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Research Article

On the Sufficient Conditions for the Univalence of Definite Integral Operators Involving Certain Functions in $\mathcal{S}$ Class

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Abstract: In general terms, integral operators play a very important role as a useful mathematical tool in order to reach the desired results and make different inferences by analyzing the relevant issues in mathematics and applied sciences. It is important to understand the conditions under which integral operators map certain analytic functions to starlike and convex functions and effectively characterizing and using them is of great importance for studies in this field. In present article, some integral operators preserving class $\mathcal{S}$ are examined from a different perspective and the relevant inequalities and equations for their univalence are determined and solved.

Keywords: Analytic function, convex function, normalized function, starlike function, univalent function.

1. Introduction

As the interaction of analysis and geometry, geometric function theory is a very interesting subbrunch of complex analysis. Perhaps the important reason for this interest is the image sets of complex functions to which certain conditions (such as being analytic, being normalized, being univalent, and being defined in the unit disc) exhibit very interesting geometric characterizations. In this sense, geometric function theory aims, in principle, to analysis the analytic properties of analytic functions depending on the geometric properties of their image sets. Moreover, geometric function theory also aims to classify functions with certain properties given above according to the common geometric characterizations exhibited by image sets (such as convex, starlike, close-to-convex, etc.). The arguments used in doing this are depends on Riemann mapping theorem in 1851 [16]. It is well known that, under certain conditions, the Riemann mapping theorem guarantees the existence of an analytic function that conformal maps a simply connected region of the complex plane to the open unit disc $|z| < 1, z \in \mathbb{C}$ (hereafter represented with $\mathcal{U}$). In more mathematical terms, where $\mathcal{D} \subset \mathbb{C}$ is a simply connected region with more than one boundary

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points, for any $z_0 \in \mathcal{D}$ there is a single function f that satisfies the conditions $f(z_0) > 0$ and $f'(z_0) > 0$ and conformally maps $\mathcal{D}$ to $\mathcal{U}$. Unfortunately, the Riemann mapping theorem in its current form creates a complicated situation for classifying analytic functions. The complicated situation is that it is very difficult or even impossible to classify the analytic functions defined on different domains according to the common geometric characterizations exhibited by the image sets. The complicated situation expressed was eliminated when Paul Koebe one of the intellectual scientists working in this field, took the open unit disc $\mathcal{U}$ as the domain in 1907, without losing generality. This idea is, in a sense, the inverse of Riemann's mapping theorem. Now, analytic functions with domains $\mathcal{U}$ can be classified [5, 6, 9].

As you may remember from the basic complex analysis information, if derivative

$$f'(z) = \lim_{\Delta \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{1}$$

exists for each $z \in \mathcal{D}$, the function $f(z)$ is said to be analytic in the set $\mathcal{D} \in \mathbb{C}$. Let us denote by $\mathcal{H}$ class formed by all complex functions thet are analytic in $\mathcal{U}$. In addition, as a subclass of the class $\mathcal{H}$, let's denote with $\mathcal{A}$ the class consisting of all functions in the class $\mathcal{H}$ that satify the conditions $f(0) = 0$ and $f'(0) = 1$, known as normalized conditions. Notice that the functions of class $\mathcal{A}$ consist of normalized analytic functions in $\mathcal{U}$. In addition to all these, if the condition of being one-to-one (that is, for all $z_1, z_2 \in \mathcal{U}, f(z_1) = f(z_2)$ implies $z_1 = z_2$(A.W. Goodman, 1983)) [7, 9] is imposed as a new condition on the functions in class $\mathcal{A}$ is formed, which is denotes by $\mathcal{S}$. In studies conducted in this field, a function that s both analytic and one-to-one in $\mathcal{U}$ is called a univalent function. Note that univalent implies being both analytic and one-to-one in $\mathcal{U}$. In the finall analysis, under the conditions given above, naturally any function $f(z)$ in the class $\mathcal{S}$ has a Taylor expansion given by

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + \ldots + a_n z^n + \ldots, z \in \mathcal{U} \text{ [9]}. \tag{2}$$

As stated above, geometric function theory focuses on the concept of univalence and analyticity. Riemann Mapping Theorem plays an important role in unifying both concepts. This combination interprets the geometric characterizations of sets of images in order to classify functions. It is well known that, $\mathcal{S}^*$ and $\mathcal{C}$ are the two usual subclasses of class $\mathcal{S}$ of starlike and convex functions, which geometric characterizations of image sets satisfy the inequalities

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{3}$$

and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \tag{4}$$

respectively [11, 17]. Therefore, these two classes can be given analytically as follows:

$$\mathcal{S}^* = \left\{ f(z) \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathcal{U} \right\} \tag{5}$$

and

$$\mathcal{C} = \left\{ f(z) \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathcal{U} \right\}, \tag{6}$$

respectively. For a better understanding of the study, the geometric definitions of starlike and convex functions can be given as follows, respectively.

Definition 1.1 *A domain $\mathcal{D} \subset \mathbb{C}$ is called starlike with respect to an interior point w_0 if the line segment connecting point w to any interior point of $\mathcal{D}$ lies entirely within $\mathcal{D}$. In this case, a function $f(z)$ is called starlike with respect to the interior point w_0 if it maps the open unit disc $\mathcal{U}$ to a region that is starlike with respect to w_0 [7].*

It is very important to know that in studies conducted in this field, starlike function expression (i.e., elements of class $\mathcal{S}^*$) are referred to functions that are starlike according to the origin (i.e., $w_0 = 0$).

Definition 1.2 *If the line segment connecting for every different pairs of points w_1 and w_2 of a region $\mathcal{D} \subset \mathbb{C}$ lies entirely in $\mathcal{D}$, $\mathcal{D}$ is called a convex region. In this case, f is called a convex function if the function f maps the open unit disc $\mathcal{U}$ to a convex region [7].*

Another well-known subclass of class $\mathcal{S}$ is the class of close-to-convex functions [8].

Definition 1.3 *A function $f \in \mathcal{A}$ is said to be close-to-convex in an open unit disc $\mathcal{U}$ if there is a function g in $\mathcal{U}$ such that*

$$\Re\left(\frac{f'(z)}{g'(z)}\right) > 0, z \in \mathcal{U}. \tag{7}$$

The class of close-to-convex functions is usually denoted by $\mathcal{K}$.

If $f = g$ is taken in (7), it can be easily seen that a function that is convex in $\mathcal{U}$ is close to convex. Similarly, it can be easily obtained that each starlike function is close to convex. For this, it will be sufficient to take a starlike function $h(z) = zg'(z)$, $z \in \mathcal{U}$.

Geometric function theory deals mostly with the study of the properties of functions belonging to class $\mathcal{S}$. As mentioned before, such functions were studied by Paul Koebe in 1907. In this sense, the function given by the

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots = z + \sum_{z=2}^{\infty} nz^n \tag{8}$$

was first introduced by Koebe and is named after him, since this function is in class $\mathcal{S}$, it means that this function is analytic, normalized, and univalent in $\mathcal{U}$ which is simple to proven [8]. Firstly, the Koebe function $k(z)$ is analytic because it is complex differentiable at every point $z \in \mathcal{U}$. Secondly, the Koebe function $k(z)$ is normalized as it satisfies the normalization conditions $k(0) = 0$ and $k'(0) = 1$ in $\mathcal{U}$, where $k'(z) = 1 + \sum_{z=2}^{\infty} n^2 z^{n-1}$. On the other hand, if the necessary algebraic operations are performed, $z_1 = z_2$ is obtained when for all $z_1, z_2 \in \mathcal{U}, k(z_1) = k(z_2)$. As a result, the Koebe function $k(z)$ is univalent since it is analytic and one-to-one in $\mathcal{U}$. In geometric sense, under its properties, $k(z)$ Koebe function maps the open unit disk $\mathcal{U}$ conformally (i.e., preserves angles and orientation) on to the complex plane $\mathbb{C}$ excluding the slit along the negative real axis from $-\infty$ to $-1/4$. The existence of the Koebe function, which is vital in the analysis of class $\mathcal{S}$, naturally caused researchers to ask themselves different questions. In this sense, perhaps the most important problem that has attracted the attention of researchers and whose solution has been bothering them for a while is whether there is a relationship between the geometric feature of the image of a function belonging to the $\mathcal{S}$ class and the coefficients of the corresponding power series. Many researchers have struggled with this issue, known as the problem (or conjecture) of finding an upper bound for the coefficients of functions in the class $\mathcal{S}$. In 1916, Bieberbach stated and proved that a_2, the second coefficient of f functions in class $\mathcal{S}$, is bounded by 2 (that is, $|a_2| \leq 2$) and that equality within inequality is valid only for the Koebe function $k(z)$. He extended this further in his paper by assuming that all coefficients a_n of functions in class $\mathcal{S}$ are not larger than n with respect to their positions. Today, this conjectur is known as the Bieberbach conjecture [2].

Conjecture 1.4 *(Bieberbach Conjecture) All coefficients a_n of functions f in class $\mathcal{S}$ satisfy the inequality $|a_n| \leq n$ for each $n \geq 2$.*

This conjecture attracted a lot of attention because it remained unsolved for a long time. However, the methodological proof was made by Louis de Branges in 1984. In 1907, using Bieberbach conjecture $|a_2| \leq 2$ for $n = 2$, Koebe concluded that every function in class $\mathcal{S}$ contains $\{w : |w| \leq 1/4\}$ of the image set. Here again, equality within inequality is valid only for the Koebe function $k(z)$. The geometric result obtained by Koebe, also which is a reference for many other important results, is today known as the Koebe's $1/4$ Theorem or the Koebe-Bieberbach Theorem [6].

Theorem 1.5 *(Koebe's $1/4$ Theorem or Koebe-Bieberbach Theorem) The image of each function f in class $\mathcal{S}$ covers the disk $\{w : |w| \leq 1/4\}$ with center at the origin $w = f(0) = 0$ and radius $1/4$.*

Koebe's $1/4$ theorem, which is valid only for functions in $\mathcal{S}$ class, also guarantees the existence of the f^{-1} inverse of a function f in class $\mathcal{S}$, given by $f^{-1}(f(z)) = z \ (z \in \mathcal{U})$, where $f^{-1}(w) = w - (a_2)w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$.

In 1921, after the two important results given above, the Bieberbach conjecture for starlike ranges fo functions in class $\mathcal{S}^*$ was proven by Rolf Nevanlinna [11].

Theorem 1.6 *The power series coefficients of a function f in class $\mathcal{S}^*$ satisfy the inequality $|a_n| \le n$ for $n = 2, 3, \dots$. Similarly, equality within inequality is valid only for the Koebe function $k(z)$.*

Corollary 1.7 *The power series coefficients of a function f in class $\mathcal{C}$ satisfy the inequality $|a_n| \le 1$ for $n = 2, 3, \dots$. Equality within inequality is valid only for the Koebe function $f(z) = z(1-z)^{-2}$.*

Theorem 1.8 *The image of each function f in class $\mathcal{C}$ covers the disk $\{w : |w| \le 1/2\}$ with center at the origin $w = f(0) = 0$ and radius $1/2$.*

At this stage, several important conclusions obtained from the given preliminary information are presented. In the light of the information given so far, naturally we can write $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A} \subset \mathcal{H}$ according to the subset relationship in the sets. If f is in class $\mathcal{S}$ then any function composed of scaling, translating, and/or rotating f is also in class $\mathcal{S}$. Then $k(z)$ Koebe function can be written as the composed of

$$w_0 = \frac{1+z}{1-z}, \ w_1 = z^2 \text{ and } w_2 = \frac{1}{4}\Big[z - 1\Big].$$

That is,

$$k(z) = (w_2 \circ w_1 \circ w_0)(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right].$$

According to the given composition operation, the graph of the $k(z)$ Koebe function can be easily drawn.

From previous section, we know that the image of Koebe function is the whole plane minus the part of the negative real axis from $1/4$ to negative infinity. This situation can be easily seen from Figure 1. Thus, it is clear that Koebe function is starlike with respect to origin and not convex.

Furthermore, in 1915, Alexander showed the existence of a very useful relationship between class $\mathcal{S}$ and class $\mathcal{C}$ [1, 10].

Theorem 1.9 *(Alexander's Theorem) Let f(z) be a function in class $\mathcal{S}$. Then, $f \in \mathcal{C}$ if and only if $zf'(z) \in \mathcal{S}^*$. So, if $f(z) \in \mathcal{S}^*$, then*

$$g(z) = \int_0^z \frac{f(z)}{z} dz \tag{9}$$

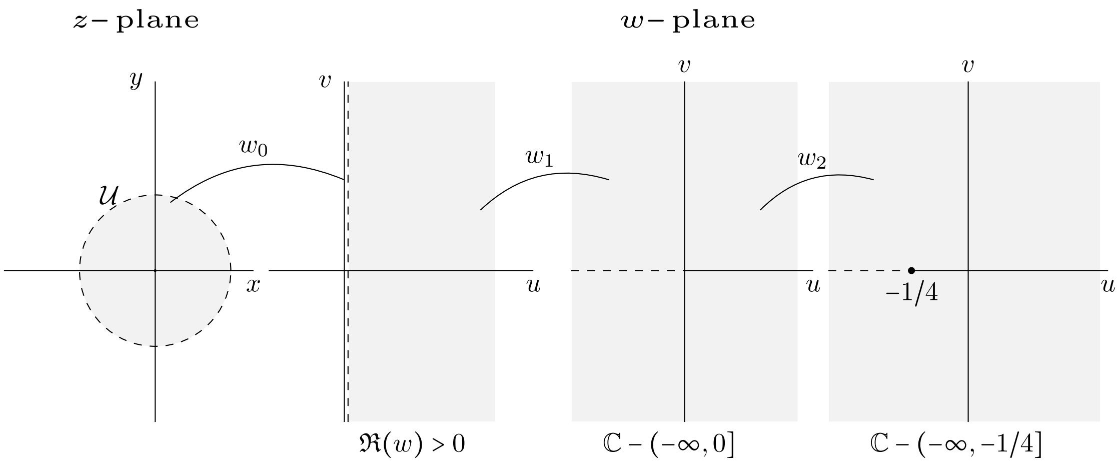

Figure 1: Image of open unit disc $\mathcal{U}$ under Koebe transform

is a convex function.

Notice that the Alexander integral operator maps functions from in class $\mathcal{S}^*$ to the class $\mathcal{C}$ of convex functions. This creative theorem, which is not difficult to prove, also accelerated the use of integral operators in geometric function theory. Some well-known integral operators in this sense are given below [3, 15].

- Alexander operator, 1915

$$g(z) = \int_0^z \frac{f(t)}{t} dt. \tag{10}$$

- Kim-Merkes operator (also atributed to Causey), 1963, α complex number

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt. \tag{11}$$

- Libera operator, 1965

$$g(z) = \frac{2}{z} \int_0^z f(t) dt. \tag{12}$$

- Bernardi operator, 1969, α complex number

$$g(z) = \frac{1+\alpha}{z^\alpha} \int_0^z f(t) t^{\alpha-1} dt. \tag{13}$$

- Pfaltzgraff operator, 1975, α complex number

$$g(z) = \int_0^z (f'(t))^\alpha dt. \tag{14}$$

Since 1907, many mathematicians have worked on integral operators that preserve class $\mathcal{S}$. In this sense, some important results can be found in [3, 12, 15]. The main purpose of these works is to determine the values of α which the functions

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} \text{ and } g(z) = \int_0^z \left(f'(t)\right)^{\alpha} dt \tag{15}$$

when $f(z)$ function in class $\mathcal{S}$ defined by certain conditions related to univalence. Also, the theorems given below can be found in [4, 13, 14, 18, 19].

Theorem 1.10 *If $f(z) \in \mathcal{S}$ is close-to-convex, then*

$$g(z) = \int_0^z \left(f'(t)\right)^{\alpha} dt \tag{16}$$

in class $\mathcal{S}$ for $\alpha \in [0,1]$.

Theorem 1.11 *If $f(z) \in \mathcal{S}$ is close-to-convex, then*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt \tag{17}$$

then $g(z)$ in class $\mathcal{S}$ for $\alpha \in [0,1]$.

Theorem 1.12 *If $f(z) \in \mathcal{S}$ and*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt, \tag{18}$$

in class $\mathcal{S}$ for $0 \le \alpha \le \left(\sqrt{1025} - 25\right)/100$.

Lemma 1.13 *Let $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}(z \in \mathcal{U}), \tag{19}$$

then $f(z)$ is in class $\mathcal{S}$.

2. Main Results

Theorem 2.1 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{C}$, and*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt. \tag{20}$$

Then, $g(z) \notin \mathcal{S}$ for $\alpha \in [0,3/2]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{C}$, where α_0 is the smallest positive root of the $\alpha(\alpha+1)(\alpha+2) = 96$.

Proof It is follows from (20) that

$$1+\frac{zg''(z)}{g'(z)}=1+\frac{z\left\{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left[\frac{zf'(z)-f(z)}{z^2}\right]\right\}}{\left(\frac{f(z)}{z}\right)^{\alpha}}$$

$$=1+\alpha\left(\frac{z}{f(z)}\right)\left[\frac{zf'(z)-f(z)}{z^2}\right]$$

$$=1+\alpha\frac{zf'(z)}{f(z)}-\alpha,$$

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right)=1-\alpha-\alpha\Re\left(\frac{zf''(z)}{f'(z)}\right)>1-\alpha\geq -0.5.$$

Thus, from Lemma 1.13, $g(z)\in\mathcal{S}$ is obtained for $\alpha\in[0,1.5]$. On the other hand, if we let $f(z)=z/(1-z)$ and $g(z)\in\mathcal{S}$, then we obtain

$$g'(z)=\left(\frac{\frac{z}{1-z}}{z}\right)^{\alpha}$$

$$=\frac{1}{(1-z)^{\alpha}}$$

$$=1+\alpha z+\frac{\alpha(\alpha+1)}{2!}z^2+\frac{\alpha(\alpha+1)(\alpha+2)}{3!}z^3+\ldots.$$

Thus,

$$\left|\alpha\right|<4,\ \left|\frac{\alpha(\alpha+1)}{2!}\right|<9 \text{ and } \left|\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right|<16 \tag{21}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $\alpha^2+\alpha-18=0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (21):

$$0<\alpha\leq\alpha_0<\frac{-1+\sqrt{73}}{2}<4,$$

where α_0 is smallest positive real number root the equation(that is, $\alpha_0=3.65165$) $\alpha(\alpha+1)(\alpha+2)-96=0$. This result ends the proof of Theorem 2.1. □

Theorem 2.2 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{S}^*$, and*

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt. \tag{22}$$

Then, $g(z) \in \mathcal{S}$ for $\alpha \in [0,3]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{S}^*$ such that $g(z) \notin \mathcal{S}$, where α_0 is the smallest positive root of the $\alpha(\alpha+1)(\alpha+2) = 96$.

Proof When the same method as applied in the proof of Theorem 2.1 is applied,

$$
\begin{aligned}
1 + \frac{zg''(z)}{g'(z)} &= 1 + \frac{z\left\{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left[\frac{zf'(z)-f(z)}{z^2}\right]\right\}}{\left(\frac{f(z)}{z}\right)^{\alpha}} \\
&= 1 + \alpha\left(\frac{z}{f(z)}\right)\left[\frac{zf'(z)-f(z)}{z^2}\right] \\
&= 1 + \alpha\frac{zf'(z)}{f(z)} - \alpha,
\end{aligned}
$$

$$
1 + \Re\left(\frac{zg''(z)}{g'(z)}\right) = 1 - \alpha - \alpha\Re\left(\frac{zf''(z)}{f'(z)}\right) > 1 - \alpha \geq -0.5,
$$

$$
\Re\left(\frac{zg''(z)}{g'(z)}\right) > 0.5
$$

is obtained for $\alpha \in [0,3]$. Letting $f(z) = z(1-z)^{-2}$ and $g(z) \in \mathcal{S}$, then we have

$$
\begin{aligned}
g'(z) &= \left(\frac{f(z)}{z}\right)^{\alpha} \\
&= 1 + 2\alpha z + \frac{2\alpha(2\alpha+1)}{2!}z^2 + \frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}z^3 + \dots .
\end{aligned}
$$

Thus,

$$
\left|2\alpha\right| < 4, \quad \left|\frac{2\alpha(2\alpha+1)}{2!}\right| < 9 \text{ and } \left|\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}\right| < 16 \tag{23}
$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $2\alpha^2 + \alpha - 9 = 0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (23):

$$
0 < \alpha \leq \alpha_0 < \frac{-1+\sqrt{73}}{2} < 4,
$$

where α_0 is smallest positive real number root the equation (that is, $\alpha_0 = 3.15717$) $\alpha(2\alpha+1)(\alpha+1) - 96 = 0$. This result ends the proof of Theorem 2.2. □

Theorem 2.3 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{C}$, and*

$$
g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt. \tag{24}
$$

Then, $g(z) \in \mathcal{S}$ for $\alpha \in [0, 1.5]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{C}$ such that $g(z) \notin \mathcal{S}$, where α_0 is the smallest positive root of the $\alpha^2 + \alpha - 18 = 0$.

Proof If algebraic operations similar to those in Theorem 2.1 and Theorem 2.2 are performed,

$$1 + \Re\left(\frac{zg''(z)}{g'(z)}\right) = 1 - \alpha + \alpha\Re\left(\frac{zf''(z)}{f'(z)}\right) > 1 - \alpha \geq -0.5$$

obtained for $\alpha \in [0, 1.5]$. Letting $f(z) = z(1-z)^{-2}$ and $g(z) \in \mathcal{S}$, then we have

$$\begin{aligned} g'(z) &= \left(\frac{f(z)}{z}\right)^{\alpha} \\ &= 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}z^3 + \dots. \end{aligned}$$

Thus,

$$\left|\alpha\right| < 4, \left|\frac{\alpha(\alpha+1)}{2!}\right| < 9 \text{ and } \left|\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right| < 16 \tag{25}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $\alpha^2 + \alpha - 18 = 0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (25):

$$0 < \alpha \leq \alpha_0 < \frac{-1+\sqrt{73}}{2} < 4,$$

where α_0 is smallest positive real number root the equation (that is, $\alpha_0 = 3.65165$) $\alpha(\alpha+1)(\alpha+1) - 96 = 0$. This result ends the proof of Theorem 2.3. $\square$

Theorem 2.4 *Let the function $f(z)$ given by (2) be a function in class $\mathcal{S}^*$, and*

$$g(z) = \int_0^z \left(f'(t)\right)^{\alpha} dt. \tag{26}$$

Then, $g(z) \in \mathcal{S}$ for $\alpha \in [0, 1.5]$ but for $\alpha_0 < \alpha$, there exists a function $f(z) \in \mathcal{C}$ such that $g(z) \notin \mathcal{S}$, where α_0 is the smallest positive root of the $\alpha^2 + \alpha - 18 = 0$.

Proof

$$1+\frac{zg''(z)}{g'(z)}=1+\frac{z\left\{\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left[\frac{zf'(z)-f(z)}{z^2}\right]\right\}}{\left(\frac{f(z)}{z}\right)^{\alpha}}$$

$$=1+\alpha\left(\frac{z}{f(z)}\right)\left[\frac{zf'(z)-f(z)}{z^2}\right]$$

$$=1+\alpha\frac{zf'(z)}{f(z)}-\alpha,$$

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right)=1-\alpha-\alpha\Re\left(\frac{zf''(z)}{f'(z)}\right)>1-\alpha\geq-0.5,$$

$$\Re\left(\frac{zg''(z)}{g'(z)}\right)>0.5$$

is obtained for $\alpha\in[0,3]$. Letting $f(z)=z(1-z)^{-2}$ and $g(z)\in\mathcal{S}$, then we have

$$g'(z)=\left(\frac{f(z)}{z}\right)^{\alpha}$$

$$=1+2\alpha z+\frac{2\alpha(2\alpha+1)}{2!}z^2+\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}z^3+\ldots.$$

Thus,

$$\left|2\alpha\right|<4,\quad\left|\frac{2\alpha(2\alpha+1)}{2!}\right|<9\text{ and }\left|\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}\right|<16 \tag{27}$$

are obtained from the Conjecture 1.4. At this stage, with a simple algebraic calculation, the positive real number root of the second degree equation $2\alpha^2+\alpha-9=0$ obtained as $\frac{-1+\sqrt{73}}{2}$. Letting α_0 be a positive real number, we must have the following inequqlity from (27):

$$0<\alpha\leq\alpha_0<\frac{-1+\sqrt{73}}{2}<4,$$

where α_0 is smallest positive real number root the equation (that is, $\alpha_0=3.15717$) $\alpha(2\alpha+1)(\alpha+1)-96=0$. This result ends the proof of Theorem 2.4. □

3. Conclusion

The meaning of the derivative of a function $w=f(z)$ defined in the complex plane at a point given by (1) is different from its meaning in real analysis. In real analysis, the derivative of a function given by $y=f(x)$ is a measure of the ratio of the change in the independent variable x to the change in the dependent variable y of the function. As you may remember, this measure represents physical information such as flux, velocity or slope at a point. However, in complex variable functions, the main priority is whether or not there is a derivative. The existence of the derivative provides information about the analytical and geometric properties of the complex function. Does the existence of a derivative of a complex valued function f at a point z_0 mean that point z_0 is an interior point of the region, where the function is defined? Or is it a border point? It varies depending on what happened. To avoid this confusion, all analytic functions are defined on an open subset of the complex plane, that is, a region. In this case, differentiability in the complex sense refers to the limitation, size and shape of the image regions of the analytical functions $w = f(z)$ as geometric characterizations. These concepts are very important for classifying analytical functions. Integral and integral operators are very useful and of great importance in geometric function theory, especially in single-valued function theory. In this sense, it has been demonstrated through wonderful studies that the integral operators introduced help in the analytical classification of univalent functions. In the presented article, various inequalities and equations were obtained in addition to the existing studies.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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