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Some Novel Fractional Integral Inequalities for Exponentially Convex Functions

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Abstract

There are several studies in the literature with the main motivation of obtaining new and general inequalities with the help of the Caputo-Fabrizio fractional integral operator, which attracts the attention of many researchers as an important concept in fractional analysis. In this study, new Hadamard type integral inequalities for exponentially convex functions are presented. The findings were obtained by the properties of the function class, the structure of the operator and the basic analysis method.

Keywords: Caputo-Fabrizio fractional integral operator, Exponentially convex functions, Hölder inequality, Young inequality

1. *Introduction*

Inequality theory is a field in which many researchers work, with new findings that can be given applications in many disciplines such as mathematical analysis, statistics, approximation theory and numerical analysis together with convex functions. Although the term of convex function is a concept intertwined with inequalities by definition, it has also formed the main motivation of many researches with its aesthetic structure, features and different types. Let's start with the definition of this important class of functions.

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2. *Materials and Methods*

Definition 2.1. Let I be on interval in R. Then $\varphi: I \to R$ is said to be convex, if

$$\varphi(tx+(1-t)y) \le t\varphi(x)+(1-t)\varphi(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$ (Pečarić et al. 1992).

Definition 2.2. A function $\varphi: I \subseteq R \to R$ is said to be exponentially convex function, if

$$\varphi((1-t)x+ty) \le (1-t)\frac{\varphi(x)}{e^{\alpha x}}+t\frac{\varphi(y)}{e^{\alpha y}}$$

for all $x, y \in I$, $\alpha \in R$ and $t \in [0,1]$(Awan et all. 2018).

Note that if α = 0, then the class of exponentially convex functions reduces to class of classical convex functions. However, the converse is not true.

Definition 2.3. Let $\varphi \in H^1(0,b), b > a, \alpha \in [0,1]$ then, the definitions of the left and right hand side of Caputo-Fabrizio fractional integral are:

$$\left({}^{CF}_{\ a}I^{\alpha}\right)(t) = \frac{1-\alpha}{B(\alpha)}\varphi(t) + \frac{\alpha}{B(\alpha)}\int_a^t \varphi(y)dy,$$

and

$$\left({}^{CF}I^{\alpha}_{b}\right)(t) = \frac{1-\alpha}{B(\alpha)}\varphi(t) + \frac{\alpha}{B(\alpha)}\int_t^b \varphi(y)dy$$

where $B(\alpha) > 0$ is normalization function (Abdeljawad and Baleanu 2017).

In the sequel of the paper, we will denote normalization function as $B(\alpha)$ with $B(0) = B(1) = 1$.

In (Tariq et al. 2022), the authors provided an integral inequality of Hermite-Hadamard type for preinvex functions via Caputo-Fabrizio fractional integral inequality as follows.

Theorem 2.1. Let $\varphi: I = [k_1, k_1 + \mu(k_2, k_1)] \to (0, \infty)$ be a preinvex function on I° and $\varphi \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $\alpha \in [0,1]$, then the following inequality holds:

$$\varphi\left(\frac{2k_1+\mu(k_2,k_1)}{2}\right) \leq \frac{B(\alpha)}{\alpha\mu(k_2,k_1)} \times\left[{}_{k_1}^{CF}I^{\alpha}\{\varphi(k)\}+{}^{CF}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{\varphi(k)\} - \frac{2(1-\alpha)}{B(\alpha)}\varphi(k)\right] \leq \frac{\varphi(k_1)+\varphi(k_2)}{2}$$

where $k\in[k_1,k_1+\mu(k_2,k_1)]$.

Atangana and Baleanu introduced a new derivative operator using Mittag-Leffler function in Caputo-Fabrizio derivative operator as the following.

Definition 2.4. (Atangana and Baleanu 2016). Let $\varphi\in H^1(0,b), b>a, \alpha\in[0,1]$ then, the definition of the new fractional derivative is given as follows:

(1.1) $$\left({}_{a}^{ABC}D_t^{\alpha}\right)[\varphi(t)] = \frac{B(\alpha)}{1-\alpha}\int_a^t \varphi'(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx.$$

Definition 2.5. (Atangana and Baleanu 2016). Let $f\in H^1(0,b), b>a, \alpha\in[0,1]$ then, the definition of the new fractional derivative is given by:

(1.2) $$\left({}_{a}^{ABR}D_t^{\alpha}\right)[\varphi(t)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t \varphi(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx.$$

Equations (1.1) and (1.2) have a non-local kernel. Also in equation (1.1) when the function is constant, we get zero.

The related fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 2.6. The fractional integral associate to the new fractional derivative with non-local kernel of a function $\varphi\in H^1(a,b)$ as defined:

$${}_{a}^{AB}I^{\alpha}\{\varphi(t)\} = \frac{1-\alpha}{B(\alpha)}\varphi(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t \varphi(y)(t-y)^{\alpha-1}dy$$

where, $b>a,\ \alpha\in[0,1]$ (Atangana and Baleanu 2016).

Abdeljawad and Baleanu introduced the right hand side of the integral operator as following;

The right fractional new integral with ML kernel of order $\alpha\in[0,1]$ is defined by

$${}^{AB}I_b^{\alpha}\{\varphi(t)\} = \frac{1-\alpha}{B(\alpha)}\varphi(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b \varphi(y)(y-t)^{\alpha-1}dy.$$

where $,b>a,\ \alpha\in[0,1]$ (Abdeljawad and Baleanu 2017).

For more information related to different kinds of fractional operators, we recommend the following articles to the readers (Abdeljawad 2015, Abdeljawad and Baleanu 2016, Akdemir et al. 2021- Akdemir et al. 2017, Aslan 2023, Butt et al. 2020, Caputo and Fabrizio 2015-Gürbüz et al. 2020, Rashid et al 2020 (a)-Samko et al. 1993, Set 2012, Set et al. 2017).

Theorem 2.2. (Awan et al. 2018) Let $\varphi: I\subseteq R\to R$ be an integrable exponentially convex function, then

$$\varphi\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\frac{\varphi(u)}{e^{\alpha u}}du \leq \frac{e^{-\alpha a}\varphi(a)+e^{-\alpha b}\varphi(b)}{2}.$$

For more information on exponentially convex functions, we recommend the following articles to the readers(Akdemir et al. 2023, Akdemir et al. 2022(a)-Akdemir et al. 2022(b), Aslan and Akdemir 2023-Aslan et al. 2022(b), Rashid et al 2019 (a), Rashid et al 2019 (b)).

3. *Results*

Theorem 3.1. Let $\mathrm{I}\subseteq\mathbb{R}$. Suppose that $\varphi:[a,b]\subseteq \mathrm{I}\longrightarrow\mathbb{R}$ is an exponentially-convex function on $[a,b]$ such that $\varphi\in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}_{a}^{CF}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I_b^{\alpha}\varphi\right)(k) \leq \frac{4(1-\alpha)\varphi(k)+\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}}+\frac{\varphi(b)}{e^{\alpha b}}\right)}{2B(\alpha)}$$

where $B(\alpha)>0$ is normalization function $\alpha\in[0,1]$.

Proof. By using the definition of exponentially-convex function, we can write:

$$\varphi\big((1-t)a+tb\big)\leq(1-t)\frac{\varphi(a)}{e^{\alpha a}}+t\frac{\varphi(b)}{e^{\alpha b}}.$$

By integrating both sides of the inequality over $[0,1]$ with respect to t, we get:

$$\int_0^1\varphi\big((1-t)a+tb\big)\,dt = \frac{\varphi(a)}{e^{\alpha a}}\int_0^1(1-t)\,dt + \frac{\varphi(b)}{e^{\alpha b}}\int_0^1 t\,dt.$$

By changing of the variable as $x = \big((1-t)a + tb\big)$ we obtain and by calculating the right hand side, we obtain:

$$\frac{1}{b-a}\int_a^b \varphi(x)\,dx = \frac{\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}}{2}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\varphi(k)$, we have:

$$\frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_a^b \varphi(x)\,dx = \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)}{B(\alpha)}\frac{\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}}{2}.$$

By simplifying the inequality, we get the result.

$$\left(\frac{1-\alpha}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_a^k \varphi(x)\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_k^b \varphi(x)\,dx\right) = \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)}{B(\alpha)}\frac{\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}}{2}.$$

$$\left({}^{CF}_{a}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{b}\varphi\right)(k) \le \frac{4(1-\alpha)\varphi(k) + \alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{2B(\alpha)}.$$

Theorem 3.2. Let $I \subseteq \mathbb{R}$. Suppose that $\varphi:[a,b] \subseteq I \longrightarrow \mathbb{R}$ is an exponentially-convex function on $[a,b]$ such that $\varphi \in L_1[a,b]$. Then ,we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}_{a}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{b}\varphi\right)(k) \le \frac{2(1-\alpha)\varphi(k)(p+1)^{\frac{1}{p}} + \alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)(p+1)^{\frac{1}{p}}},$$

where $B(\alpha) > 0$ is normalization function $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

Proof. By using the definition of exponantially-convex function, we can write:

$$\varphi\big((1-t)a + tb\big) \le (1-t)\frac{\varphi(a)}{e^{\alpha a}} + t\frac{\varphi(b)}{e^{\alpha b}}.$$

By integrating both sides of the inequality over $[0,1]$ with respect to t, we get:

$$\int_0^1 \varphi\big((1-t)a + tb\big)\,dt = \frac{\varphi(a)}{e^{\alpha a}}\int_0^1 (1-t)\,dt + \frac{\varphi(b)}{e^{\alpha b}}\int_0^1 t\,dt.$$

If we apply the Hölder's inequality to the right hand side of the inequality, we get:

$$\int_0^1 \varphi\big((1-t)a + tb\big)\,dt \le \frac{\varphi(a)}{e^{\alpha a}}\left(\int_0^1 |1-t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1dt\right)^{\frac{1}{q}} + \frac{\varphi(b)}{e^{\alpha b}}\left(\int_0^1 |t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1dt\right)^{\frac{1}{q}}.$$

By changing of the variable as $x = (1-t)a + tb$ and calculating the right hand side, we obtain:

$$\frac{1}{b-a}\int_a^b \varphi(x)\,dx = \frac{\varphi(a)}{e^{\alpha a}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} + \frac{\varphi(b)}{e^{\alpha b}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\varphi(k)$, we have:

$$\frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_a^b \varphi(x)\,dx = \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

$$\left(\frac{1-\alpha}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_a^k \varphi(x)\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_k^b \varphi(x)\,dx\right) = \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

By simplifying the inequality, we get the result.

$$\left({}^{CF}_{a}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{b}\varphi\right)(k) \le \frac{2(1-\alpha)\varphi(k)(p+1)^{\frac{1}{p}} + \alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)(p+1)^{\frac{1}{p}}}.$$

Theorem 3.3. Let $I \subseteq \mathbb{R}$. Suppose that $\varphi: [a,b] \subseteq I \longrightarrow \mathbb{R}$ is an exponentially convex function on $[a,b]$ such that $\varphi \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}_{a}^{CF}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I_{b}^{\alpha}\varphi\right)(k) \leq \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)}\left(\frac{q+p(p+1)}{qp(p+1)}\right)$$

where $B(\alpha) > 0$ is normalization function $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

Proof : By using the definition of exponantiallyconvex function, we can write:

$$\varphi\big((1-t)a + tb\big) \leq (1-t)\frac{\varphi(a)}{e^{\alpha a}} + t\frac{\varphi(b)}{e^{\alpha b}}.$$

By integrating both sides of the inequality over $[0,1]$ with respect to t, we get

$$\int_0^1 \varphi\big((1-t)a + tb\big)\,dt = \frac{\varphi(a)}{e^{\alpha a}}\int_0^1 (1-t)\,dt + \frac{\varphi(b)}{e^{\alpha b}}\int_0^1 t\,dt.$$

If we apply the Young's inequality to the right hand side of the inequality, we get:

$$\int_0^1 \varphi\big((1-t)a + tb\big)\,dt \leq \frac{\varphi(a)}{e^{\alpha a}}\left(\frac{1}{p}\left(\int_0^1 |1-t|^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q dt\right)\right) + \frac{\varphi(b)}{e^{\alpha b}}\left(\frac{1}{p}\left(\int_0^1 |t|^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q dt\right)\right).$$

By changing of the variable as $x = (1-t)a + tb$ and calculating the right hand side, we obtain:

$$\frac{1}{b-a}\int_a^b \varphi(x)\,dx = \frac{\varphi(a)}{e^{\alpha a}}\left(\frac{1}{p(p+1)} + \frac{1}{q}\right) + \frac{\varphi(b)}{e^{\alpha b}}\left(\frac{1}{p(p+1)} + \frac{1}{q}\right).$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\varphi(k)$, we have:

$$\frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_a^b \varphi(x)\,dx = \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)}\left(\frac{1}{p(p+1)} + \frac{1}{q}\right).$$

By simplifying the inequality, we get the result.

$$\left(\frac{1-\alpha}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_a^k \varphi(x)\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)}\varphi(k) + \frac{\alpha}{B(\alpha)}\int_k^b \varphi(x)\,dx\right) = \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)}\left(\frac{q+p(p+1)}{qp(p+1)}\right).$$

$$\left({}_{a}^{CF}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I_{b}^{\alpha}\varphi\right)(k) \leq \frac{2(1-\alpha)}{B(\alpha)}\varphi(k) + \frac{\alpha(b-a)\left(\frac{\varphi(a)}{e^{\alpha a}} + \frac{\varphi(b)}{e^{\alpha b}}\right)}{B(\alpha)}\left(\frac{q+p(p+1)}{qp(p+1)}\right).$$

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