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RESEARCH ARTICLE

LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION

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Abstract

In the present study, we give the proofs about important properties of Lambert azimuthal projection, for instance conformality, preserve area that important points for characterization. While there are some kind of Lambert projections in the literature (for instance standard, cylindrical), we utilize from the south polar aspect. In our proofs, we use the south polar aspect, and finally we give some visualizations of the inverse of projection as an example.

Keywords

Möbius transformation,
Lambert azimuthal projection,
Visualization

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1. INTRODUCTION

Möbius transformations are very useful tool for understanding patterns in Euclidean space. Especially in extended complex plane many kind of curves can be modelled. Möbius transformations also in use tesellations and calculations of distances in hyperbolic geometry. Also it maps circles to circles and preserves cross-ratio, angles and symmetry [1]. Since the angles preserved, Möbius transformation is a conformal map. Moreover, one can compose the Möbius transformations by basic type transformations like translation, rotation, scaling and inversion. Möbius transformations are connected to non-Euclidean geometries (in some models of hyperbolic geometry (e.g. Poincare's disk), isometries are represented by Möbius transforms) and these transformations are connected to Einstein's theory of relativity (via Lorenz transforms) [1]. Closely related to spherical geometry is the geometry of Möbius transformations on the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. This connection is provided by the stereographic projection map $\pi: S^2 \to \mathbb{C}_\infty$ [2]. Many geometricians have studied on Möbius transformation in order to tesellations and create patterns by using different kind of approaches [3,4]. On the other hand, the Lambert azimuthal equal-area projection is one of the normal cylindrical projection who defined by the Swiss mathematician and cartographer Johann Heinrich Lambert [5].

Let us imagine a tangent plane to the unit sphere with center $O(0,0,0)$ at some point S south pole (see, Figure 1). Let P be any point on the sphere differ from antipodal of S and d be the Euclidean distance between S and P in three-dimensional space. Then, the Lambert azimuthal equal-area projection sends point P to only one point P' on the plane that is a distance d from S. Hence, the Lambert projection maps point P to P' at equal distance to S, i.e. $|PS| = |P'S|$. In the general case P' lies on open disk of radius 2 centered at the origin $(0,0)$ in the plane. Moreover, it lies on circle of radius $\sqrt{2}$ centered at

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(0,0) when P lies on equator. Notice that, while the Lambert projection preserves the area, it is not conformal, i.e. angles are not preserved under the projection. For more details, we refer [6-8].
In the present study, we use the polar aspect of Lambert azimuthal equal-area projection. We choose the projection center as $S(0,0,-1)$ on unit sphere. In section 3, we give the proofs which characterize the important properties of the projection. In the last section, under the inverse Lambert projection, we give some visualizations rely on fixed points of the Möbius transformation.

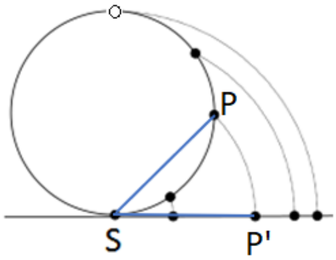


Figure 1: The Lambert azimuthal equal-area projection.

2. PRELIMINARIES

In this section, some basics related with the Möbius transformation are given.

Definition 2.1 Any mapping of the form

$$\mathcal{M}(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}$$

where $ad-bc \neq 0$, is called a Möbius transformation. If $c=0$, $\mathcal{M}(z)$ is defined for all $z \in \mathbb{C}$; if $c \neq 0$ it is defined for all $z \neq -d/c$. We can avoid this dichotomy by extending $\mathcal{M}$ to a map on the extended complex plane $C_\infty = C \cup \{\infty\}, \mathcal{M}(\infty) = \infty$ if $c=0, \mathcal{M}(-d/c) = \infty, \mathcal{M}(\infty) = a/c$ if $c \neq 0$ [9].

If $ad-bc=0$ then $\mathcal{M}$ is constant. Note that $\mathcal{M}$ is conformal, i.e., it preserves angles.

The coefficients of $\mathcal{M}$ are not unique. For any $\lambda \neq 0$ real number $\mathcal{M}(z) = \frac{(\lambda a)z+\lambda b}{(\lambda c)z+\lambda d}$. Let the Möbius transformation is not constant. Then, $\lambda = \frac{1}{\sqrt{ad-bc}}$ gives

$$\mathcal{M}(z) = \frac{a'z+b'}{c'z+d'},$$

such that $a'd'-b'c'=1$. In this case, $\mathcal{M}$ is in normalized form [9].

Lemma 2.2 Let $\mathcal{M}$ be a Möbius transformation in normalized form, i.e., $ad-bc=1$. Then the fixed points of $\mathcal{M}$ are,

$$\xi_{1,2} = \frac{(a-d) \mp \sqrt{(a+d)^2-4}}{2c}.$$

In particular, $|fix(\mathcal{M})| = 1$ if and only if $a+d = \mp 2$. Otherwise, $|fix(\mathcal{M})| = 2$ when $\mathcal{M} \neq i.\ \infty \in fix(\mathcal{M})$ if and only if $c=0$.

Proof. Obviously, $z \in fix(\mathcal{M})$ if and only if $\frac{az+b}{cz+d} = z$ which yields the following:

$$cz^2 + (d-a)z - b = 0. \tag{2.1}$$

Complex roots of the Eq. (2.1) as follows

$$z = \frac{(a-d) \mp \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Since $(a-d)^2 + 4bc = (a+d)^2 - 4$, we get

$$z = \frac{(a-d) \mp \sqrt{(a+d)^2 - 4}}{2c}. \tag{2.2}$$

Hence $z \in fix(\mathcal{M})$ if and only if z is written by Eq. (2.2). Remain of the proof is obvious.

Let $\infty \in fix(\mathcal{M})$ or $c = 0$. In this case, the Möbius transformation $\mathcal{M}$ can be written by

$$\mathcal{M}(z) = \omega z + \Omega, \tag{2.3}$$

where $\omega = a/d,\ \Omega = b/d$. By the help of Eqs. (2.1) and (2.3), the following table can be given:

Table 1. Möbius transformation and its fixed points depends on coefficients.

$\boldsymbol{\Omega}$	$\boldsymbol{\omega}$	$\boldsymbol{\mathcal{M}(z)}$	$\boldsymbol{fix(\mathcal{M})}$
$\Omega \neq 0$	$\omega = 1$	$\mathcal{M}(z) = z + \Omega$ translation on $\mathbb{C}$	$\{\infty\}$
$\Omega = 0$	$\|\omega\| = 1$	$\mathcal{M}(z) = e^{i\theta} z$ rotation on $\mathbb{C}$	$\{0, \infty\}$
$\Omega = 0$	$\omega = r \in \mathbb{R}$	$\mathcal{M}(z) = rz$ expansion on $\mathbb{C}$	$\{0, \infty\}$
$\Omega = 0$	$\omega = re^{i\theta} \in \mathbb{C}$	$\mathcal{M}(\mathrm{z}) = re^{i\theta} z$ spiralization on $\mathbb{C}$	$\{0, \infty\}$

3. LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION

In this section, we give the proofs which characterize the Lambert azimuthal equal-area projection.

Theorem 3.1 The Lambert azimuthal equal-area projection maps points (x, y, z) of unit sphere to the points (X, Y) of plane such that

$$X = \sqrt{\frac{2}{1-z}}x, \qquad Y = \sqrt{\frac{2}{1-z}}y.$$

On the other hand, the inverse of Lambert projection maps points (X, Y) of plane to the points (x, y, z) of unit sphere with centre $O(0,0,0)$ such that

$$x = \sqrt{1 - \frac{X^2 + Y^2}{4}}X, \qquad y = \sqrt{1 - \frac{X^2 + Y^2}{4}}Y, \qquad z = -1 + \frac{X^2 + Y^2}{2},$$

where $X^2 + Y^2 < 4$ and $z \neq 1$.

Proof. In order to calculations, we utilize from Figure 2 (see, [1, p.250]). Notice that the notations we use x, y, z for coordinates of sphere, and capital letters X, Y for coordinates of plane. Let us consider the tangent plane π of unit sphere $\mathbb{S}^2(1)$ with center $O(0,0,0)$ at point S (south pole). Since the plane is tangent to sphere at $S(0,0,-1)$, it is more useful to consider π as $z = -1$. Thus, all $P' \in \pi$ points can be written as $P' = P'(X, Y, -1)$. Note that P and P' lie on the same arc with center S. Therefore, $|PS| = |P'S|$ which yields

$$X^2 + Y^2 = x^2 + y^2 + (z+1)^2. \tag{3.1}$$

Moreover, $P \in \mathbb{S}^2(1)$, that is

$$x^2 + y^2 + z^2 = 1. \tag{3.2}$$

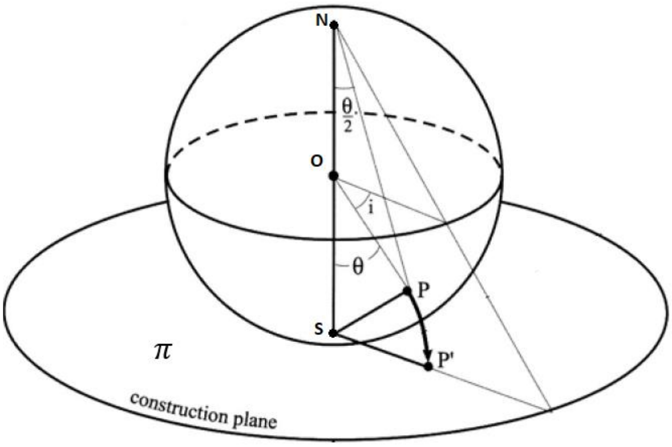


Figure 2: Lambert azimuthal equal-area projection

By Eqs. (3.1) and (3.2) we obtain $z = -1 + \frac{X^2+Y^2}{2}$. On the other hand, well-known cosine theorem in $\overset{\triangle}{POS}$ gives $|PS|^2 = 2 - 2\cos\theta = 4sin^2\frac{\theta}{2}$. Thus, $|P'S|^2 = |PS|^2 = 4sin^2\frac{\theta}{2}$ and it follows

$$\cos\frac{\theta}{2} = \sqrt{1 - \frac{X^2 + Y^2}{4}}, \tag{3.3}$$

where $0 < \theta < \pi$. Now we orthogonal project P to $[SP']$ at corresponding point $K(x, y, -1)$ on this segment (see Figure 3). Since $\overset{\triangle}{POS}$ is isosceles triangle and $[NS]$ orthogonal to $[SP']$, we obtain $m(\widehat{PSP'}) = \frac{\theta}{2}$. Hence,

$$\cos\frac{\theta}{2} = \frac{|KS|}{|PS|} = \frac{\sqrt{x^2 + y^2}}{|PS|}. \tag{3.4}$$

Since $|PS| = |P'S| = \sqrt{X^2 + Y^2}$, Eqs. (3.3) and (3.4) gives

$$x^2 + y^2 = (X^2 + Y^2)\left(1 - \frac{X^2 + Y^2}{4}\right).$$

Now, we consider triangle $\overset{\triangle}{PSP'}$ (triangle with P, S, P' vertexes). From cosine theorem it is written, $|PP'|^2 = |PS|^2 + |P'S|^2 - 2|PS||P'S|\cos\frac{\theta}{2}$. If sub $|PP'|^2 = (X - x)^2 + (Y - y)^2 + (z + 1)^2$, $|PS| = |P'S| = 2\,sin\frac{\theta}{2}$, and $z + 1 = \frac{X^2+Y^2}{2}$ in equation of cosine theorem, it follows

$$(X - x)^2 + (Y - y)^2 + \left(\frac{X^2 + Y^2}{2}\right)^2 = 8\,sin^2\frac{\theta}{2}\left(1 - \cos\frac{\theta}{2}\right). \tag{3.5}$$

From Eq. (3.3), it is easily seen $X^2 + Y^2 = 4\,sin^2\frac{\theta}{2}$. If we expand the left-hand side of Eq. (3.5) we obtain

$$X^2 - 2xX + x^2 + Y^2 - 2yY + y^2 + \frac{(X^2+Y^2)^2}{4} = 8\, sin^2\frac{\theta}{2}\left(1 - \cos\frac{\theta}{2}\right)$$

and $X^2 + Y^2 = 4\, sin^2\frac{\theta}{2}$ gives

$$4\, sin^2\frac{\theta}{2} + 4\, sin^4\frac{\theta}{2} - 2xX + x^2 - 2yY + y^2 = 8\, sin^2\frac{\theta}{2}\left(1 - \cos\frac{\theta}{2}\right). \tag{3.6}$$

Now we recall the projection point K.

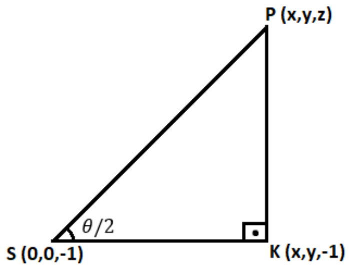


Figure 3: Orthogonal projection of $P(x,y,z)$ onto segment $[SP']$.

Since $|PS| = 2\, sin\frac{\theta}{2}$, it is obvious from Figure 3 that $|KS| = sin\theta$. Also, $|KS| = \sqrt{x^2+y^2}$. Then, we have $x^2 + y^2 = sin^2\theta$. Let sub this identity in Eq. (3.6) we get

$$4\, sin^2\frac{\theta}{2} + 4\, sin^4\frac{\theta}{2} - 2xX - 2yY + sin^2\theta = 8sin^2\frac{\theta}{2} - 8sin^2\frac{\theta}{2}\cos\frac{\theta}{2}.$$

By straightforward calculations we obtain

$$2xX + 2yY = 4sin\frac{\theta}{2}sin\theta + sin^2\theta - 4\, sin^2\frac{\theta}{2} + 4\, sin^4\frac{\theta}{2}.$$

Since $-4\, sin^2\frac{\theta}{2} + 4\, sin^4\frac{\theta}{2} = -sin^2\theta$, the last equation gives

$$2xX + 2yY = 4sin\frac{\theta}{2}sin\theta = \frac{2sin^2\theta}{cos\frac{\theta}{2}}. \tag{3.7}$$

By $x^2 + y^2 = sin^2\theta$, and Eq. (3.7) it follows

$$xX + yY = \frac{x^2}{cos\frac{\theta}{2}} + \frac{y^2}{cos\frac{\theta}{2}}. \tag{3.8}$$

Eq. (3.8) is true if and only if

$$X = \frac{x}{cos\frac{\theta}{2}} \text{ and } Y = \frac{y}{cos\frac{\theta}{2}}. \tag{3.9}$$

It follows from Eqs. (3.3), (3.9), and the identity $z = -1 + \frac{X^2+Y^2}{2}$ that

$$X = \sqrt{\frac{2}{1-z}}x, \qquad Y = \sqrt{\frac{2}{1-z}}y.$$

Hence, we get the Lambert azimuthal equal-area projection. On the other hand, from Eqs. (3.3) and (3.9) we obtain

$$x = \sqrt{1 - \frac{X^2 + Y^2}{4}} X, \qquad y = \sqrt{1 - \frac{X^2 + Y^2}{4}} Y, \quad z = -1 + \frac{X^2 + Y^2}{2}.$$

Thus, we get the inverse projection, and this completes the proof.

By the help of [8, p.27-28], the following Cauchy-Riemann like condition can be given:

Lemma 3.2 A map projection of the sphere to be equal-area if and only if

$$\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda} - \frac{\partial y}{\partial \lambda}\frac{\partial x}{\partial \varphi} = scos\varphi, \tag{3.10}$$

where s is constant. Here, φ represents latitude, λ represents longitude and x and y are the projected coordinates for a given (φ, λ) pair.

In spherical coordinates (ψ, θ) on the sphere with ψ the colatitude (complement of the latitude) and θ the longitude, and polar coordinates (R, Θ) on the disk, the map and its inverse are given by [8]:

$$(R, \Theta) = \left(2cos\frac{\psi}{2}, -\theta\right) \text{ and } (\psi, \theta) = \left(2arccos\frac{R}{2}, -\Theta\right). \tag{3.11}$$

By Eqs. (3.10) and (3.11), the following theorem can be given:

Theorem 3.3 The Lambert azimuthal projection preserves the area. Furthermore, the constant $s = 1$ in Eq. (3.10).

Proof. Let the first deal with the notations by mentioned above. Here, we show the latitude by φ and longitude by λ in the Lambert projection. By Eq. (3.11), the polar coordinates correspond to $\left(2cos\frac{\psi}{2}, -\theta\right)$ on the plane which gives us the cartesian coordinates as

$$x = 2cos\frac{\psi}{2}\cos(-\theta) \text{ and } y = 2cos\frac{\psi}{2}\sin(-\theta). \tag{3.12}$$

Since ψ is the colatitude, and $\lambda = \theta$, we have

$$x = 2cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) cos\,\lambda, \qquad y = -2cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) sin\,\lambda.$$

If sub x and y in Eq. (3.10), after the simplifying some calculations, we obtain the following:

$$\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda} - \frac{\partial y}{\partial \lambda}\frac{\partial x}{\partial \varphi} = sin\left(\frac{\pi}{2} - \varphi\right) = cos\varphi.$$

Hence, the Lambert azimuthal projection satisfies Eq. (3.10) with constant $s = 1$. Thus, the map preserves the area and is equal-area projection.

Proposition 3.4 Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be the points on unit sphere and the corresponding points under the Lambert azimuthal equal-area projection be $P_1{}'$ and $P_2{}'$ on the plane. Then the following reads:

$$\theta'' = \mp\theta' + 2k\pi, k \in \mathbb{Z},$$

where θ' is angle between $P_1{}'$ and $P_2{}'$, and θ'' is angle between the orthogonal projection points $P_1{}''(x_1, y_1, 0)$ and $P_2{}''(x_2, y_2, 0)$.

Proof. By Theorem 3.1, it is easily seen that

$$cos\theta' = \frac{x_1x_2 + y_1y_2}{\sqrt{({x_1}^2 + {y_1}^2)({x_2}^2 + {y_2}^2)}}. \tag{3.13}$$

On the other hand, from the orthogonal projection points we have,

$$cos\theta'' = \frac{x_1x_2 + y_1y_2}{\sqrt{({x_1}^2 + {y_1}^2)({x_2}^2 + {y_2}^2)}}. \tag{3.14}$$

From the Eqs. (3.13) and (3.14), we get the intended.

Theorem 3.5 The Lambert azimuthal equal-area projection is not conformal.

Proof. Let us consider two different points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ on unit sphere with center O, and assume that the Lambert azimuthal projection is conformal. At least one of z_1 and z_2 differ from zero. Since P_1 and P_2 are unit vectors, it follows:

$$P_1 \bullet P_2 = cos\theta = x_1x_2 + y_1y_2 + z_1z_2, \tag{3.15}$$

where θ is angle between OP_1 and OP_2, and "$\bullet$ " is Euclidean inner product. On the other hand,

$$cos\theta' = \frac{x_1x_2 + y_1y_2}{\sqrt{({x_1}^2 + {y_1}^2)({x_2}^2 + {y_2}^2)}}, \tag{3.16}$$

where θ' is angle between projected points $SP_1{}'$ and $SP_2{}'$, S is the projection center. From Eqs. (3.15) and (3.16) it follows:

$$cos\theta = \sqrt{(1 - {z_1}^2)(1 - {z_2}^2)}\, cos\theta' + z_1z_2, \tag{3.17}$$

where ${x_1}^2 + {y_1}^2 + {z_1}^2 = {x_2}^2 + {y_2}^2 + {z_2}^2 = 1$. From assumption and Eq. (3.18), $\theta = \theta'$ and so

$$z_1z_2 = 0 \text{ and } (1 - {z_1}^2)(1 - {z_2}^2) = 1.$$

The last condition implies $z_1 = z_2 = 0$ which is contradiction. This completes the proof.

4. EXAMPLES

In this section, by the help of Eq. (2.3) and inverse Lambert equal-area projection, we illustrate some samples of surface.

Example 4.1 Recall Eq. (2.3). If $\Omega \neq 0$, $\omega = 1$, then we can write $\Omega = \Omega_r + \Omega_i$ where Ω_r and Ω_i are real and imaginer part of Ω, respectively. If we take $\Omega_r = \Omega_i = 1$ then, inverse Lambert equal-area projection maps points $(X, Y) \in M(z)$ to the following points on unit sphere:

$$(x, y, z) = \left(\sqrt{1 - \frac{(X+1)^2 + (Y+1)^2}{4}}(X + 1), \sqrt{1 - \frac{(X+1)^2 + (Y+1)^2}{4}}(Y + 1), -1 + \frac{(X+1)^2 + (Y+1)^2}{2}\right).$$

This transformation forms Figure 4.

If we take $\Omega_r = -1$, $\Omega_i = 1$ then, inverse Lambert equal-area projection maps points $(X, Y) \in \mathcal{M}(z)$ to the following points on unit sphere:

$$(x, y, z) = \left(\sqrt{1 - \frac{(X-1)^2 + (Y+1)^2}{4}}(X - 1), \sqrt{1 - \frac{(X-1)^2 + (Y+1)^2}{4}}(Y + 1), -1 + \frac{(X-1)^2 + (Y+1)^2}{2}\right).$$

This transformaton forms Figure 5.

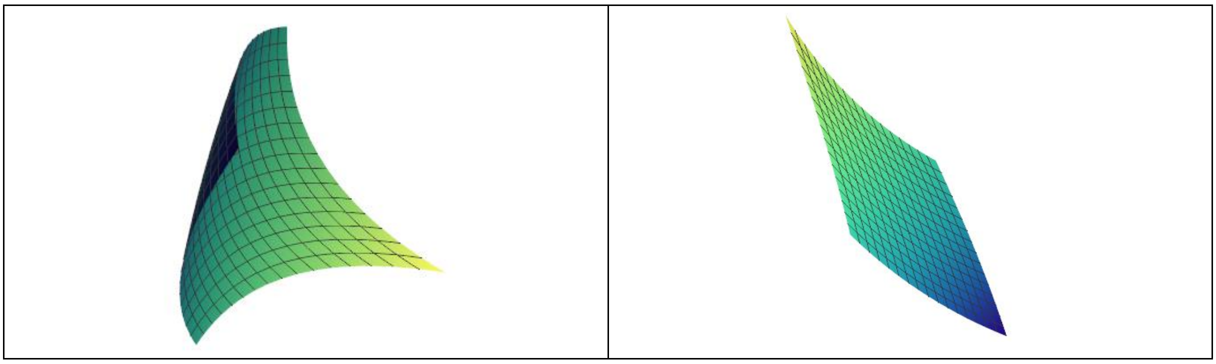


Figure 4: Lines on leaf surfaces of sphere

Remark. Figure 4 and Figure 5 gives us a clue about that not only translations forms family of parallel of lines in plane but also it forms family of parallel circular arcs on the sphere, under the inverse Lambert azimuthal equal-area projection.

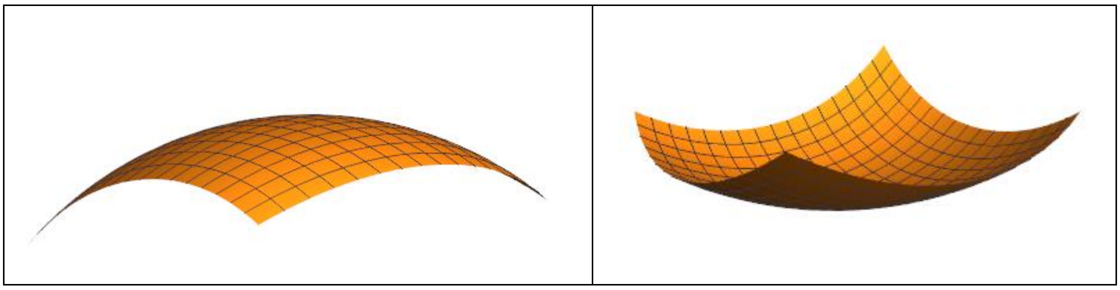
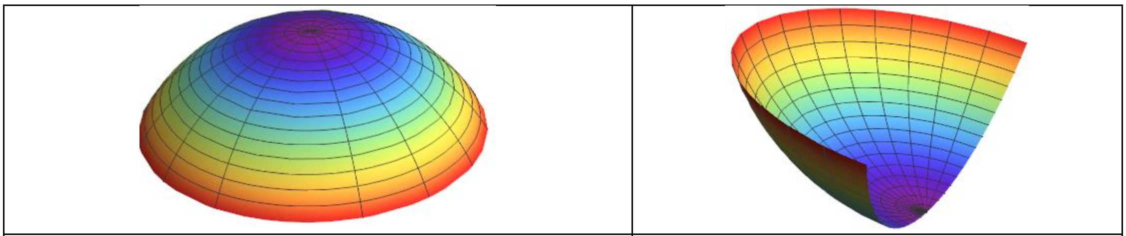


Figure 5: Arcs on fluffy patches

Example 4.2 $\Omega = 0$, and $|\omega| = 1$. Then, $\mathcal{M}(z) = e^{i\theta}z$ or explicitly $\mathcal{M}(z) = re^{i(\theta_1+\theta_2)}$ where $\theta = \theta_1$, $z = re^{i\theta_2}$. Thus, inverse Lambert projection maps points $(X, Y) \in \mathcal{M}(z)$ to the following points:

$$(r,\phi) = \left(r\sqrt{1-\frac{r^2}{4}}cos\phi, r\sqrt{1-\frac{r^2}{4}}sin\phi, -1+\frac{r^2}{2}\right), \tag{3.14}$$

where $\phi = \theta_1 + \theta_2$. Depend on (r, ϕ), Eq. (3.14) represents the parts of sphere.



$(r,\phi): 0 \le r \le 2, 0 \le \phi \le 2\pi$	$(r,\phi): 0 \le r \le 1, 0 \le \phi \le \pi;$

Figure 6: Hemisphere and quarter part

Remark. Figure 6 gives us a clue such that the rotations in the plane corresponds to circles on the sphere $\mathbb{S}^2$, under the inverse Lambert azimuthal equal-area projection.

Example 4.3 Let us consider the points $P_1(\frac{2}{7},\frac{3}{7},\frac{6}{7})$ and $P_2(-\frac{3}{\sqrt{17}},-\frac{2}{\sqrt{17}},\frac{2}{\sqrt{17}})$ on unit sphere. It is easily seen that $OP_1 \bullet OP_2 = 0$ where $\bullet$ is inner product of Euclidean space. Thus, $OP_1 \perp OP_2$. Lambert projection sends points P_1 and P_2 to

$$P_1{}'(\frac{2\sqrt{14}}{7},\frac{3\sqrt{14}}{7}) \text{ and } P_2{}'\left(-3\sqrt{\frac{2}{221}\left(17+2\sqrt{17}\right)},-2\sqrt{\frac{2}{221}\left(17+2\sqrt{17}\right)}\right)$$

respectively. It is obvious that $P_1{}'$ is not orthogonal to $P_2{}'$. That is, Lambert projection is not conformal.

5. CONCLUSION

In the recent literature, there is not enough paper about Lambert azimuthal equal-area projection. However, a few papers in cartography, especially in [7] and [8] some basic theorems about the projection are given. In this study, we give elementary and differential geometric proofs of theorems related with the projection. We discuss the projection by using south pole projection. Moreover, the fundamental properties of the projection are examined. On the other hand, by using Möbius transformation and the Lambert azimuthal equal-area projection, some patterns on the sphere are given. We hope that this paper will be useful for further studies about the projection. Also, it will fill a void in the literature.

CONFLICT OF INTEREST

The author(s) stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Emre Öztürk: Formal analysis, Writing - original draft, Conceptualization, Investigation, Supervision, Visualization.

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