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The Linear Algebra of the Pell-Lucas Matrix

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Abstract

In this paper, we introduce the Pell-Lucas and the symmetric Pell-Lucas matrices. The study delves into the linear algebra aspects of these matrices, analyzing their mathematical properties and relationships. We construct decompositions for both the Pell-Lucas matrix and its inverse matrix. We present the Cholesky factorization of the symmetric Pell-Lucas matrices. Furthermore, we derive some valuable identities and bounds for the eigenvalues of these symmetric matrices through the application of majorization notation.

1. Introduction

Numerous researchers in the disciplines of calculus, applied mathematics, and linear algebra, as well as other branches of mathematics, have been interested in the Fibonacci and Lucas numbers. There are also other relationships that are written and new number sequences, such as Pell and Pell-Lucas number sequences, are derived that are similar to the recurring relationships of the Fibonacci and Lucas number sequences. The Pell numbers P_n and the Pell-Lucas numbers Q_n are defined by

$$P_{n+1} = 2P_n + P_{n-1}, \quad for \quad n \geq 1,$$

where $P_0 = 0$ and $P_1 = 1$, and

$$Q_{n+1} = 2Q_n + Q_{n-1}, \quad for \quad n \geq 1,$$

where $Q_0 = 2$ and $Q_1 = 2$, respectively. In addition, we present several identities associated with the Pell-Lucas numbers and relationship between the Pell numbers and the Pell-Lucas numbers for $k \in \mathbb{N}$.

$$Q_k + Q_{k+1} = 4P_{k+1}, \tag{1.1}$$
$$Q_k + Q_{k+2} = 8P_{k+1}, \tag{1.2}$$
$$Q_1^2 + Q_2^2 + \cdots + Q_k^2 = \frac{Q_k Q_{k+1} - 4}{2}. \tag{1.3}$$

We refer to [1, 2, 3] for further information on the Pell and the Pell-Lucas numbers.

M_n denotes the set of all $n \times n$ matrices. If any matrix $P \in M_n$ may be written as $P = RR^T$ or $P = R^T R$, where $R \in M_n$ is a lower triangular matrix with diagonal entries that are not negative, then this factorization is known as a Cholesky factorization. Moreover, this factorization is unique if R is nonsingular.

A matrix $S \in M_n$ of the form

$$S = \begin{bmatrix} S_{11} & 0 & \cdots & 0 \\ 0 & S_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{nn} \end{bmatrix}$$

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in which $S_{ii} \in M_{n_i}$, $i = 1,2,\ldots,k$, and $\sum_{i=1}^{k} n_i = n$, is called block diagonal. This matrix is frequently described as $S = S_{11} \oplus S_{22} \oplus \cdots \oplus S_{nn}$.

Many issues resulting from linear recurrence relations can be resolved using matrix methods, which are a significant instrument (see, for example, [4]). Before we go on to matrix factorization, we need to first grasp Cholesky factorization of the Pascal matrix (see, for example, [5]). Furthermore, factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices were presented by Lee et al. in [6]. The authors [7] discussed linear algebra of the k-Fibonacci and the symmetric k-Fibonacci matrix. In addition to [7], a factorization of the Pascal matrix are provided in [8]. Zhang [9] also researched the Pascal matrix and its generalization. Irmak and Köme [10] investigated the factorizations of the Lucas and the symmetric Lucas matrix. In [11], factorizations and inverse factorizations of generalized k-Fibonacci matrices were proposed. The authors [12] discussed the decomposition of Jacobsthal matrix and Jacobsthal-Lucas symmetric matrix, along with the inverses of these matrices. Kılıç and Taşcı [13] gave the factorizations and eigenvalues of Pell and symmetric Pell matrices. Furthermore, for the eigenvalues of the symmetric Pell matrix, they provided some relations and boundaries. Motivated by this paper, we define a new matrix as follows. Then, in this paper we consider the factorizations and eigenvalues of Pell-Lucas and symmetric Pell-Lucas matrices.

Definition 1.1. *Let $i, j = 1,2,\ldots,n$. Then, we define the Pell-Lucas matrix such that*

$$A_n = [a_{ij}] = \begin{cases} Q_{i-j+1}\,, & i-j+1>0 \\ 0\,, & i-j+1 \le 0 \end{cases}.$$

Example 1.2. *For $n = 6$ in Definition 1.1, then we have*

$$A_6 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ 14 & 6 & 2 & 0 & 0 & 0 \\ 34 & 14 & 6 & 2 & 0 & 0 \\ 82 & 34 & 14 & 6 & 2 & 0 \\ 198 & 82 & 34 & 14 & 6 & 2 \end{bmatrix},$$

and the first column of A_6 is the vector $(2,6,14,34,82,198)^T$. As a result, the matrix A_n reveals a variety of interesting facts.

2. Factorizations

This section discusses the creation and factorization of our Pell-Lucas matrix of order n using the $(0,1,2)$-matrix, which is defined as a matrix whose elements are all either 0, 1 or 2. Let I_n represents the order n identity matrix. Further, we define the $n \times n$ matrices L_n, $\overline{A_n}$ and X_k by

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\; L_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and $L_k = L_0 \oplus I_k$, $k = 1,2,\ldots$, $\overline{A_n} = [1] \oplus A_{n-1}$, $X_1 = I_n$, $X_2 = I_{n-3} \oplus L_{-1}$, for $3 \le k < n$, $X_k = I_{n-k} \oplus L_{k-3}$, and $X_n = L_{n-3}$. Then we reach the following lemma.

Lemma 2.1. *For $k \ge 3$, we have $\overline{A_k} \cdot L_{k-3} = A_k$.*

Proof. For $k = 3$, we have $\overline{A_3} \cdot L_0 = A_3$. Let $k > 3$. By using the familiar Pell-Lucas sequences, and matrix product definition, we get the following conclusion. □

For $i, j = 1,2,\ldots,n$, we define a matrix

$$\Gamma_n = [\gamma_{ij}] = \begin{cases} 2, & i = j \\ 2, & i = j+1 \\ 0, & otherwise \end{cases}. \tag{2.1}$$

Also we can give the inverse of matrix Γ_n as follows:

$$\Gamma_n^{-1} = [\gamma_{ij}] = \begin{cases} (-1)^{i-j}\dfrac{1}{2}, & i \ge j \\ 0, & otherwise \end{cases}. \tag{2.2}$$

We can obtain the following theorem by using Lemma 2.1 and equation (2.1).

Theorem 2.2. *The X_k's and Γ_n can factor the Pell-Lucas matrix A_n in the following way:*

$$A_n = X_1 X_2 \cdots X_n \Gamma_n = \Gamma_n X_1 X_2 \cdots X_n.$$

Now we give the factorization of A_6 in Example 1.2.

Example 2.3. *From Theorem 2.2, for $n=6$, we have*

$$\begin{aligned} A_6 &= X_1X_2X_3X_4X_5X_6\Gamma_6 \\ &= I_6\,(I_3\oplus L_{-1})\,(I_3\oplus L_0)\,(I_2\oplus L_1)\,(I_1\oplus L_2)\,L_3\Gamma_6 \\ &= \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}\cdot\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&2&1 \end{bmatrix}\cdot\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&2&1&0\\ 0&0&0&1&0&1 \end{bmatrix}\cdot\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&2&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix} \\ &\cdot\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&2&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}\cdot\begin{bmatrix} 1&0&0&0&0&0\\ 2&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}\cdot\begin{bmatrix} 2&0&0&0&0&0\\ 2&2&0&0&0&0\\ 0&2&2&0&0&0\\ 0&0&2&2&0&0\\ 0&0&0&2&2&0\\ 0&0&0&0&2&2 \end{bmatrix}. \end{aligned}$$

Now, we give another factorization of A_n. For $i,j=1,2,\dots,n$, we define a matrix

$$V_n=[v_{ij}]=\begin{cases} Q_i\,, & j=1\,,\\ 1\,, & i=j\,,\\ 0\,, & otherwise \end{cases}\quad,\quad i.e,\quad V_n=\begin{bmatrix} Q_1&0&\cdots&0\\ Q_2&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ Q_n&0&\cdots&1 \end{bmatrix}.$$

An elementary calculation leads to the next theorem.

Theorem 2.4. *For $n\geq 1$, $A_n=V_n(I_1\oplus V_{n-1})(I_2\oplus V_{n-2})\cdots(I_{n-1}\oplus V_1)$.*

The inverse of the Pell-Lucas matrix A_n is easily found. We know that

$$L_0^{-1}=\begin{bmatrix} 1&0&0\\ -2&1&0\\ -1&0&1 \end{bmatrix},\quad L_{-1}^{-1}=\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&-2&1 \end{bmatrix},\quad\text{and}\quad L_k^{-1}=L_0^{-1}\oplus I_k.$$

For $k=1,2,\dots,n$, we define $Y_k=X_k^{-1}$. Then $Y_1=X_1^{-1}=I_n$,

$Y_2=X_2^{-1}=I_{n-3}\oplus L_{-1}^{-1}=I_{n-2}\oplus\begin{bmatrix} 1&0\\ -2&1 \end{bmatrix}$, and $Y_n=L_{n-3}^{-1}$. Also we can derive

$$V_n^{-1}=\begin{bmatrix} Q_1/4&0&0&\cdots&0\\ -Q_2/2&1&0&\cdots&0\\ -Q_3/2&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ -Q_n/2&0&0&\cdots&1 \end{bmatrix},\quad\text{and}\quad (I_k\oplus V_{n-k})^{-1}=I_k\oplus V_{n-k}^{-1}.$$

Utilizing Theorem 2.2 and Theorem 2.4, we derive the subsequent corollary.

Corollary 2.5. *The inverse of the Pell-Lucas matrix A_n^{-1} can be factored by the Y_k's and Γ_n^{-1} as follows:*

$$\begin{aligned} A_n^{-1} &= \Gamma_n^{-1}Y_nY_{n-1}\dots Y_2Y_1=Y_nY_{n-1}\dots Y_2Y_1\Gamma_n^{-1} \\ &= (I_{n-1}V_1)^{-1}\cdots(I_2\oplus V_{n-2})^{-1}(I_1\oplus V_{n-1})^{-1}V_n^{-1} \end{aligned}$$

By Corollary 2.5, we get

$$A_n^{-1}=[\alpha_{ij}]=\begin{cases} 1/2\,, & i=j\\ -3/2\,, & i-j=1\\ (-1)^{i-j}\,, & i-j\geq 2\\ 0, & otherwise \end{cases}. \tag{2.3}$$

Example 2.6. *By (2.3), the inverse of A_6 in Example 1.2 is*

$$A_6^{-1}=\begin{bmatrix}1/2&0&0&0&0&0\\-3/2&1/2&0&0&0&0\\1&-3/2&1/2&0&0&0\\-1&1&-3/2&1/2&0&0\\1&-1&1&-3/2&1/2&0\\-1&1&-1&1&-3/2&1/2\end{bmatrix}.$$

Definition 2.7. *For $i,j=1,2,\ldots,n$, we define the symmetric Pell-Lucas matrix such that*

$$B_n=[b_{ij}]=[b_{ji}]=\begin{cases}\sum_{k=1}^{i}Q_k^2, & i=j\\ b_{i,j-2}+2b_{i,j-1}+4, & i+1=j\,,\\ b_{i,j-2}+2b_{i,j-1}, & i+1<j\end{cases}$$

where $b_{1,0}=4$.

So we get

$$b_{1j}=b_{j1}=2Q_j,\quad for\ \ j\geq 1 \tag{2.4}$$
$$b_{2j}=b_{j2}=8P_{j+1},\quad for\ \ j\geq 2. \tag{2.5}$$

Example 2.8. *For $n=6$ in Definition 2.7, then we get*

$$B_6=\begin{bmatrix}4&12&28&68&164&396\\12&40&96&232&560&1352\\28&96&236&572&1380&3332\\68&232&572&1392&3360&8112\\164&560&1380&3360&8116&19596\\396&1352&3332&8112&19596&47320\end{bmatrix}.$$

According to the Definition 2.7, the following lemma is derived.

Lemma 2.9. *For $j\geq 3$, we get $b_{3,j}=P_{j-3}\left(8P_4+4\right)+P_{j-2}\left(\frac{Q_3Q_4-4}{2}\right)$.*

Proof. From (1.3), we know that $b_{3,3}=Q_1^2+Q_2^2+Q_3^2=\frac{Q_3Q_4-4}{2}$. On the other hand, since $P_0=0$, and $P_1=1$, then we have $b_{3,3}=\frac{Q_3Q_4-4}{2}=P_0\left(8P_4+4\right)+P_1\left(\frac{Q_3Q_4-4}{2}\right)$. By induction, the proof is completed. □

We know that $b_{3,1}=b_{1,3}=2Q_3$ and $b_{3,2}=b_{2,3}=8P_4$ by (2.4) and (2.5). In addition, we get that $b_{4,1}=b_{1,4}$, $b_{4,2}=b_{2,4}$, and $b_{4,3}=b_{3,4}$. By induction, the following lemma is reaced.

Lemma 2.10. *For $j\geq 4$, we have $b_{4,j}=P_{j-4}\left(8P_4+4+Q_3Q_4\right)+P_{j-3}\left(\frac{Q_4Q_5-4}{2}\right)$.*

By using Lemmas 2.9 and 2.10, we can derive $b_{5,1},b_{5,2},b_{5,3}$, and $b_{5,4}$. From these conclusions and Definition 2.7, we reach the following lemma.

Lemma 2.11. *For $j\geq 5$, we get*

$$b_{5,j}=P_{j-5}\left(8P_4+4+Q_3Q_4+Q_4Q_5\right)+P_{j-4}\left(\frac{Q_5Q_6-4}{2}\right).$$

Proof. From (1.3), and Definition 2.7, since $b_{5,5}=Q_1^2+Q_2^2+Q_3^2+Q_4^2+Q_5^2=\frac{Q_5Q_6-4}{2}$, by induction, the proof is completed. □

Utilizing Definition 2.7, Lemmas 2.9, 2.10 and 2.11, we arrive at the following lemma through induction on the variable i.

Lemma 2.12. *For $j\geq i\geq 6$, we have*

$$b_{i,j}=P_{j-i}\left(8P_4+4+\sum_{k=4}^{i}Q_{k-1}Q_k\right)+P_{j-i+1}\left(\frac{Q_iQ_{i+1}-4}{2}\right).$$

We can easily obtain the following corollary by using Pell numbers and Pell-Lucas numbers.

Corollary 2.13. *For the symmetric Pell-Lucas matrix B_n, we get*

$$B_n=[b_{ij}]=\begin{cases}\frac{1}{2}Q_{i+j+1}-\frac{1}{2}Q_{j-i+1+(-1)^{i+1}}+(-1)^{i+1}2P_{j-i}, & i\leq j\\ \frac{1}{2}Q_{i+j+1}-\frac{1}{2}Q_{i-j+1+(-1)^{j+1}}+(-1)^{j+1}2P_{i-j}, & i>j\end{cases}.$$

Lemma 2.14. *Let $i,j\in\mathbb{Z}^+$ and $i\geq 3$. Then we have*

$$\sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}=\begin{cases}Q_{j-i+1}, & i\leq j\\ 0, & i>j\end{cases}. \tag{2.6}$$

Proof. Assume that $i\leq j$. Now, we prove the theorem by the induction method on i. Let $i=3$. From Corollary 2.13, we can derive

$$\begin{aligned} b_{1,j}-\frac{3}{2}b_{2,j}+\frac{1}{2}b_{3,j} &= \left(\frac{1}{2}Q_{j+2}-\frac{1}{2}Q_{j+1}+2P_{j-1}\right)+\left(-\frac{3}{4}Q_{j+3}+\frac{3}{4}Q_{j-2}+3P_{j-2}\right)+\left(\frac{1}{4}Q_{j+4}-\frac{1}{4}Q_{j-1}+P_{j-3}\right)\\ &= Q_{j-2}. \end{aligned}$$

Suppose that the hypothesis is true for i. For $i+1$, by using equations (1.1), (1.2) and Corollary 2.13, we find

$$\begin{aligned} \sum_{k=1}^{i-1}(-1)^{i-1-k}b_{k,j}-\frac{3}{2}b_{i,j}+\frac{1}{2}b_{i+1,j} &= b_{i-1,j}-\sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i,j}+\frac{1}{2}b_{i+1,j}\\ &= b_{i-1,j}+\left(-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}-Q_{j-i+1}\right)-\frac{3}{2}b_{i,j}+\frac{1}{2}b_{i+1,j}\\ &= -\frac{1}{2}b_{i-1,j}-b_{i,j}+\frac{1}{2}b_{i+1,j}-Q_{j-i+1}\\ &= \left(-\frac{1}{4}Q_{i+j}+\frac{1}{4}Q_{j-i+2+(-1)^i}-(-1)^iP_{j-i+1}\right)\\ &\quad+\left(-\frac{1}{2}Q_{i+j+1}+\frac{1}{2}Q_{j-i+1+(-1)^{i+1}}+(-1)^i2P_{j-i}\right)\\ &\quad+\left(\frac{1}{4}Q_{i+j+2}-\frac{1}{4}Q_{j-i+(-1)^i}+(-1)^iP_{j-i-1}\right)-Q_{j-i+1}\\ &= \frac{1}{4}\left(-Q_{i+j}-2Q_{i+j+1}+Q_{i+j+2}\right)+\frac{1}{4}\left(Q_{j-i+2+(-1)^i}+2Q_{j-i+1+(-1)^{i+1}}-Q_{j-i+(-1)^i}\right)\\ &\quad+(-1)^i\left(-P_{j-i+1}+2P_{j-i}+P_{j-i-1}\right)-Q_{j-i+1}\\ &= \frac{1}{4}\left(2Q_{j-i+1+(-1)^i}+2Q_{j-i+1+(-1)^{i+1}}\right)-Q_{j-i+1}\\ &= 4P_{j-i+1}-Q_{j-i+1}\\ &= Q_{j-i}+Q_{j-i+1}-Q_{j-i+1}\\ &= Q_{j-i}. \end{aligned}$$

The proof for $i>j$ can be completed in a similar way. □

Theorem 2.15. *For $n\in\mathbb{Z}^+$, we have $Y_nY_{n-1}\ldots Y_2Y_1\Gamma_n^{-1}B_n=A_n^T$ and the Cholesky factorization of B_n is given by $B_n=A_nA_n^T$.*

Proof. By Corollary 2.5, $Y_nY_{n-1}\ldots Y_2Y_1\Gamma_n^{-1}=A_n^{-1}$. So, if we get $A_n^{-1}B_n=A_n^T$, then the theorem holds. Let $A_n^{-1}B_n=[c_{ij}]$. So, from (1.1), (2.3), (2.4), (2.5) and Lemma 2.14, we find the following:

$$
\begin{aligned}
A_n^{-1}B_n = [c_{ij}] &= \begin{cases} \frac{1}{2}b_{1j}, & i=1 \\ -\frac{3}{2}b_{11}+\frac{1}{2}b_{21}, & i=2, j=1 \\ -\frac{3}{2}b_{1j}+\frac{1}{2}b_{2j}, & i=2, j\geq 2 \\ \sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}, & i\geq 3 \end{cases} \\
&= \begin{cases} Q_j, & i=1 \\ -3Q_1+Q_2, & i=2, j=1 \\ -3Q_j+4P_{j+1}, & i=2, j\geq 2 \\ \sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}, & i\geq 3 \end{cases} \\
&= \begin{cases} Q_j, & i=1 \\ 0, & i=2, j=1 \\ Q_{j-1}, & i=2, j\geq 2 \\ Q_{j-i+1}, & i\geq 3, i\leq j \\ 0, & i\geq 3, i>j \end{cases} \\
&= \begin{cases} Q_{j-i+1}, & i\leq j \\ 0, & i>j \end{cases} \\
&= A_n^T.
\end{aligned}
$$

Hence, the Cholesky factorization of B_n is given by $B_n = A_nA_n^T$. □

Now we give the Cholesky factorization of B_6 by using A_6 in Example 1.2.

Example 2.16. *By Theorem 2.15, since the Cholesky factorization of B_6 is $A_6A_6^T$, then we get*

$$
B_6 = \begin{bmatrix} 4 & 12 & 28 & 68 & 164 & 396 \\ 12 & 40 & 96 & 232 & 560 & 1352 \\ 28 & 96 & 236 & 572 & 1380 & 3332 \\ 68 & 232 & 572 & 1392 & 3360 & 8112 \\ 164 & 560 & 1380 & 3360 & 8116 & 19596 \\ 396 & 1352 & 3332 & 8112 & 19596 & 47320 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ 14 & 6 & 2 & 0 & 0 & 0 \\ 34 & 14 & 6 & 2 & 0 & 0 \\ 82 & 34 & 14 & 6 & 2 & 0 \\ 198 & 82 & 34 & 14 & 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 & 14 & 34 & 82 & 198 \\ 0 & 2 & 6 & 14 & 34 & 82 \\ 0 & 0 & 2 & 6 & 14 & 34 \\ 0 & 0 & 0 & 2 & 6 & 14 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.
$$

Moreover, since $B_n^{-1} = \left(A_n^T\right)^{-1} A_n^{-1}$, we obtain

$$B_n^{-1} = [\beta_{ij}] = [\beta_{ji}] = \begin{cases} \dfrac{2(n+1-i)+1}{2}, & i=j<n \\ \dfrac{1}{4}, & i=j=n \\ -\dfrac{4(n-i)+1}{4}, & i+1=j<n \\ -\dfrac{3}{4}, & i+1=j=n \\ (-1)^{j-i}(n+1-j), & i+1<j<n \\ \dfrac{(-1)^{j-i}}{2}, & i+1<j=n \end{cases}. \tag{2.7}$$

Example 2.17. *By (2.7), the inverse of B_6 in Example 2.8 is*

$$B_6^{-1} = \begin{bmatrix} 13/2 & -21/4 & 4 & -3 & 2 & -1/2 \\ -21/4 & 11/2 & -17/4 & 3 & -2 & 1/2 \\ 4 & -17/4 & 9/2 & -13/4 & 2 & -1/2 \\ -3 & 3 & -13/4 & 7/2 & -9/4 & 1/2 \\ 2 & -2 & 2 & -9/4 & 5/2 & -3/4 \\ -1/2 & 1/2 & -1/2 & 1/2 & -3/4 & 1/4 \end{bmatrix}.$$

From Theorem 2.15, we get the following corollary.

Corollary 2.18. *For $n \in \mathbb{Z}^+$, we get*

$$Q_{n+1}Q_n + Q_nQ_{n-1} + \cdots + Q_2Q_1 = \begin{cases} \dfrac{(Q_{n+1})^2}{2} - 2, & if\ n\ is\ even \\ \dfrac{(Q_{n+1})^2}{2} - 6, & if\ n\ is\ odd \end{cases}.$$

3. Eigenvalues of the symmetric Pell-Lucas matrix B_n

In this section, we consider the eigenvalues of the symmetric Pell-Lucas matrix B_n.

Let $W = \{r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n \,:\, r_1 \geq r_2 \geq \cdots \geq r_n\}$. For $r, s \in W$, $r \prec s$ if $\sum_{i=1}^{t} r_i \leq \sum_{i=1}^{t} s_i$, $t = 1, 2, \ldots, n-1$, and if $t = n$, then equality holds.It is stated that s majorizes r or that r is majorized by s when $r \prec s$. The condition for majorization can be written as follows: for $r, s \in W$, $r \prec s$ if $\sum_{i=0}^{t} r_{n-i} \geq \sum_{i=0}^{t} s_{n-i}$, $t = 0, 1, \ldots, n-2$, and if $t = n-1$, then equality holds.

The following is an exciting simple fact:

$$(\bar{r}, \bar{r}, \ldots, \bar{r}) \prec (r_1, r_2, \ldots, r_n), \quad \text{where} \quad \bar{r} = \frac{\sum_{i=1}^{n} r_i}{n}.$$

We refer to [14] and [15] for more information about majorizations.

An $n \times n$ matrix $D = [d_{ij}]$ is doubly stochastic if $d_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$, $\sum_{i=1}^{n} d_{ij} = 1$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} d_{ij} = 1$, $i = 1, 2, \ldots, n$. Hardy et al. [16] show that there must exist a doubly stochastic matrix D such that r=sD. This is the necessary and sufficient condition for $r \prec s$.

It is a well-known fact that the eigenvalues and the main diagonal components of a real symmetric matrix are both real numbers. The concept of majorization provides the precise link between the main diagonal components and the eigenvalues. The diagonal components symmetric matrix majorize the vector of eigenvalues of the matrix.

By Definition 1.1, we have $det(A_n) = 2^n$. Also by Theorem 2.15, since $B_n = A_n A_n^T$, we have $det(B_n) = 2^{2n}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of B_n. Since $B_n = A_n A_n^T$ and $\sum_{i=1}^{k} Q_i^2 = \dfrac{Q_{k+1}Q_k}{2} - 2$ by (1.3), the eigenvalues of B_n are all positive and

$$\left(\frac{Q_{n+1}Q_n}{2} - 2, \frac{Q_nQ_{n-1}}{2} - 2, \ldots, \frac{Q_2Q_1}{2} - 2\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{3.1}$$

In [17], we arrive at the combinatorial property

$$Q_n = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-m}\binom{n-m}{m} 2^{n-2m}, \ for\ n \neq 0. \tag{3.2}$$

Hence, we obtain the following corollaries.

Corollary 3.1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of B_n. Then we have*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \begin{cases} \dfrac{\left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \dfrac{n+1}{n-m+1}\dbinom{n-m+1}{m} 2^{n-2m+1}\right)^2}{4} - 2n - 1, & if\ n\ is\ even \\ \dfrac{\left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \dfrac{n+1}{n-m+1}\dbinom{n-m+1}{m} 2^{n-2m+1}\right)^2}{4} - 2n - 3, & if\ n\ is\ odd \end{cases}.$$

Proof. From (3.1), and Corollary 2.18, we find

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n &= \frac{Q_{n+1}Q_n + Q_nQ_{n-1} + \cdots + Q_2Q_1}{2} - 2n \\ &= \begin{cases} \dfrac{(Q_{n+1})^2}{4} - 2n - 1, & if\ n\ is\ even \\ \dfrac{(Q_{n+1})^2}{4} - 2n - 3, & if\ n\ is\ odd \end{cases}. \end{aligned}$$

By (3.2), the proof is completed. □

Corollary 3.2. *If n is an even number, then we have*

$$4n\lambda_n \leq \left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n-m+1}\binom{n-m+1}{m} 2^{n-2m+1}\right)^2 - 8n - 4 \leq 4n\lambda_1.$$

If n is an odd number, then we have

$$4n\lambda_n \leq \left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n-m+1}\binom{n-m+1}{m} 2^{n-2m+1}\right)^2 - 8n - 12 \leq 4n\lambda_1.$$

Proof. Let $S_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since

$$\left(\frac{S_n}{n}, \frac{S_n}{n}, \ldots, \frac{S_n}{n}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{3.3}$$

we have $\lambda_n \leq \frac{S_n}{n} \leq \lambda_1$. Then by Corollary 3.1, the proof is completed. □

From (2.7), we have

$$\left(\frac{2n+1}{2}, \frac{2n-1}{2}, \ldots, \frac{7}{2}, \frac{5}{2}, \frac{1}{4}\right) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \frac{1}{\lambda_{n-2}}, \ldots, \frac{1}{\lambda_3}, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right). \tag{3.4}$$

Therefore, there exists a doubly stochastic matrix $H = [h_{ij}]$ such that

$$\left(\frac{2n+1}{2}, \frac{2n-1}{2}, \ldots, \frac{7}{2}, \frac{5}{2}, \frac{1}{4}\right) = \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_3}, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right) \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{bmatrix}.$$

That is, we find $\frac{1}{\lambda_n}h_{1n} + \frac{1}{\lambda_{n-1}}h_{2n} + \cdots + \frac{1}{\lambda_1}h_{nn} = \frac{1}{4}$ and $h_{1n} + h_{2n} + \cdots + h_{nn} = 1$.

Lemma 3.3. *For all $i = 1, 2, \ldots, n$, we get $h_{n-(i-1),n} \leq \frac{\lambda_i}{n-1}$.*

Proof. Assume that $h_{n-(i-1),n} > \frac{\lambda_i}{n-1}$. So

$$\begin{aligned} h_{1n} + h_{2n} + \cdots + h_{nn} &> \frac{\lambda_1}{n-1} + \frac{\lambda_2}{n-1} + \cdots + \frac{\lambda_n}{n-1} \\ &= \frac{1}{n-1}\left(\lambda_1 + \lambda_2 + \cdots + \lambda_n\right). \end{aligned}$$

Since $h_{1n} + h_{2n} + \cdots + h_{nn} = 1$ and $\sum_{i=1}^{n} \lambda_i \geq n$, this yields a contradiction, then $h_{n-(i-1),n} \leq \frac{\lambda_i}{n-1}$. □

For $k \in \mathbb{Z}^+$, we define

$$\begin{aligned} T_k &= \sum_{i=1}^{k} \frac{1}{\lambda_i} \\ &= \frac{2k+1}{2} + \frac{2k-1}{2} + \frac{2k-3}{2} + \cdots + \frac{7}{2} + \frac{5}{2} + \frac{1}{4} \\ &= \frac{2k^2 + 4k - 5}{4}. \end{aligned} \tag{3.5}$$

Hence we obtain

$$\left(\frac{T_n}{n}, \frac{T_n}{n}, \ldots, \frac{T_n}{n}, \frac{T_n}{n}\right) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right).$$

Theorem 3.4. *Let $2 \leq n \in \mathbb{Z}^+, S_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $U_n = \frac{1}{n-1}\left(S_n - \frac{n}{T_n}\right)$. Then we have*

$$\left(\frac{n}{T_n}, U_n, U_n, \ldots, U_n\right) \prec \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right).$$

.

Proof. For $i, j = 1, 2, \ldots, n$, we define an $n \times n$ matrix

$$G_n = [g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{12} & \cdots & g_{12} \\ g_{21} & g_{22} & g_{22} & \cdots & g_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} & g_{n2} & g_{n2} & \cdots & g_{n2} \end{bmatrix}, \tag{3.6}$$

where for $i = 1, 2, \ldots, n$, $g_{i1} = \dfrac{1}{T_n \lambda_i}$ and $g_{i2} = \dfrac{1 - g_{i1}}{n-1}$.
From (3.5) and (3.6), for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} g_{11} + g_{21} + \cdots + g_{n1} &= \frac{1}{T_n \lambda_1} + \frac{1}{T_n \lambda_2} + \cdots + \frac{1}{T_n \lambda_n} = 1, \\ g_{12} + g_{22} + \cdots + g_{n2} &= \frac{1 - g_{11}}{n-1} + \frac{1 - g_{21}}{n-1} + \cdots + \frac{1 - g_{n1}}{n-1} = 1, \\ g_{i1} + (n-1)\, g_{i2} &= g_{i1} + (n-1)\, \frac{1 - g_{i1}}{n-1} = 1, \end{aligned}$$

where $g_{i1} \geq 0$ and $g_{i2} \geq 0$. Then, G_n is a doubly stochastic matrix. Also, we get

$$\begin{aligned} \lambda_1 g_{11} + \lambda_2 g_{21} + \cdots + \lambda_n g_{n1} &= \lambda_1 \frac{1}{T_n \lambda_1} + \lambda_2 \frac{1}{T_n \lambda_2} + \cdots + \lambda_n \frac{1}{T_n \lambda_n} = \frac{n}{T_n}, \\ \lambda_1 g_{12} + \lambda_2 g_{22} + \cdots + \lambda_n g_{n2} &= \lambda_1 \left(\frac{1 - g_{11}}{n-1}\right) + \lambda_2 \left(\frac{1 - g_{21}}{n-1}\right) + \lambda_n \left(\frac{1 - g_{n1}}{n-1}\right) = U_n. \end{aligned}$$

Therefore, we have

$$\left(\frac{n}{T_n}, U_n, U_n, \ldots, U_n\right) = \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right) G_n,$$

and so, we obtain

$$\left(\frac{n}{T_n}, U_n, U_n, \ldots, U_n\right) \prec \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right).$$

□

Lemma 3.5. *For $k = 2, 3, \ldots, n$, we get*

$$\lambda_k \geq \frac{1}{T_k},$$

where $T_k = \frac{2k^2+4k-5}{4}$.

Proof. By using (3.4), for $k \geq 2$, we have

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_{k-1}} + \frac{1}{\lambda_k} \leq \frac{1}{4} + \frac{5}{2} + \frac{7}{2} + \cdots + \frac{2k-1}{2} + \frac{2k+1}{2} = T_k$$

Therefore, we have

$$\frac{1}{\lambda_k} \leq T_k - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_{k-1}} \right) \leq T_k,$$

and so, the proof is completed. □

Theorem 3.6. *Let $2 \leq n \in \mathbb{Z}^+, S_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $U_n = \frac{1}{n-1}\left(S_n - \frac{n}{T_n}\right)$. Then for $k \leq n-2$, we have*

$$\begin{aligned} \lambda_1 &\leq 2^{2n} \prod_{i=2}^{n} T_i, \\ \lambda_{n-k} &\leq (k+1)\, U_n - \sum_{i=0}^{k-1} \frac{1}{T_{n-i}}. \end{aligned}$$

Proof. By Theorem 2.15, we know that $det\,(B_n) = 2^{2n} = \lambda_1 \lambda_2 \cdots \lambda_n$. By Lemma 3.5, we get

$$2^{2n} = \lambda_1 \lambda_2 \cdots \lambda_n \geq \lambda_1 \prod_{i=2}^{n} \frac{1}{T_i},$$

and so, we obtain $\lambda_1 \leq 2^{2n} \prod_{i=2}^{n} T_i$. By Theorem 3.4, for $k \leq n-2$, we have

$$\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-(k-1)} + \lambda_{n-k} \leq (k+1)\, U_n,$$

and so, by Lemma 3.5, we get

$$\begin{aligned} \lambda_{n-k} &\leq (k+1)\, U_n - \left(\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-(k-1)} \right) \\ &\leq (k+1)\, U_n - \sum_{i=0}^{k-1} \frac{1}{T_{n-i}}. \end{aligned}$$

Then the proof is completed. □

By applying Theorem 3.4 and Lemma 3.5, we can readily derive the subsequent corollary.

Corollary 3.7. *Let $2 \leq n \in \mathbb{Z}^+$ and $k \leq n-2$. Then we have*

$$\begin{aligned} &\frac{n}{T_n} \leq \lambda_1 \leq 2^{2n} \prod_{i=2}^{n} T_i, \\ &\frac{1}{T_{n-k}} \leq \lambda_{n-k} \leq (k+1)\, U_n - \sum_{i=0}^{k-1} \frac{1}{T_{n-i}}, \\ &\frac{1}{T_n} \leq \lambda_n \leq U_n. \end{aligned}$$

4. Conclusions

In this article, we introduce the Pell-Lucas A_n and the symmetric Pell-Lucas B_n matrices. We consider the linear algebra of these matrices. Firstly, we construct two different factorizations of Pell-Lucas matrices by the new matrix Γ_n. We find the inverse of the Pell-Lucas matrix A_n^{-1}, and present the factorization of A_n^{-1}. Then, we derive the components $[b_{ij}]$ of the Pell-Lucas matrix B_n, and construct the Cholesky factorization of B_n. This factorization is $A_n A_n^T$. We determine the inverse of the symmetric Pell-Lucas matrix B_n^{-1}. We give some interesting relations which include the eigenvalues of Pell-Lucas matrices. Moreover, we obtain the lower and upper boundaries for the eigenvalues of B_n by majorizations.

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