ISSN: 2458-8989

Natural and Engineering Sciences

NESciences, 2017, 2 (3): 37-44

-RESEARCH ARTICLE-

Haar wavelet collocation method for the approximate solutions of Emden-Fowler type equations

Sertan Alkan*

Department of Computer Engineering, Iskenderun Technical University, Hatay, Turkey

Abstract

This paper investigates the Haar wavelet collocation method (HWCM) to obtain approximate solution of the linear Emden-Fowler type equations. To show the efficiency and accuracy of the proposed method, some problems are solved and the obtained solutions are compared with the approximate solutions obtained by using the other numerical methods as well as the exact solutions of the problems.

Keywords:
Haar wavelet collocation method, linear Emden-Fowler type equations, initial value problems

Article history:
Received 15 September 2017, Accepted 17 October 2017, Available online 30 October 2017

Introduction

Singular initial value problems are widely used in mathematical physics and astrophysics. One of this problems is initial value problems (IVPs) of Emden–Fowler type equations. Many methods havebeen developed for solving these equations numerically. Some numerical methods in the literature include the following: the homotopy perturbation method (Chowdhury & Hashim, 2009, Ahamed et al., 2017), Adomian decomposition method (Wazwaz, 2005; Wazwaz, 2005; Wazwaz et al., 2014) the variational iteration method (Wazwaz, 2015), differential transform method (Ibis, 2012), homotopy analysis method (Bataineh et al., 2009), hybrid functions method (Tabrizidooz et al., 2017).

In this paper, Haar collocation method is presented to obtain the approximate solution of initial value problems (IVPs) of generalized Emden–Fowler type equations in the following form

* *Corresponding Author: Sertan Alkan, e-mail: sertan.alkan@iste.edu.tr*

$$y'' + \frac{n}{x}y' + \mu f(x)g(y) = r(x), \qquad 0 < x \le 1$$

subject to the conditions

$$y(0) = \alpha, \qquad y'(0) = \beta$$

where n, α, β and μ are constants, and $f(x), g(y)$ and $r(x)$ continuous functions. Also, especially, $g(y)$ is a linear function.

The paper organized as follows. In section 2, we have given some definitions for Haar wavelet collocation method. In section 3, we use Haar wavelet collocation method to obtain approximate solution of the class of equations given by (1). In section 4, by using tables and graphs, some test problems are given to show the abilities of present method. Finally, in section 5, we have completed the paper with a conclusion.

Haar Wavelets

Haar wavelet collocation method is frequently used in the literature. For more details, see (Li & Zhao, 2010; Lepik, 2009; Rehman & Khan, 2012; Chang & Piau, 2008; Lepik, 2005; Lepik, 2008; Hsiao, 2004). The Haar wavelet family for $t \in [0,1]$ is defined as follows:

$$h_i(t) = \begin{cases} 1, & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1, & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0, & \text{elsewhere.} \end{cases}$$

Integer $m = 2^j$ $(j = 0,1,2, \dots, J)$ indicates the level of the wavelet; $k = 0,1,2, \dots, m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$; in the case of minimal values. $m = 1, k = 0$, we have $i = 2$, the maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1$ in [0, 1]. Let us define the collocation points $t_l = (l - 0.5)/2M, (l = 1,2, \dots, 2M)$ and discretise the Haar function $h_i(t)$; in this way we get the coefficient matrix $H(i,l) = \left(h_i(t_l)\right)$, which has the dimension $2M \times 2M$.

The operational matrix of integration P, which is a $2M$ square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t)dt$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^{t} h_i(t)dt$$

The elements of the matrices H, P and Q can be evaluated

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$H_4=\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&0&0\\0&0&1&-1\end{pmatrix},\qquad P_4=\frac{1}{16}\begin{pmatrix}8&-4&-2&-2\\4&0&-2&2\\1&1&0&0\\1&-1&0&0\end{pmatrix}$$

Chen and Hsiao [Chen & Hsiao, 1997] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)}=\frac{1}{2m}\begin{pmatrix}2mP_{\left(\frac{m}{2}\right)} & -H_{\left(\frac{m}{2}\right)}\\ H^{-1}_{\left(\frac{m}{2}\right)} & O\end{pmatrix}$$

where O is a null matrix of order $\frac{m}{2}\times\frac{m}{2}$,

$$H_{m\times m}=[h_m(t_0)\ h_m(t_1)\ \dots\ h_m(t_{m-1})]$$

and $\frac{i}{m}\le t<i+\frac{1}{m}$ and $H^{-1}_{m\times m}=\frac{1}{m}H^T_{m\times m}diag(r)$.

Function approximation

Any function $y(x)\in L^2[0,1)$ is given by

$$y(x)=\sum_{n=0}^{\infty}c_nh_n(x)$$

where the coefficients c_n are calculated by

$$c_n=2^j\int_0^1 y(x)h_n(x)dx$$

where $n=2^j+k, j\ge 0, 0\le k<2^j$. Specially $c_0=\int_0^1 y(x)dx$. If $y(x)$ is piecewise constant by itself, then $y(x)$ will be terminated at finite terms, that is

$$y(x)=\sum_{n=0}^{m-1}c_nh_n(x)=c^T_{(m)}h_{(m)}(x) \tag{1}$$

where the coefficients $c^T_{(m)}$ and the Haar function vector $h_{(m)}(x)$ are defined for $m=2^j$ as

$$c^T_{(m)}=[c_0,c_1,\dots,c_{m-1}]$$

and

$$h_{(m)}(x)=[h_0(x),h_1(x),\dots,h_{m-1}(x)]^T.$$

Method of solution of linear Emden-Fowler equation

Consider linear Emden-Fowler type equation

$$y'' + \frac{n}{x}y' + \mu f(x)y = r(x), \qquad 0 < x \leq 1 \tag{2}$$

with initial conditions $y(0) = \alpha,\ y'(0) = \beta$. If terms of equation (2) using (1) expand in terms of Haar wavelets, then we obtain

$$y''(x) = c_{(m)}^T h_{(m)}(x) \tag{3}$$

$$y'(x) = c_{(m)}^T P_{(m)} h_{(m)}(x) + \beta \tag{4}$$

$$y(x) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + \alpha \tag{5}$$

and, similarly, $r(x)$ may be expanded by the Haar functions as follows:

$$r(x) = r_{(m)}^T h_{(m)}(x) \tag{6}$$

where $r_{(m)}^T$ is a known constant vector. Substituting equations (3)-(6) into the equation (2), we obtain,

$$c_{(m)}^T h_{(m)}(x) + \frac{n}{x}\left(c_{(m)}^T P_{(m)} h_{(m)}(x) + \beta\right) + \mu f(x)\left(c_{(m)}^T Q_{(m)} h_{(m)}(x) + \alpha\right) = r_{(m)}^T h_{(m)}(x) \tag{7}$$

In equation (7), if we write collocation points t_l, it is obtained that linear Haar matrix system. When this system is solved, we obtain unknown coefficients c_n in (1).

Numerical examples

In this section, we present several numerical examples for the Emden–Fowler type equations to show the accuracy of the introduced method. We use Mathematica10 for all calculations.

Example1 Consider linear Emden-Fowler equation given by [Chowdhury & Hashim, 2009, Iqbal & Javed, 2011] in the following form

$$y'' + \frac{8}{x}y' + xy = r(x), \qquad y(0) = y'(0) = 0 \tag{8}$$

where $r(x) = x^5 - x^4 + 44x^2 - 30x$ and the exact solution of equation (8) is $y(x) = x^4 - x^3$. The numerical solutions obtained by using the present method for this problem are presented in Table 1. In Table 2, it is given that the comparisons of the present method with optimal homotopy asymptotic method (OHAM). Additionally, the graphics of the exact and approximate solutions for different values of m for Example1 are given in Figure 1.

Table 1. Absolute errors for different values of m

x	$m = 16$	$m = 64$	$m = 256$
0.0	0.0	0.0	0.0
0.1	1.094×10^{-4}	5.219×10^{-6}	3.213×10^{-7}
0.2	1.343×10^{-4}	8.380×10^{-6}	5.263×10^{-7}
0.3	1.600×10^{-4}	9.745×10^{-6}	6.116×10^{-7}
0.4	1.470×10^{-4}	9.313×10^{-6}	5.771×10^{-7}
0.5	1.090×10^{-4}	6.813×10^{-6}	4.258×10^{-7}
0.6	5.003×10^{-5}	2.331×10^{-6}	1.551×10^{-7}
0.7	8.216×10^{-5}	3.557×10^{-6}	2.483×10^{-7}
0.8	1.695×10^{-4}	1.267×10^{-5}	7.612×10^{-7}
0.9	3.776×10^{-4}	2.257×10^{-5}	1.429×10^{-6}
1.0	5.702×10^{-4}	3.564×10^{-5}	2.227×10^{-6}

Table 2. Comparison of HWCM with OHAM

x	Exact Solution	Abs. Error for HCM ($m = 512$)	Abs. Error for OHAM(Zeroth order)
0.0	0.0	0.0	0.0
0.1	-0.0009	8.075×10^{-8}	1.180×10^{-8}
0.2	-0.0064	1.314×10^{-7}	6.899×10^{-7}
0.3	-0.0189	1.528×10^{-7}	7.114×10^{-6}
0.4	-0.0384	1.447×10^{-7}	3.579×10^{-5}
0.5	-0.0625	1.064×10^{-7}	1.206×10^{-4}
0.6	-0.0864	3.776×10^{-8}	3.125×10^{-4}
0.7	-0.1029	6.112×10^{-8}	6.679×10^{-4}
0.8	-0.1024	1.914×10^{-7}	1.220×10^{-3}
0.9	-0.0729	3.556×10^{-7}	1.932×10^{-3}
1.0	0.0	5.569×10^{-7}	2.616×10^{-3}

Example2 Consider linear Emden-Fowler equation given by Ahamed et al. (2017) in the following form;

$$y'' + \frac{2}{x}y' + y = r(x), \qquad y(0) = y'(0) = 0 \tag{9}$$

where $r(x) = x^3 + x^2 + 12x + 6$ and the exact solution of equation (9) is $y(x) = x^2 + x^3$. The numerical solutions obtained by using the present method for this problem are presented in Table 3. Additionally, the graphics of the exact and approximate solutions for different values of m for Example2 are given in Figure 2.

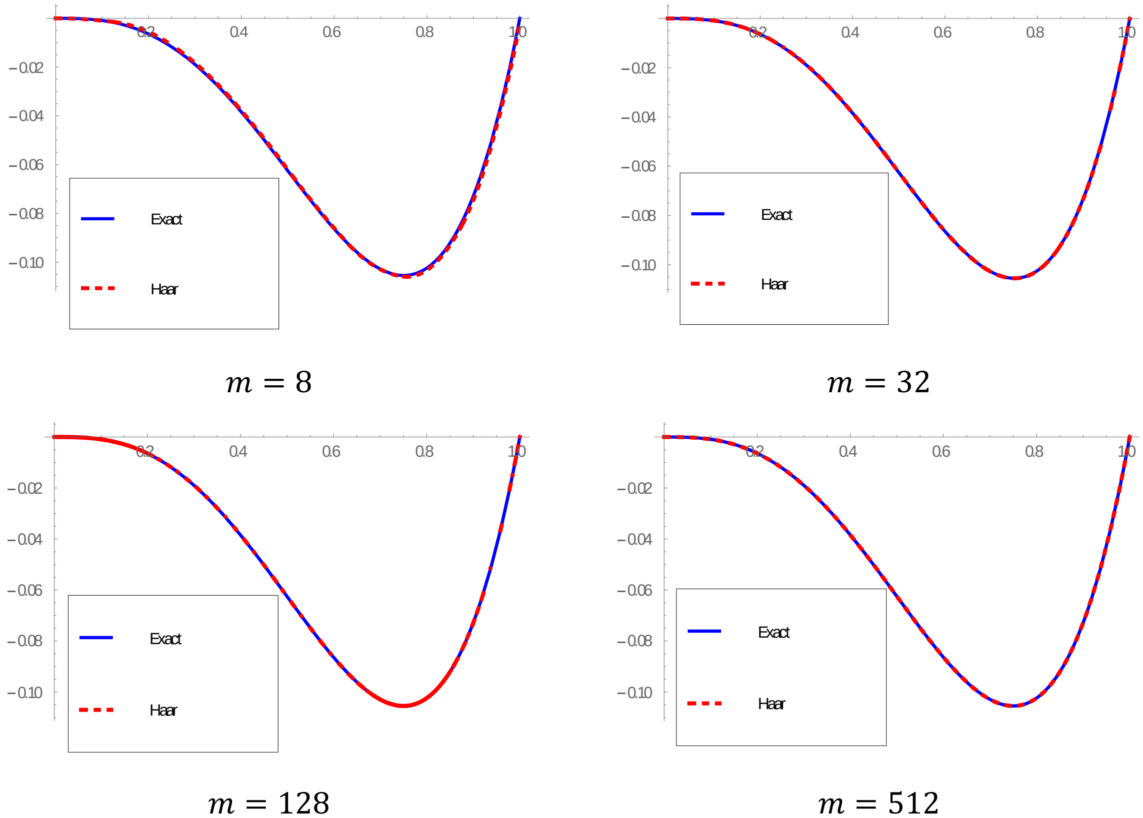

Figure 1. The graphics of the exact and approximate solutions for different values of m for example1.

Table 3. Absolute errors for different values of m

x	$m = 16$	$m = 64$	$m = 256$
0.0	0.0	0.0	0.0
0.1	6.042×10^{-5}	6.194×10^{-6}	3.865×10^{-7}
0.2	2.003×10^{-4}	1.245×10^{-5}	7.960×10^{-7}
0.3	2.909×10^{-4}	1.964×10^{-5}	1.215×10^{-6}
0.4	4.259×10^{-4}	2.655×10^{-5}	1.668×10^{-6}
0.5	5.434×10^{-4}	3.427×10^{-5}	2.143×10^{-6}
0.6	6.682×10^{-5}	4.246×10^{-5}	2.647×10^{-6}
0.7	8.247×10^{-5}	5.082×10^{-5}	3.191×10^{-6}
0.8	9.493×10^{-4}	6.040×10^{-5}	3.761×10^{-6}
0.9	1.124×10^{-4}	6.996×10^{-5}	4.380×10^{-6}
1.0	1.287×10^{-3}	8.054×10^{-5}	5.034×10^{-6}

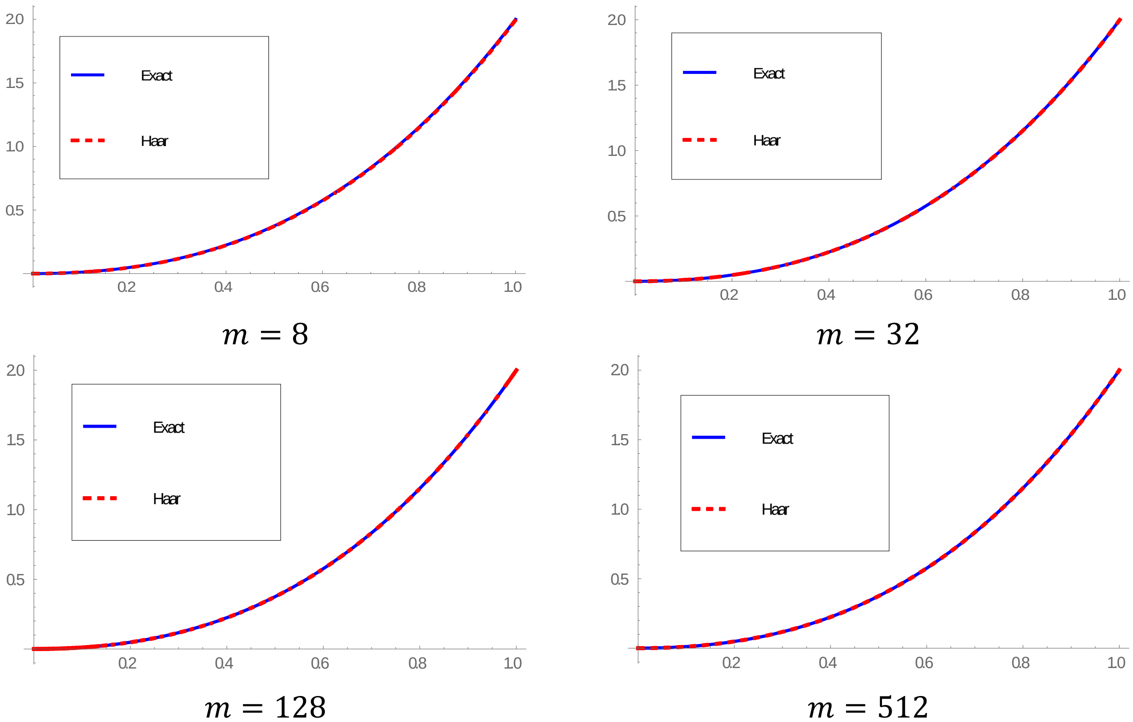

Figure 2. The graphics of the exact and approximate solutions for different values of m for example2.

Conclusion

In this paper, Haar wavelet collocation method is applied to linear Emden-Fowler type equations with initial conditions. All computations associated with the examples are done using Mathematica. The exact and approximate solutions are compared for all the examples. It can be concluded that Haar wavelet collocation method is a quite effective and accurate method. We aim to apply Haar wavelet collocation method to nonlinear Emden-Fowler type equations in a furure study.

References

Ahamed, M. S., Hasan, M.K. , Alam, M. S., (2017) A New Approach to Homotopy Perturbation Method for Solving Emden–Fowler Equations., *Applied Mathematical Sciences* 11.40: 1955-1964.

Bataineh, A. S., Mohd, S. M. N., Hashim, I., (2009) Homotopy analysis method for singular IVPs of Emden–Fowler type. *Communications in Nonlinear Science and Numerical Simulation* 14.: 1121-1131.

Chang, P., Piau, P, (2008) Haar wavelet matrices designation in numerical solution of ordinary differential equations, *IAENG International Journal of Applied Mathematics* 38.3 (2008): 164-168.

Chen, C.F., Hsiao, C.H., (1997) Haar wavelet method for solving lumped and distributed-parameter systems, *IEE Proc.: Part D* 144 (1) 87–94.

Chowdhury, M. S. H., Hashim, I., (2009) Solutions of Emden–Fowler equations by homotopy-perturbation method. *Nonlinear Analysis: Real World Applications* 10.1: 104-115.

Hsiao, C. H., (2004) Haar wavelet approach to linear stiff systems. *Mathematics and Computers in Simulation* vol 64, pp. 561-567.

Ibis, B., (2012) Approximate analytical solutions for nonlinear Emden-Fowler type equations by differential transform method. *arXiv preprint arXiv*:1211.3521.

Iqbal, S., Javed, A., (2011) Application of optimal homotopy asymptotic method for the analytic solution of singular Lane–Emden type equation. *Applied Mathematics and Computation* 217.19: 7753-7761.

Lepik, U., (2008) Haar wavelet method for solving higher order differential equations *Int. J. Math. Comput* 1 : 84-94.

Lepik, U., (2005) Numerical solution of differential equations using Haar wavelets. *Mathematics and Computers in Simulation* vol 68, pp. 127-143.

Lepik, U., (2009) Solving fractional integral equations by the Haar wavelet method, *Appl. Math. Comput*. 214: 468–478.

Li, Y., Zhao, W., (2010) Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, *Appl. Math. Comput*. 216 2276–2285.

Rehman, M. U., Khan, R. A., A., (2012) numerical method for solving boundary value problems for fractional differential equations. *Applied Mathematical Modelling*, 36(3), 894-907.

Tabrizidooz, H. R., Marzban, H. R., Razzaghi, M., (2009) Solution of the generalized Emden–Fowler equations by the hybrid functions method. *Physica Scripta* 80.2: 025001.

Wazwaz, A. M., Rach, R., Bougoffa, L., Duan, J. S., (2014) Solving the Lane–Emden–Fowler type equations of higher orders by the Adomian decomposition method. *Comput. Model. Eng. Sci.(CMES)* 100.6: 507-529.

Wazwaz, A.M., (2005) Adomian decomposition method for a reliable treatment of the Emden–Fowler equation”, *Appl. Math. Comput*. 161, 543–560.

Wazwaz, A.M., (2005) Analytical solution for the time-dependent Emden–Fowler type of equations by Adomian decomposition method, *Appl. Math.Comput*. 166 (2005) 638–651.

Wazwaz, A.M., (2015) Solving Two Emden-Fowler Type Equations of Third Order by the Variational Iteration Method. *Applied Mathematics & Information Sciences* 9.5: 2429.