvatures are found. ■

Corollary 23. *If $\theta=\dfrac{\pi}{4}+k\pi$ $(k\in\mathbb{N})$, the ruled surface $\Gamma(s,v)$ is a developable surface.*

Example 24. *Let Salkowski curve β [31] be the Successor curve of α. The equation of this curve for $m=\frac{1}{3}$ is as follows:*

$$\beta(s)=\frac{3}{\sqrt{10}}\left(\begin{array}{c}-\frac{\sqrt{10}-1}{4\sqrt{10}+8}\big(\sin(\frac{\sqrt{10}+2}{\sqrt{10}})s\big)-\frac{\sqrt{10}-1}{4\sqrt{10}-8}\big(\sin(\frac{\sqrt{10}-2}{\sqrt{10}})s\big)-\frac{1}{2}\sin s,\\ \\ -\frac{\sqrt{10}-1}{4\sqrt{10}+8}\big(\cos(\frac{\sqrt{10}+2}{\sqrt{10}})s\big)+\frac{\sqrt{10}-1}{4\sqrt{10}-8}\big(\cos(\frac{\sqrt{10}-2}{\sqrt{10}})s\big)+\frac{1}{2}\cos s,\quad \frac{3}{4}\cos(\frac{2s}{\sqrt{10}})\end{array}\right).$$

The Successor frames $\{T_1,N_1,B_1\}$ of curve β are as follows:

$$\begin{cases}T_1(s) &= \Big(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\quad -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}},\quad \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\Big),\\ N_1(s) &= \Big(\frac{3}{\sqrt{10}}\sin s,\quad -\frac{3}{\sqrt{10}}\cos s,\quad -\frac{1}{\sqrt{10}}\Big),\\ B_1(s) &= \Big(-\cos s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}},\quad -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}},\quad \frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\Big).\end{cases}$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 1- 7;

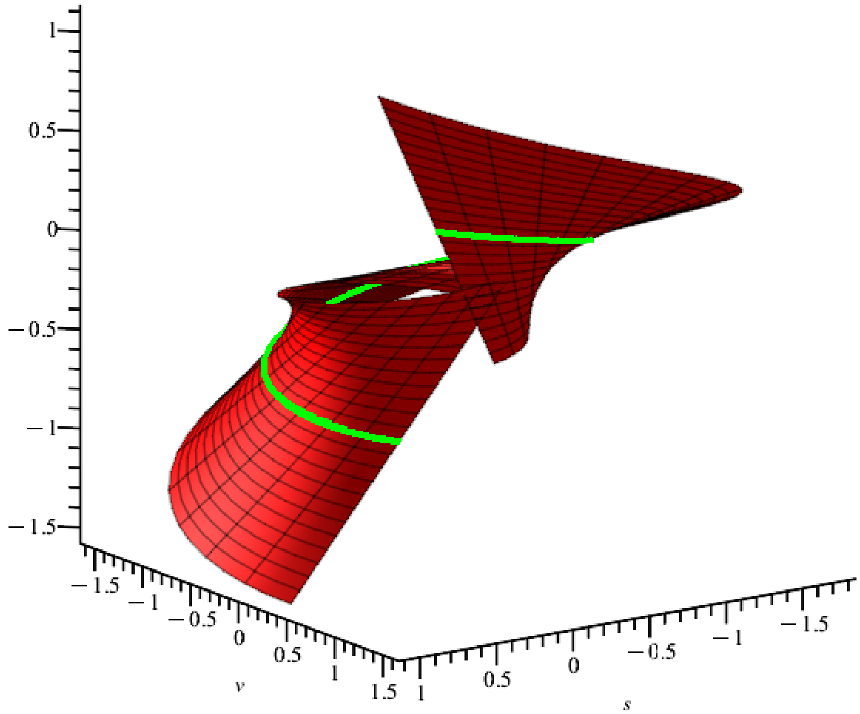

Fig. 1. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vT_1$

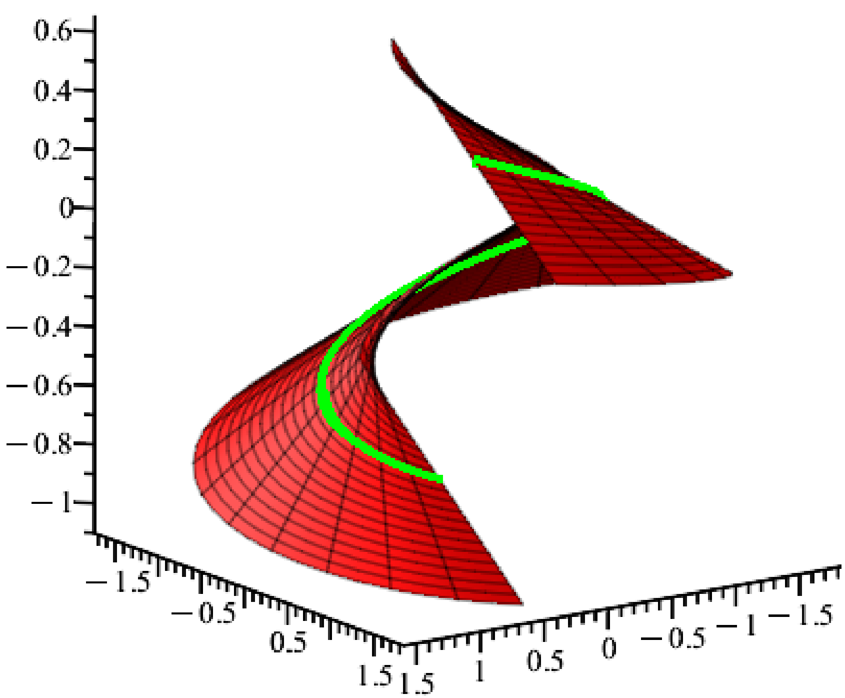

Fig. 2. The ruled surface $Q(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vN_1$

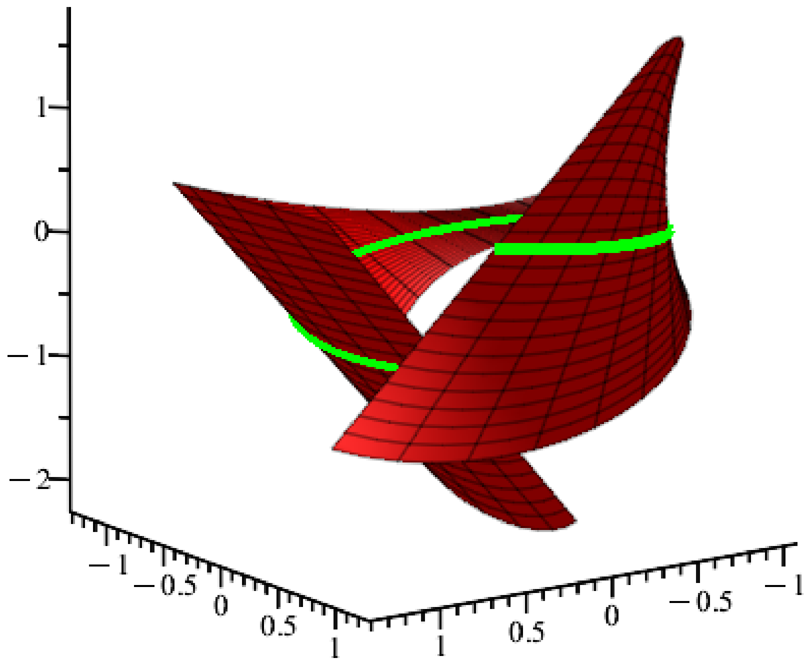

Fig. 3. The ruled surface $M(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + vB_1$

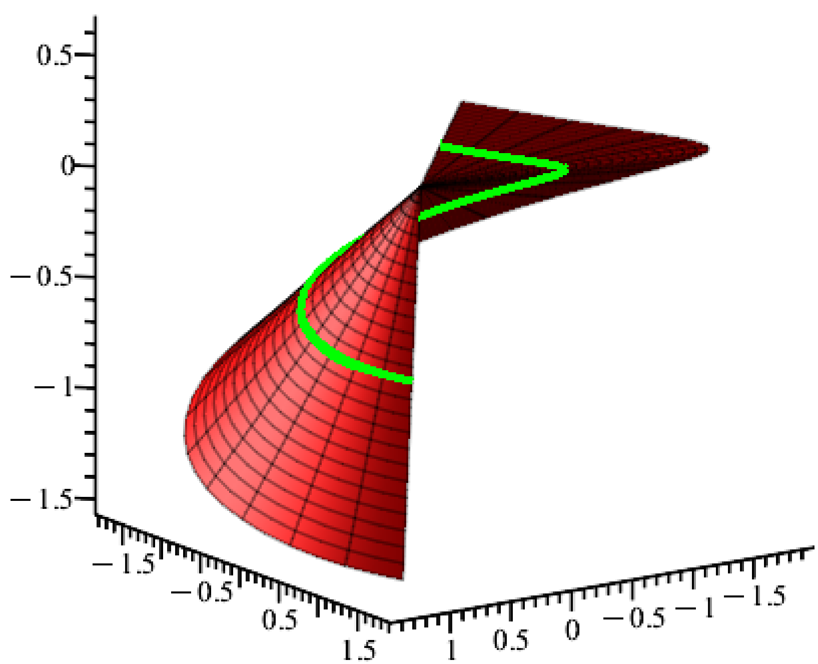

Fig. 4. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

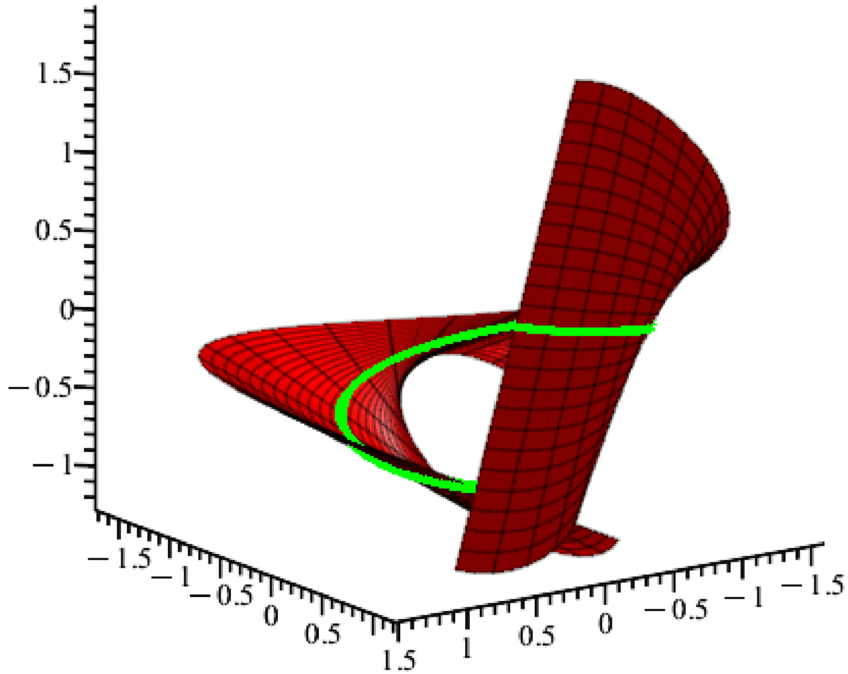

Fig. 5. The ruled surface $\psi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

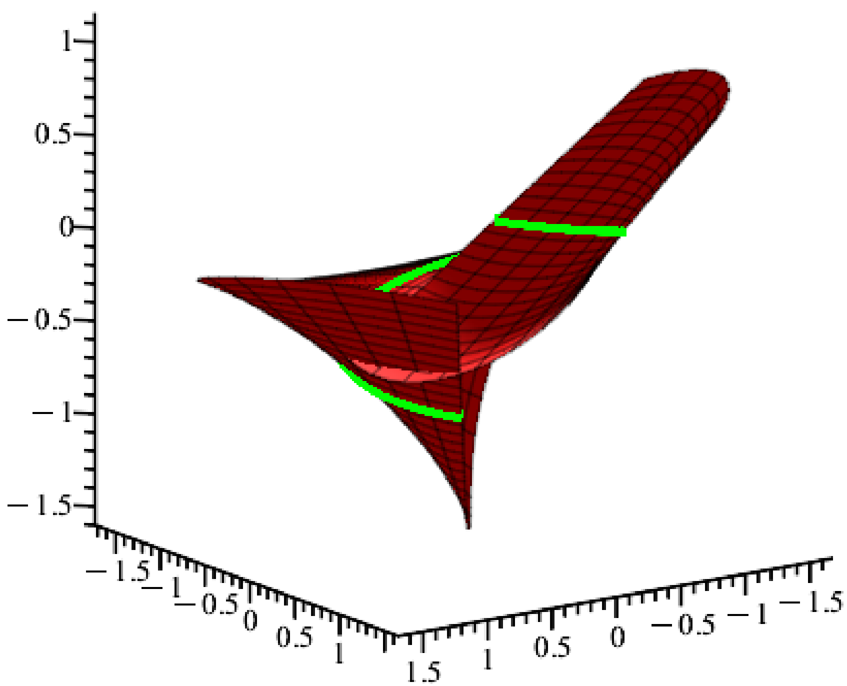

Fig. 6. The ruled surface $\eta(s,v)\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

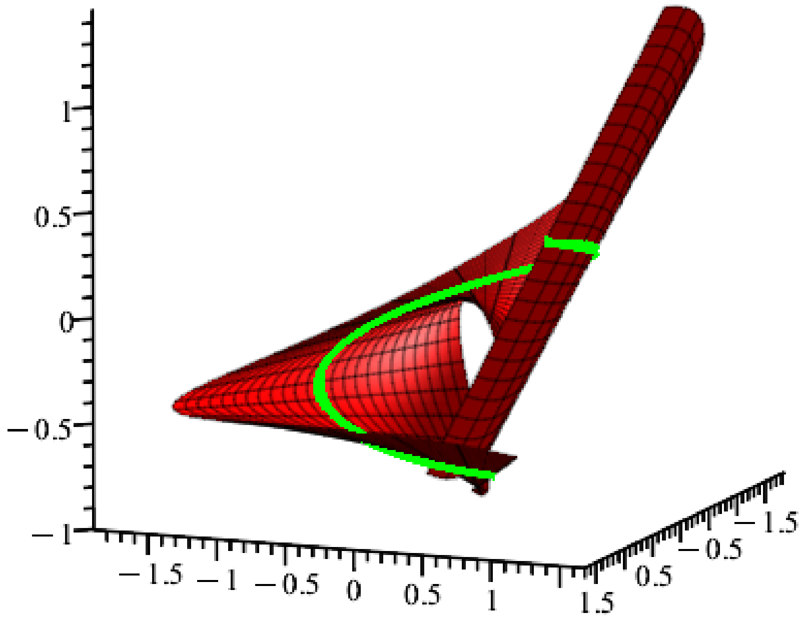

Fig. 7. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 25. *Let the Salkowski curve in Example 24 be the main curve. Theorem 2 the successor frames are as follows:*

$$\begin{cases} T_1(s) &= \begin{pmatrix} -\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right)+\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\cos s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right), \\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right)-\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\sin s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right), \\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}}+\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right) \end{pmatrix}, \\ N_1(s) &= \left(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\right), \\ B_1(s) &= \begin{pmatrix} \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right)-\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right), \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right)-\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right), \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}}+\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right) \end{pmatrix}. \end{cases}$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 8-14;

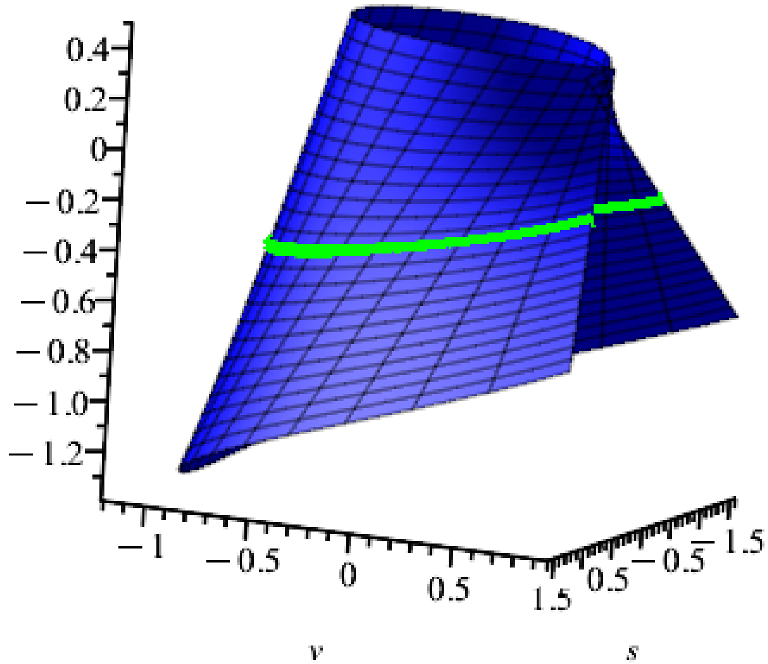

Fig. 8. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vT_1$

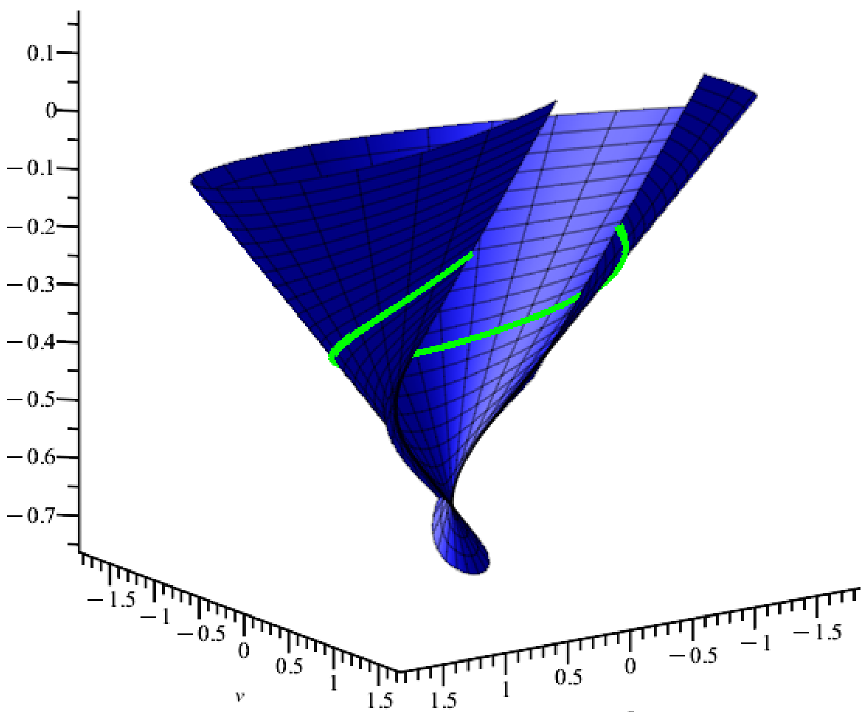

Fig. 9. The ruled surface $Q(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vN_1$

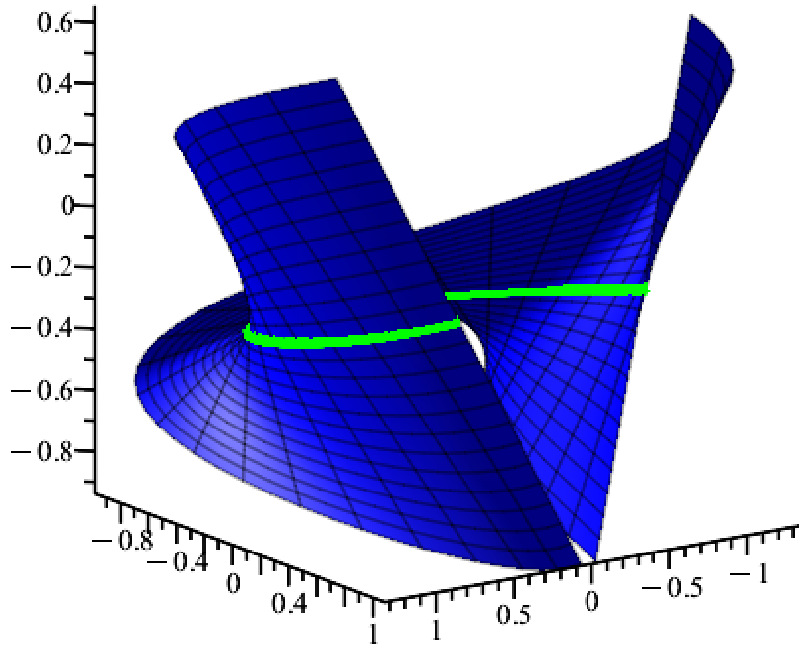

Fig. 10. The ruled surface $M(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vB_1$

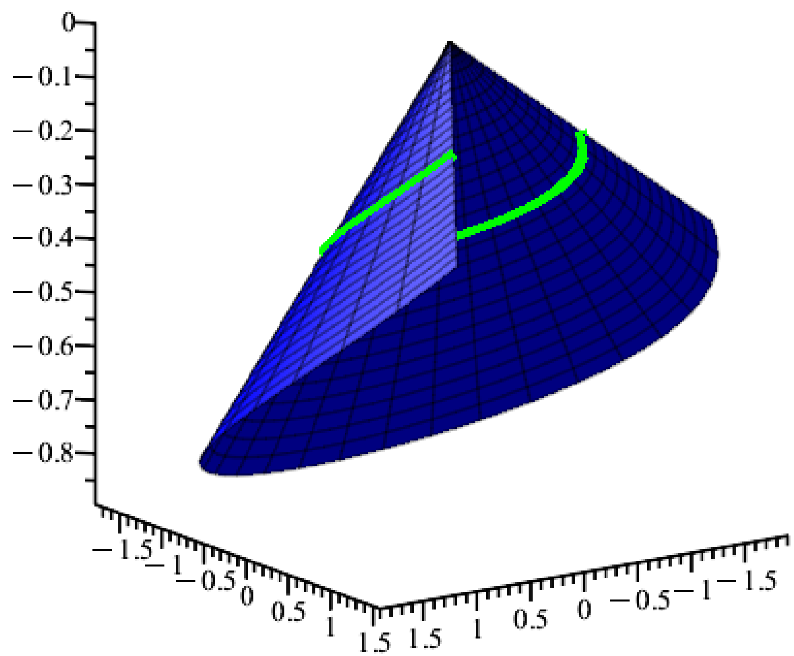

Fig. 11. The ruled surface $\mu(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+N_1)$

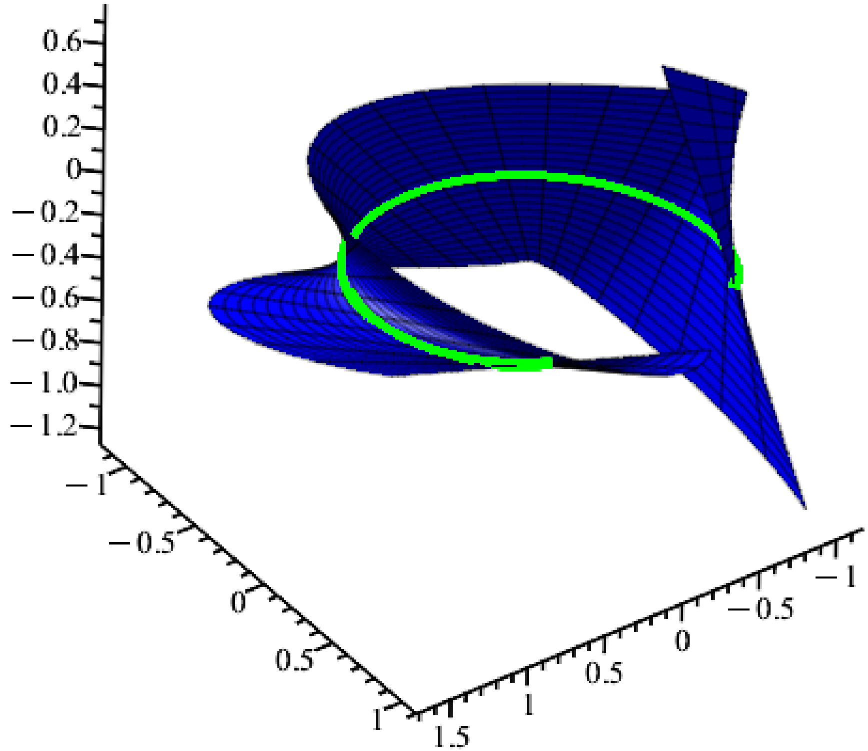

Fig. 12. The ruled surface $\psi(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

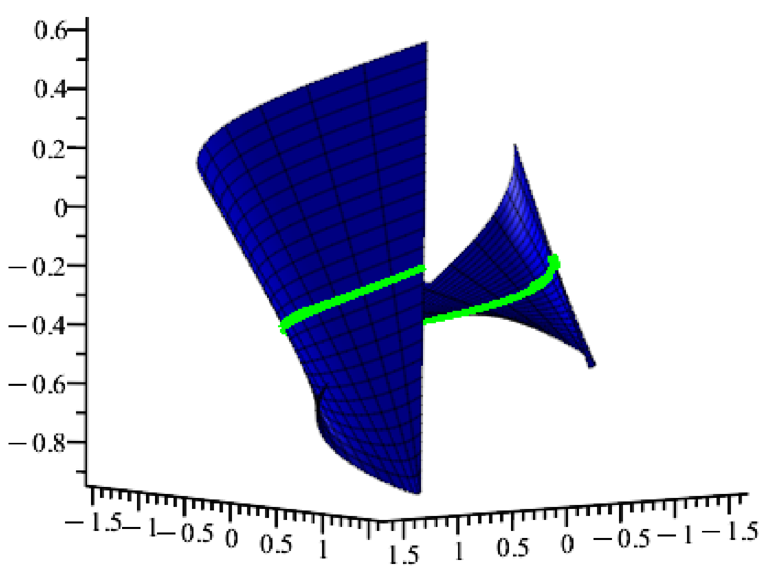

Fig. 13. The ruled surface $\eta(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + \frac{v}{\sqrt{2}}(N_1 + B_1)$

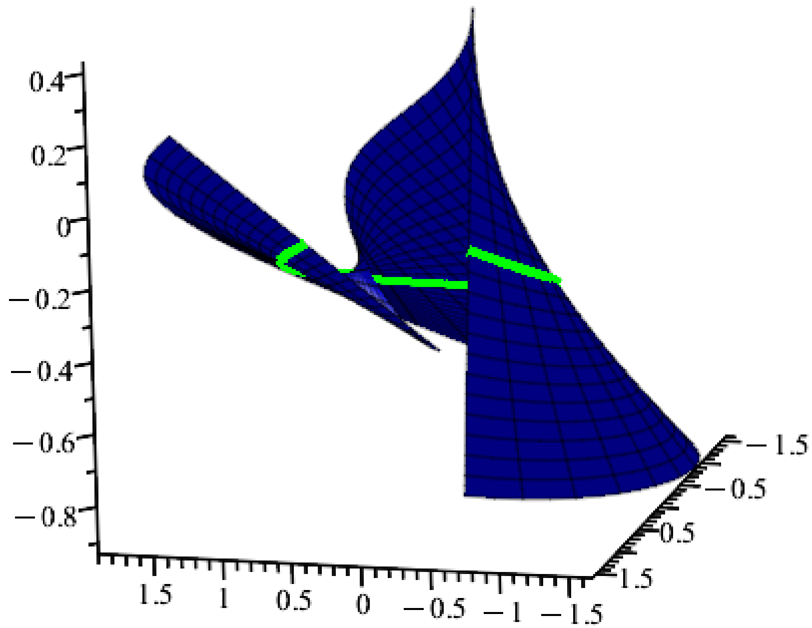

Fig. 14. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 26. *Let anti Salkowski curve β^* [31] be the Successor curve of α. The equation of this curve for $m=\frac{1}{3}$ is as follows:*

$$\beta^*(s)=\frac{\sqrt{10}}{40}\begin{pmatrix} -\frac{5}{2\sqrt{10}}\left(\frac{3}{\sqrt{10}}\cos(\frac{1}{5}+\cos(\frac{2}{\sqrt{10}})s)\right)+\frac{6}{5}\sin s\sin\frac{2}{\sqrt{10}}s, \\ -\frac{5}{2\sqrt{10}}\left(\frac{3}{\sqrt{10}}\sin(\frac{1}{5}+\cos(\frac{2}{\sqrt{10}})s)\right)+\frac{6}{5}\cos s\sin\frac{2}{\sqrt{10}}s, \quad -\frac{9\sqrt{10}}{40}\left(\frac{2}{\sqrt{10}}s+\sin(\frac{2}{\sqrt{10}})s\right) \end{pmatrix}.$$

The Successor frames $\{T_1^*,N_1^*,B_1^*\}$ of curve β^* are as follows:

$$\begin{cases} T_1^*(s) &= \left(-\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}},\ -\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right), \\ N_1^*(s) &= \left(\frac{3}{\sqrt{10}}\sin s,\ -\frac{3}{\sqrt{10}}\cos s,\ \frac{1}{\sqrt{10}}\right), \\ B_1^*(s) &= \left(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}},\ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\right). \end{cases}$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figure 15-21;

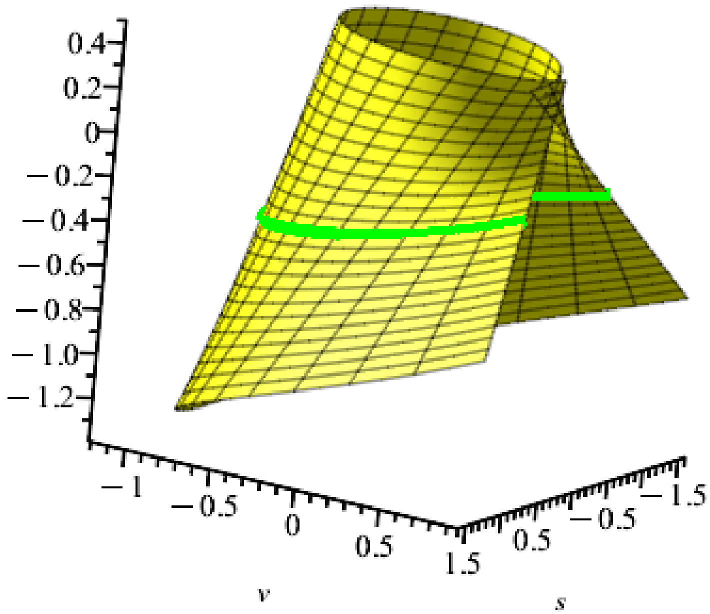

Fig. 15. The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + vT_1$

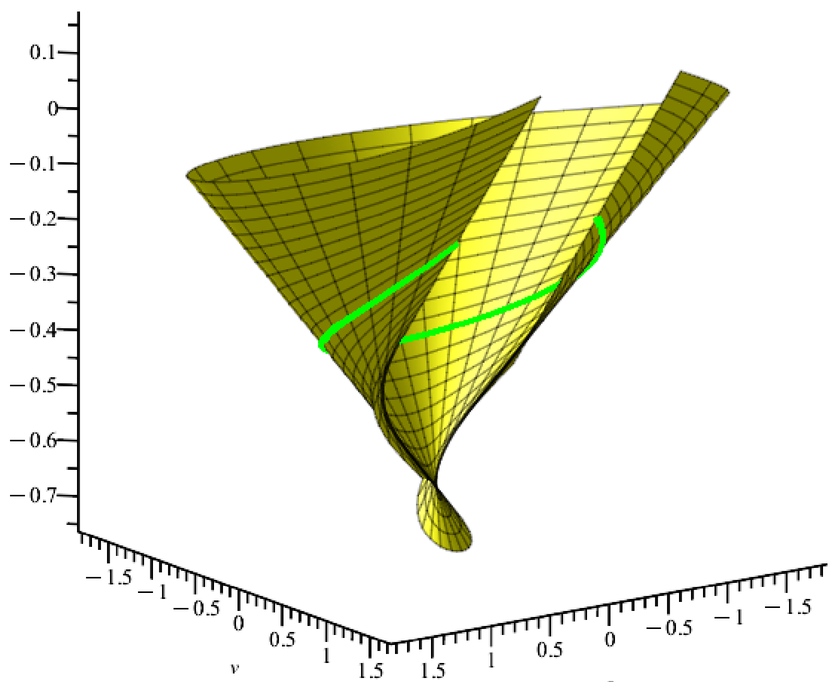

Fig. 16. The ruled surface $Q(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + vN_1$

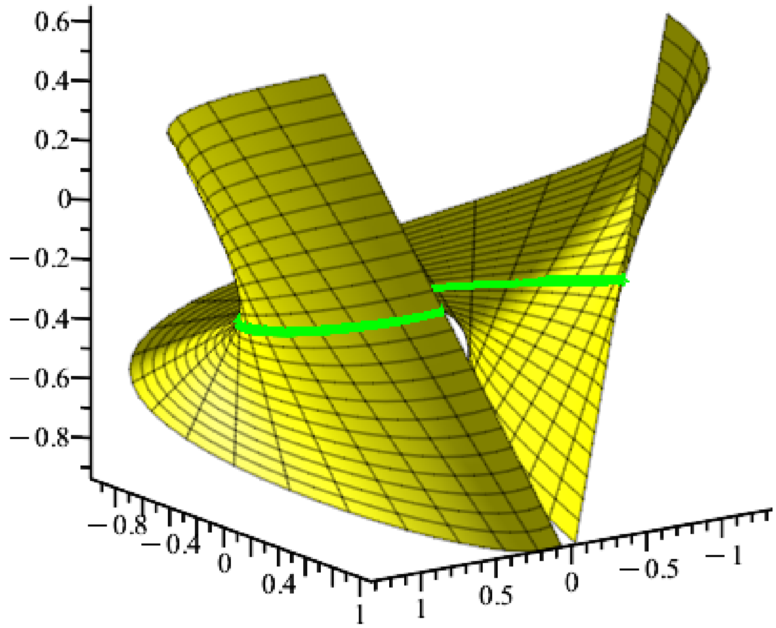

Fig. 17. The ruled surface $M(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + vB_1$

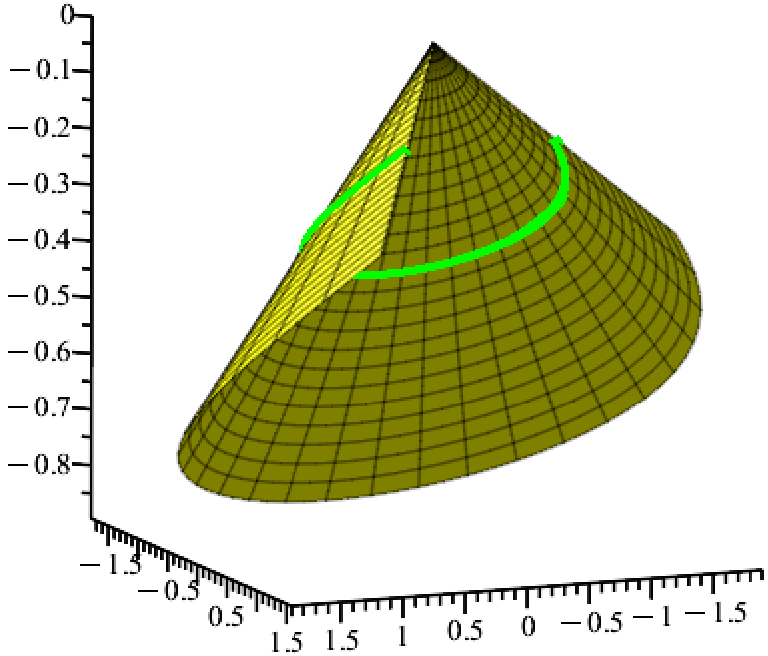

Fig. 18. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

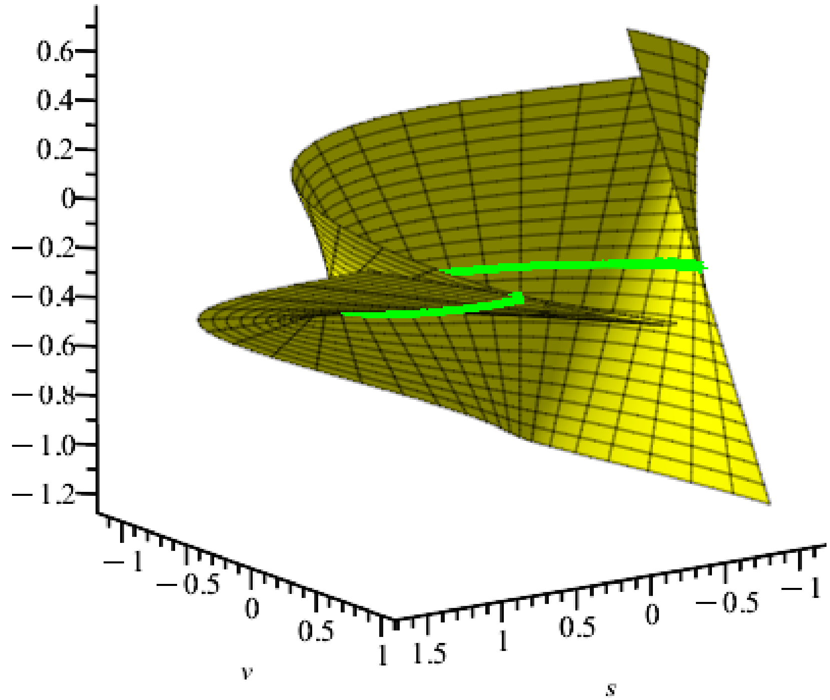

Fig. 19. The ruled surface $\psi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

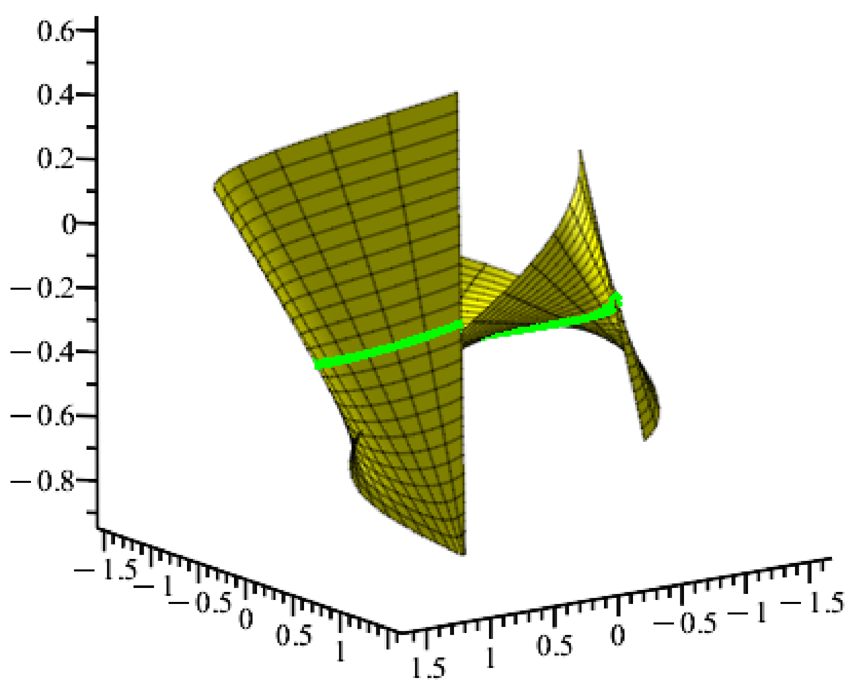

Fig. 20. The ruled surface $\eta(s,v)\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

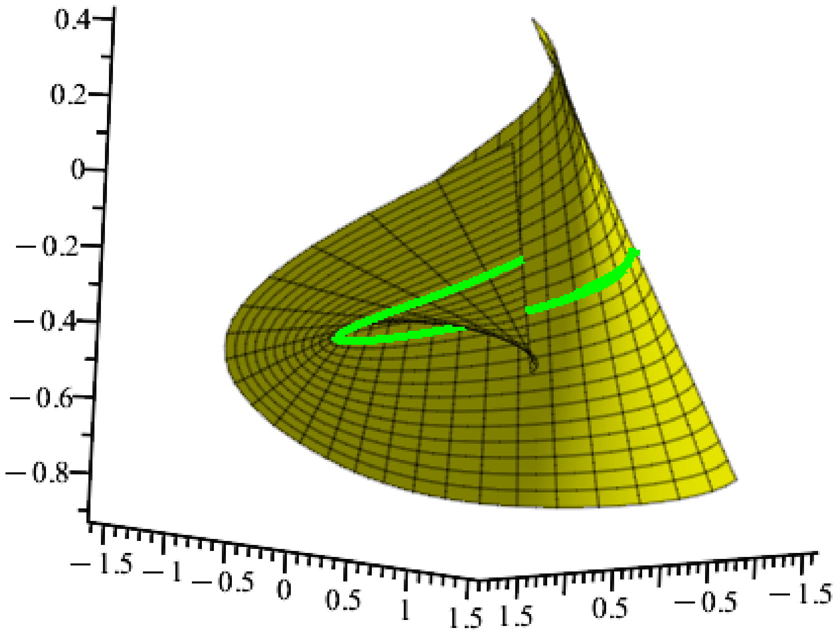

Fig. 21. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 27. *Let the Salkowski curve in Example 26 be the main curve. Theorem 2 the Successor frames are as follows:*

$$\begin{cases} T_1^*(s) &= \begin{pmatrix} -\cos(s+c)\left(\frac{3}{\sqrt{10}}\sin s\right)+\sin(s+c)\left(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\right), \\ \cos(s+c)\left(\frac{3}{\sqrt{10}}\cos s\right)+\sin(s+c)\left(-\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}}\right),\ -\cos(s+c)\frac{1}{\sqrt{10}}+\sin(s+c)\left(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\right) \end{pmatrix}, \\ N_1^*(s) &= \left(-\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}},\ -\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{3}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right), \\ B_1^*(s) &= \begin{pmatrix} \sin(s+c)\left(\frac{3}{\sqrt{10}}\sin s\right)+\cos(s+c)\left(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\right), \\ \sin(s+c)\left(\frac{3}{\sqrt{10}}\cos s\right)+\cos(s+c)\left(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\right),\ \sin(s+c)\frac{1}{\sqrt{10}}+\cos(s+c)\left(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\right) \end{pmatrix}. \end{cases}$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 22-28;

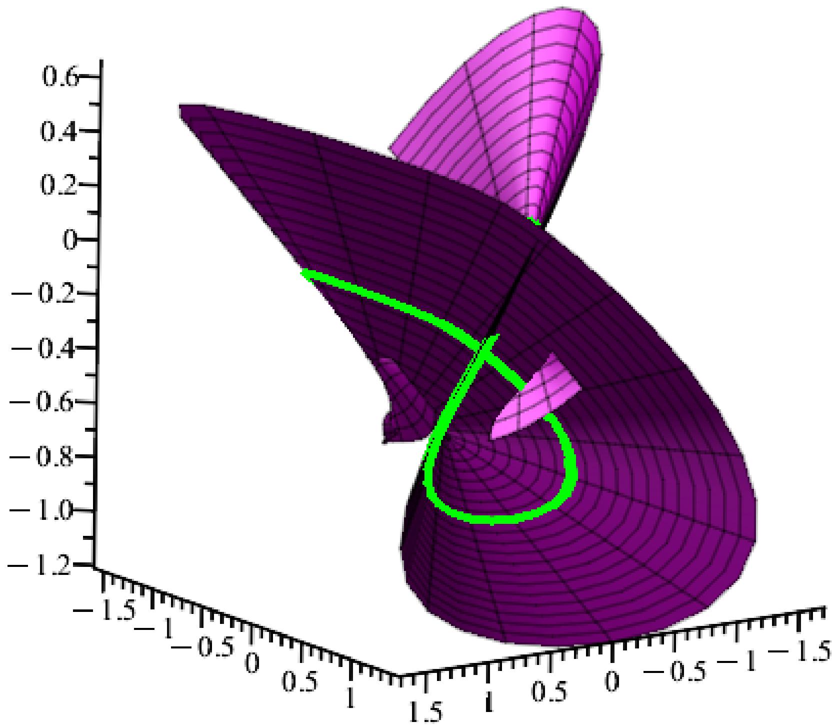

Fig. 22. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vT_1$

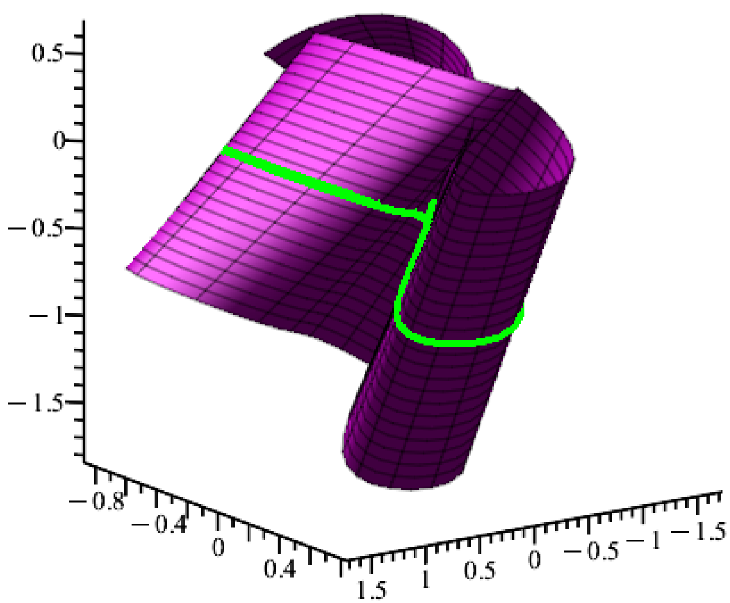

Fig. 23. The ruled surface $Q(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vN_1$

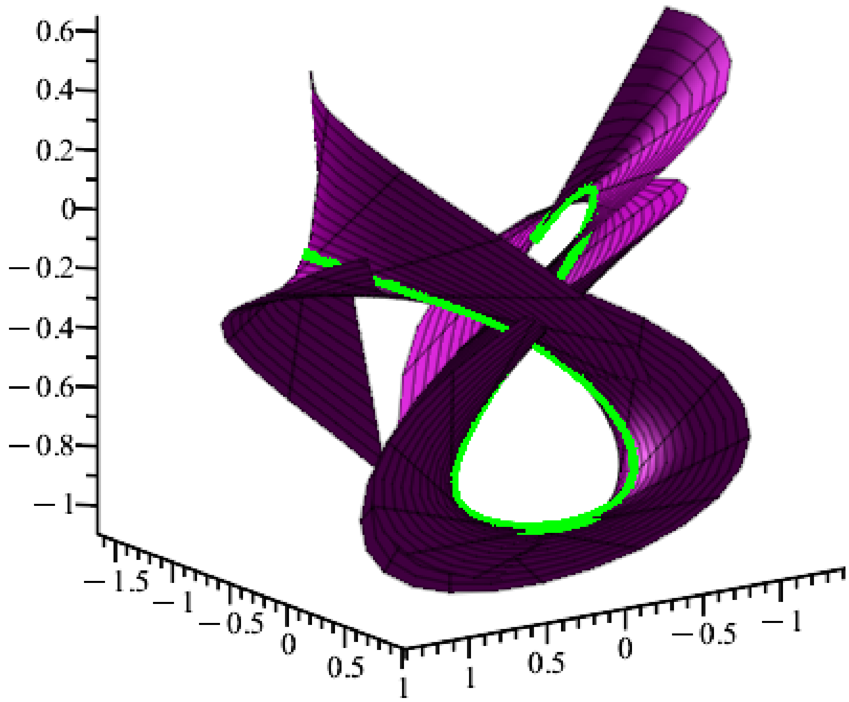

Fig. 24. The ruled surface $M(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+vB_1$

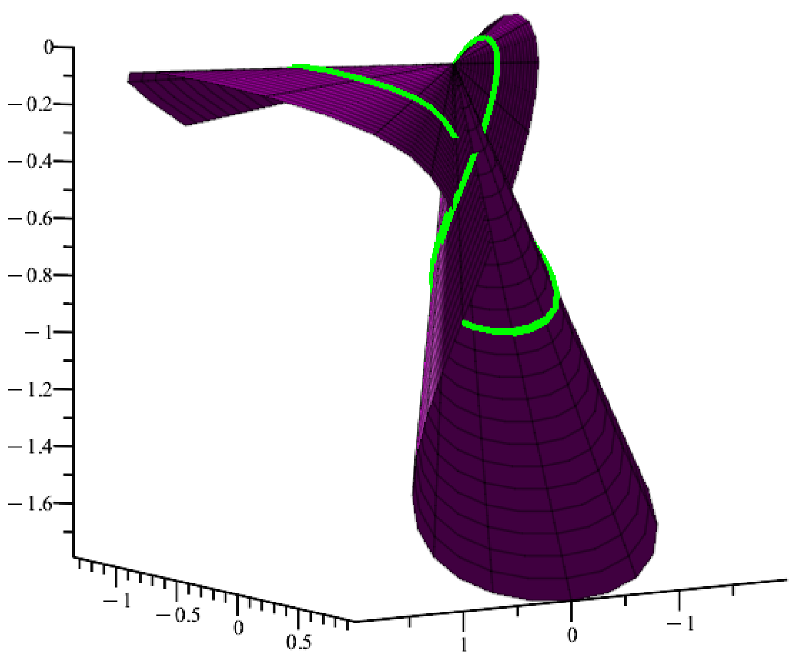

Fig. 25. The ruled surface $\mu(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+N_1)$

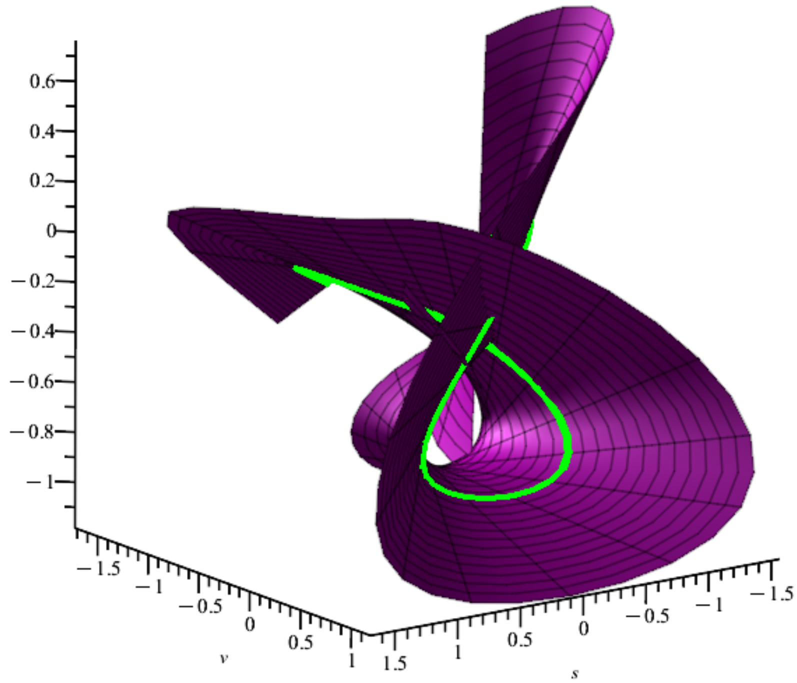

Fig. 26. The ruled surface $\psi(s,v)=\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

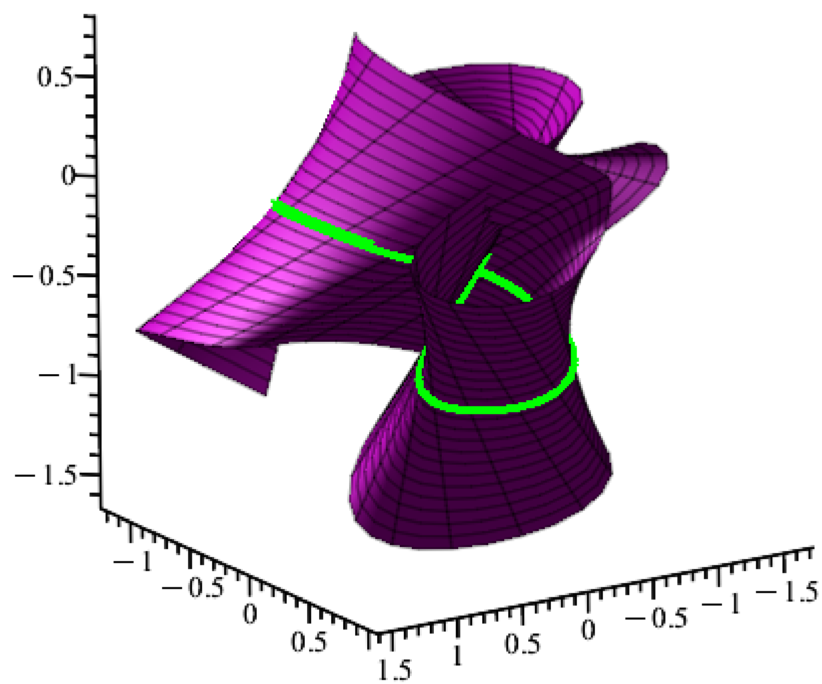

Fig. 27. The ruled surface $\eta(s,v)\frac{1}{\sqrt{2}}(T_1+N_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

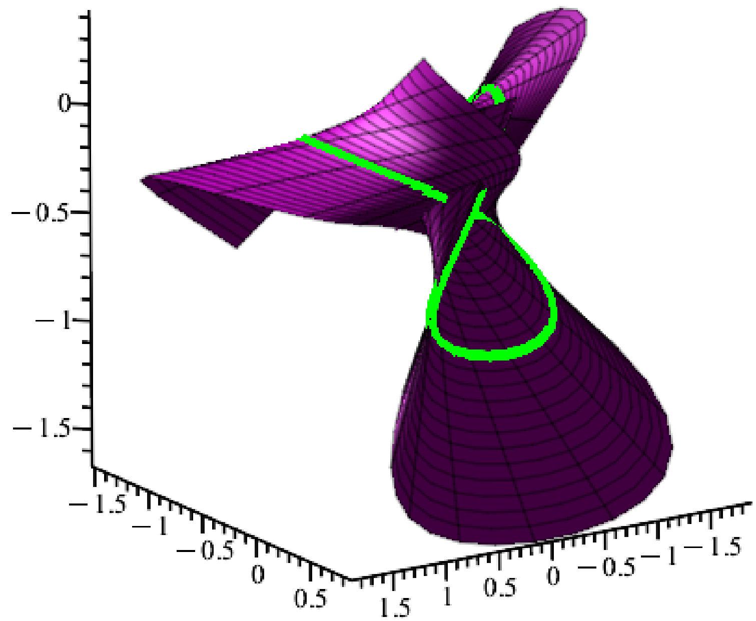

Fig. 28. The ruled surface $\Gamma(s,v) = \frac{1}{\sqrt{2}}(T_1 + N_1) + \frac{v}{\sqrt{3}}(T_1 + N_1 + B_1)$

4. Conclusions

In this study, the ruled surfaces obtained from the tangent and principal normal vectors of the successor curve were defined. The Gaussian and mean curvatures of the surfaces were calculated by using the coefficients of the first and the second fundamental forms. This work can be studied in the Euclidean and dual space by changing the curve. Also, similar work can be done in the Lorentz space.

5. Acknowledgements

We, as the authors, thank the referees who reviewed our study and made significant contributions with their opinions, and the journal editors for their valuable time.

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