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# On The Konhauser Biorthogonal Polynomials

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Abstract. In this study, we give new properties of the Konhauser biorthogonal polynomials $Y_n^\alpha(x;k)$. Various families of multilinear, multilateral generating functions, some special cases and several recurrence relations for these polynomials are obtained.

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**Keywords:** Konhauser biorthogonal polynomials, generating function, multilinear and multilateral generating function, recurrence relations.

## 1. Introduction

The Konhauser biorthogonal polynomials $Y_n^\alpha(x;k)$ are defined by the generating relation (see, for example, [11], p. 432)

$$\sum_{n=0}^{\infty} Y_n^\alpha(x;k)\,t^n = (1-t)^{\frac{-(\alpha+1)}{k}} \exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \tag{1.1}$$

where, $\alpha > -1$, $k$ is a positive integer.

It is from (1.1) that [11],

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n, \tag{1.2}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of gamma function) by

$$\begin{aligned}(\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-)\\ &= \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}\\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}\end{cases}\end{aligned}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

These polynomials have the following generating relation [11]:

$$\sum_{n=0}^{\infty}\binom{n+m}{n}Y_{n+m}^\alpha(x;k)t^n = (1-t)^{\frac{-(\alpha+mk+1)}{k}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\}Y_m^\alpha\left[x(1-t)^{\frac{-1}{k}};k\right]. \tag{1.3}$$

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In addition, we have the following relationships between the Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$ and Srivastava-Singhal polynomials $G_n^{(\alpha+1)}(x,1,1,k)$ and the Laguerre polynomials $L_n^{(\alpha)}(x)$ [11]:

$$\begin{aligned} Y_n^{\alpha}(x;1) &= L_n^{(\alpha)}(x), \\ Y_n^{\alpha}(x;k) &= k^{-n} G_n^{(\alpha+1)}(x,1,1,k). \end{aligned}$$

A few of Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$ defined by (1.1) and (1.2) are:

$$\begin{aligned} Y_0^{\alpha}(x;1) &= 1, \\ Y_1^{\alpha}(x;1) &= -x+\alpha+1, \\ Y_2^{\alpha}(x;1) &= \frac{1}{2}\left(x^2 - x(2\alpha+4) + \alpha^2 + 3\alpha + 2\right), \\ Y_3^{\alpha}(x;1) &= \frac{1}{6}[-x^3 + 3x^2(\alpha+3) - 3x(\alpha^2+5\alpha+6) + \alpha^3 + 6\alpha^2 + 11\alpha + 6]. \end{aligned}$$

The graphs of these polynomials (up to $(n,k) = (2,1)$ in special case $\alpha = 0,1,2,3,4$ are shown below:

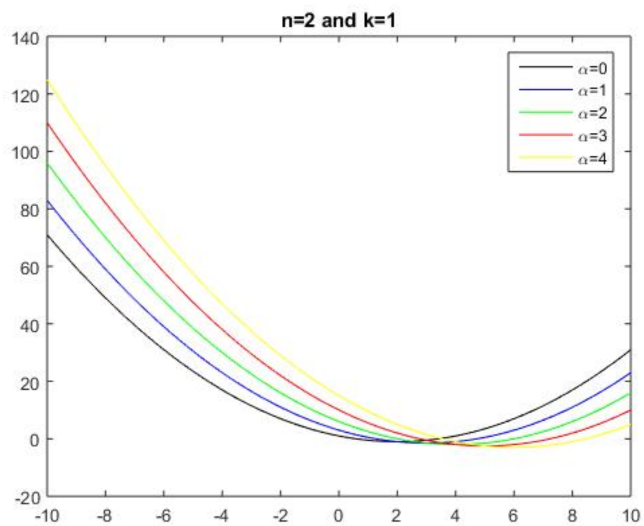


FIGURE 1. $Y_2^{\alpha}(x;1) = \frac{1}{2}\left(x^2 - x(2\alpha+4) + \alpha^2 + 3\alpha + 2\right), \quad x \in [-10,10]$

## 2. GENERATING FUNCTIONS

In this section, we give a theorem about the addition formula for Konhauser biorthogonal polynomials:

**Theorem 2.1.** *We have*

$$Y_n^{\alpha_1+\alpha_2+1}(x_1+x_2;k) = \sum_{m=0}^{n} Y_{n-m}^{\alpha_1}(x_1;k) Y_m^{\alpha_2}(x_2;k). \tag{2.1}$$

*Proof.* Replacing $\alpha$ by $\alpha_1+\alpha_2+1$ and $x$ by $x_1+x_2$ in (1.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Y_n^{\alpha_1+\alpha_2+1}(x_1+x_2;k)t^n &= (1-t)^{-\frac{(\alpha_1+\alpha_2+2)}{k}}\exp\left\{(x_1+x_2)\left[1-(1-t)^{\frac{-1}{k}}\right]\right\}\\ &= (1-t)^{\frac{-(\alpha_1+1)}{k}}\exp\left(x_1\left[1-(1-t)^{\frac{-1}{k}}\right]\right)\\ &\quad\times(1-t)^{\frac{-(\alpha_2+1)}{k}}\exp\left(x_2\left[1-(1-t)^{\frac{-1}{k}}\right]\right)\\ &= \sum_{n=0}^{\infty} Y_n^{\alpha_1}(x_1;k)\,t^n\sum_{m=0}^{\infty} Y_m^{\alpha_2}(x_2;k)\,t^m\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Y_n^{\alpha_1}(x_1;k)\,Y_m^{\alpha_2}(x_2;k)\,t^{m+n}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} Y_{n-m}^{\alpha_1}(x_1;k)\,Y_m^{\alpha_2}(x_2;k)\,t^n.\end{aligned}$$

From the coefficients of $t^n$ on the both sides of the last equality, one can get the desired result. □

The main object of this paper to study different properties of the Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given. Nowadays, there are a lot of works related to Konhauser biorthogonal polynomials theory and its applications (see, [12, 13, 15]).

## 3. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$ which are generated by (1.1) and given explicitly by (1.2) using the similar method considered in (see, [1–3, 6–10, 14]).

We begin by stating the following theorem.

**Theorem 3.1.** *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of $r$ complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order $\mu,\psi$, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k\quad(a_k\neq 0)$$

*and*

$$\Theta_{n,p}^{\mu,\psi}(x;u;y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_kY_{n-pk}^{\alpha}(x;u)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

*Then, for $p\in\mathbb{N}$, we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x;u;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n=(1-t)^{\frac{-(\alpha+1)}{u}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\tag{3.1}$$

*provided that each member of (3.1) exists.*

*Proof.* For convenience, let $S$ denote the first member of the assertion (3.1) of Theorem 3.1. Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kY_{n-pk}^{\alpha}(x;u)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^kt^{n-pk}.$$

Replacing $n$ by $n+pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\, Y_n^{\alpha}(x;u)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty} Y_n^{\alpha}(x;u)t^n \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k \\ &= (1-t)^{\frac{-(\alpha+1)}{u}} \exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

**Theorem 3.2.** *Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order $\mu,\psi,\alpha,\beta$ let*

$$\Lambda^{n,p}_{\mu,\psi,\alpha,\beta}(x_1+x_2;u;y_1,...,y_r;t) := \sum_{k=0}^{[n/p]} a_k Y^{\alpha+\beta+1}_{n-pk}(x_1+x_2;u)\Omega_{\mu+\psi l}(y_1,...,y_r)t^k,$$

*where $a_k\neq 0$, $n,p\in\mathbb{N}$ and the notation $[n/p]$ means the greatest integer less than or equal $n/p$.*
*Then, for $p\in\mathbb{N}$, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l Y^{\alpha}_{n-k}(x_1;u)Y^{\beta}_{k-pl}(x_2;u)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l = \Lambda^{n,p}_{\mu,\psi,\alpha,\beta}(x_1+x_2;u;y_1,...,y_r;z) \tag{3.2}$$

*provided that each member of (3.2) exists.*

*Proof.* For convenince, let $T$ denote the first member of the assertion (3.2) of Theorem 3.2. Then, upon subsituting for the polynomials $Y_n^{\alpha}(x_1+x_2;k)$ from the (2.1) into the left-hand side of (3.2), we obtain

$$\begin{aligned} T &= \sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l Y^{\alpha}_{n-k}(x_1;u)Y^{\beta}_{k-pl}(x_2;u)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l Y^{\alpha}_{n-k-pl}(x_1;u)Y^{\beta}_{k+pl-pl}(x_2;u)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]} a_l\left(\sum_{k=0}^{n-pl} Y^{\alpha}_{n-pl-k}(x_1;u)Y^{\beta}_{k}(x_2;u)\right)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]} a_l Y^{\alpha+\beta+1}_{n-pl}(x_1+x_2;u)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \Lambda^{n,p}_{\mu,\psi,\alpha,\beta}(x_1+x_2;u;y_1,...,y_r;z). \end{aligned}$$

□

**Theorem 3.3.** *Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order $\mu$, let*

$$\Lambda_{\mu,p,q}\left[x;u;y_1,...,y_r;z\right] := \sum_{i=0}^{\infty} a_i Y^{\alpha}_{m+qi}(x;u)\Omega_{\mu+pi}(y_1,...,y_r)z^i$$

*where $a_i\neq 0$ and*

$$\theta_{\mu,p,q}(y_1,...,y_r;z) := \sum_{j=0}^{[i/q]}\binom{m+i}{i-qj} a_j\Omega_{\mu+pj}(y_1,...,y_r)z^j.$$

*Then, for* $p,q\in\mathbb{N}$; *we have*

$$\sum_{i=0}^{\infty} Y_{i+m}^{\alpha}(x;u)\,\theta_{\mu,p,q}(y_1,...,y_r;z)t^i = (1-t)^{\frac{-(\alpha+mu+1)}{u}} \exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\} \times\Lambda_{\mu,p,q}\left(x(1-t)^{\frac{-1}{u}};u;y_1,...,y_r;z(\frac{t}{1-t})^q\right) \tag{3.3}$$

*provided that each member of (3.3) exists.*

*Proof.* For convenience, let $H$ denote the first member of the assertion (3.3) of Theorem 3.3. Then,

$$H=\sum_{i=0}^{\infty} Y_{i+m}^{\alpha}(x;u)\sum_{j=0}^{[i/q]}\binom{m+i}{i-qj}a_j\Omega_{\mu+pj}(y_1,...,y_r)z^jt^i.$$

Replacing $i$ by $i+qj$ and then using (1.3), we may write that

$$\begin{aligned} H &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+i+qj}{i}Y_{i+m+qj}^{\alpha}(x;u)a_j\,\Omega_{\mu+pj}(y_1,...,y_r)z^jt^{i+qj}\\ &= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}\binom{m+i+qj}{i}Y_{i+m+qj}^{\alpha}(x;u)t^i\right)a_j\Omega_{\mu+pj}(y_1,...,y_r)(zt^q)^j\\ &= \sum_{j=0}^{\infty}a_j(1-t)^{\frac{-(\alpha+(m+qj)u+1)}{u}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\\ &\quad\times Y_{m+qj}^{\alpha}\left[x(1-t)^{\frac{-1}{u}};u\right]\Omega_{\mu+pj}(y_1,...,y_r)(zt^q)^j\\ &= (1-t)^{\frac{-(\alpha+mu+1)}{u}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\\ &\quad\times\sum_{j=0}^{\infty}a_jY_{m+qj}^{\alpha}\left[x(1-t)^{\frac{-1}{u}};u\right]\Omega_{\mu+pj}(y_1,...,y_r)(\frac{zt^q}{(1-t)^q})^j\\ &= (1-t)^{\frac{-(\alpha+mu+1)}{u}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\\ &\quad\times\Lambda_{\mu,p,q}\left(x(1-t)^{\frac{-1}{u}};u;y_1,...,y_r;z(\frac{t}{1-t})^q\right) \end{aligned}$$

which completes the proof. □

## 4. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 3.1 , where the multivariable extension of the Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(x_1,...,x_r)$ [4], generated by

$$\prod_{j=1}^{r}\left\{(1-x_jt^j)^{-\alpha_j}\right\}=\sum_{n=0}^{\infty}h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)\,t^n,\ \left(\alpha\in\mathbb{C}\,;\ |t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},...,|x_r|^{-1/r}\right\}\right). \tag{4.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for the multivariable extension of the Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)$ and the Konhauser biorthogonal polynomials.

**Corollary 4.1.** *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_kh_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)\zeta^k,\ \ (a_k\neq0,\ \ \mu,\psi\in\mathbb{C}),$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k Y_{n-pk}^{\alpha}(x;u)h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)\frac{\zeta^k}{t^{pk}}t^n = (1-t)^{\frac{-(\alpha+1)}{u}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) \tag{4.2}$$

*provided that each member of (4.2) exists.*

**Remark 4.2.** Using the generating relation (4.1) for the multivariable polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(x_1,...,x_r)$ and getting $a_k = 1, \mu = 0, \psi = 1$, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} Y_{n-pk}^{\alpha}(x;u)h_{k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)\zeta^k t^{n-pk} = (1-t)^{\frac{-(\alpha+1)}{u}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\}\prod_{j=1}^{r}\left\{(1-y_j\zeta^j)^{-\alpha_j}\right\}$$

where $|\zeta| < \min\left\{|y_1|^{-1}, |y_2|^{-1/2}, ..., |y_r|^{-1/r}\right\}$ and $|t| < 1$.

If we set $r = 1$ and

$$\Omega_{\mu+\psi k}(y_1) = Y_{\mu+\psi k}^{\omega}(y_1;u)$$

in Theorem 3.2, we have the following bilinear generating functions for the Konhauser biorthogonal polynomials.

**Corollary 4.3.** *If*

$$\Lambda_{\mu,\psi,\alpha,\beta}^{n,p}(x_1+x_2;u;y_1;u;t) := \sum_{k=0}^{[n/p]} a_k Y_{n-pk}^{\alpha+\beta+1}(x_1+x_2;u)Y_{\mu+\psi k}^{\omega}(y_1;u)t^k, \quad (a_k \neq 0,\ \mu,\psi \in \mathbb{C})$$

*then, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l Y_{n-k}^{\alpha}(x_1;u)Y_{k-pl}^{\beta}(x_2;u)Y_{\mu+\psi l}^{\omega}(y_1;u)z^l = \Lambda_{\mu,\psi,\alpha,\beta}^{n,p}(x_1+x_2;u;y_1;u;z) \tag{4.3}$$

*provided that each member of (4.3) exists.*

**Remark 4.4.** Using (4.3) and taking $a_l = 1, \mu = 0, \psi = 1, p = 1, z = 1$ we have

$$\begin{aligned}\sum_{k=0}^{n}\sum_{l=0}^{k} Y_{n-k}^{\alpha}(x_1;u)Y_{k-l}^{\beta}(x_2;u)Y_{l}^{\omega}(y_1;u) &= \sum_{k=0}^{n} Y_{n-k}^{\alpha}(x_1;u)\sum_{l=0}^{k} Y_{k-l}^{\beta}(x_2;u)Y_{l}^{\omega}(y_1;u)\\ &= \sum_{k=0}^{n} Y_{n-k}^{\alpha}(x_1;u)Y_{k}^{\beta+\omega+1}(x_2+y_1;u)\\ &= Y_{n}^{\alpha+\beta+\omega+2}(x_1+x_2+y_1;u).\end{aligned}$$

If we set

$$r = 1 \text{ and } \Omega_{\mu+pj}(y_1) = P_{\mu+pj}^{(\alpha,\beta)}(y)$$

in Theorem 3.3, where the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$ is generated by [5],

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n = \frac{2^{\alpha+\beta}}{\rho}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}, \quad \left\{\rho = (1-2xt+t^2)^{1/2}, \quad |\rho| < 1\right\}$$

we get a family of the bilateral generating functions for the classical Jacobi polynomials and the Konhauser biorthogonal polynomials as follows:

**Corollary 4.5.** *If*

$$\Lambda_{\mu,p,q}[x;u;y;z] := \sum_{i=0}^{\infty} a_i Y_{m+qi}^{\alpha}(x;u)P_{\mu+pi}^{(\alpha,\beta)}(y)\,z^i, \quad (a_i \neq 0,\ m \in \mathbb{N}_0,\ \mu,\psi \in \mathbb{C})$$

*and*

$$\theta_{\mu,p,q}(y;z) := \sum_{j=0}^{[i/q]} \binom{m+i}{i-qj} a_j P_{\mu+pj}^{(\alpha,\beta)}(y)\,z^j$$

*where $i, p \in \mathbb{N}$, then we have*

$$\sum_{i=0}^{\infty} Y_{i+m}^{\alpha}(x;u)\theta_{\mu,p,q}(y;z)t^{i} = (1-t)^{\frac{-(\alpha+mu+1)}{u}} \exp\left\{x\left[1-(1-t)^{\frac{-1}{u}}\right]\right\} \times \Lambda_{\mu,p,q}\left(x(1-t)^{\frac{-1}{u}};u;y;z(\frac{t}{1-t})^{q}\right) \tag{4.4}$$

*provided that each member of (4.4) exists.*

Furthermore, for every suitable choice of the coefficients $a_j$ $(j \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+pj}(y_1, ..., y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 3.1, Theorem 3.2, Theorem 3.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the Konhauser biorthogonal polynomials given explicitly by (1.2).

## 5. Recurrence Relations

We now discuss some miscellaneous recurrence relations of the Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$ given by (1.1). By differentiating each member of the generating function relation (1.1) with respect to $x$ and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

we arrive at the following (differential) recurrence relation for the Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$ given explicitly by (1.1):

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\partial}{\partial x} Y_n^{\alpha}(x;k)\, t^n &= (1-t)^{\frac{-(\alpha+1)}{k}}\left(1-(1-t)^{\frac{-1}{k}}\right)\exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \\
&= \left(1-(1-t)^{\frac{-1}{k}}\right)\sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, t^n \\
&= \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, t^n - (1-t)^{\frac{-1}{k}} \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, t^n \\
&= \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, t^n - \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{\left(\frac{1}{k}\right)_m}{m!} Y_{n-m}^{\alpha}(x;k)\, t^n.
\end{aligned}$$

$$\frac{\partial}{\partial x} Y_n^{\alpha}(x;k) = Y_n^{\alpha}(x;k) - \sum_{m=0}^{n} \frac{\left(\frac{1}{k}\right)_m}{m!} Y_{n-m}^{\alpha}(x;k).$$

By differentiating each member of the generating function relation (1.1) with respect to $\alpha$, we have the following another recurrence relation for these polynomials:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\partial}{\partial \alpha} Y_n^{\alpha}(x;k)t^n &= \left(\frac{-1}{k}\right)(1-t)^{\frac{-(\alpha+1)}{k}} \ln(1-t) \exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \\
&= -\frac{\ln(1-t)}{k} \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, t^n \\
&= \frac{1}{k} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Y_n^{\alpha}(x;k)\left(\frac{1}{m+1}\right) t^{n+m+1} \\
&= \frac{1}{k} \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \frac{Y_{n-m-1}^{\alpha}(x;k)}{m+1} t^n.
\end{aligned}$$

$$\frac{\partial}{\partial \alpha} Y_n^{\alpha}(x;k) = \frac{1}{k}\sum_{m=0}^{n-1} \frac{Y_{n-m-1}^{\alpha}(x;k)}{m+1}, \qquad (n \geq 1).$$

Finally, by differentiating each member of the generating function relation (1.1) with respect to $t$, we have the following another recurrence relation for these polynomials:

$$\begin{aligned}
\sum_{n=1}^{\infty} nY_n^{\alpha}(x;k)t^{n-1} &= \frac{(\alpha+1)}{k}(1-t)^{-\frac{(\alpha+1)}{k}-1}\exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \\
&\quad -\frac{x}{k}(1-t)^{\frac{-1}{k}-1}\exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\}(1-t)^{-\frac{(\alpha+1)}{k}} \\
&= \frac{(\alpha+1)}{k}(1-t)^{-1}(1-t)^{-\frac{(\alpha+1)}{k}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \\
&\quad -\frac{x}{k}(1-t)^{\frac{-1-k}{k}}(1-t)^{-\frac{(\alpha+1)}{k}}\exp\left\{x\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \\
&= \frac{(\alpha+1)}{k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(1)_m\frac{1}{m!}Y_n^{\alpha}(x;k)t^{n+m} \\
&\quad -\frac{x}{k}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{1+k}{k}\right)_p\frac{1}{p!}Y_n^{\alpha}(x;k)t^{n+p}.
\end{aligned}$$

$$(n+1)\,Y_{n+1}^{\alpha}(x;k) = \frac{(\alpha+1)}{k}\sum_{m=0}^{n}Y_{n-m}^{\alpha}(x;k) - \frac{x}{k}\sum_{p=0}^{n}\left(\frac{1+k}{k}\right)_p\frac{Y_{n-p}^{\alpha}(x;k)}{p!}.$$

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