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On Suzuki$-$Proinov Type Contractions in Modular $b-$Metric Spaces with an Application

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Abstract
In this paper, by taking $\mathscr{C}_{\mathscr{A}}-$simulation function and Proinov type function into account, we set up a new contraction mapping called Suzuki$-$Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction, including both rational expressions that possess quadratic terms and $\mathcal{E}-$type contractions. Furthermore, we demonstrate a common fixed point theorem through the mappings endowed with triangular $\alpha-$admissibility in the setting of modular $b-$metric spaces. Besides that, we achieve some new outcomes that contribute to the current ones in the literature through the main theorem, and, as an application, we examine the existence of solutions to a class of functional equations emerging in dynamic programming.

Keywords: Common fixed point, Dynamic programming, Modular $b-$metric space, Proinov type mappings, Simulation functions
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1. Introduction and Preliminaries

The symbol $\mathbb{N}$ is used throughout the research to represent all positive natural numbers, whereas $\mathbb{R}^+$ represents the set of all non-negative real numbers.

Fixed point theory is a significant mathematical technique that finds applications in various scientific research areas. This theory has played a crucial role in creating several significant concepts and approaches and is an exciting area of ongoing study and advancement, which acts as an intermediary connecting topology and analysis and is commonly used in pure and applied mathematics. For the past several years, researchers in this field have been exploring potential applications of this field to a wide range of physically relevant engineering challenges. On the other hand, the metric fixed point theory is very attractive on account of the Banach Fixed Point Theorem or Banach Contraction Principle, which was conferred by S. Banach [1] in 1922. In this theorem, there is an answer about the existence and uniqueness of fixed point of contraction mappings in the setting of complete metric space. Further, many studies have been done to enhance this theorem's impressiveness, and it underwent several changes and generalizations as time progressed, see [2]-[5]. Simultaneously, in this direction, many authors try to obtain a more general metric space structure and diverse contractive conditions or both of them. Herewith, many new topological structures and contraction mappings have emerged. The notation of b-metric is one of the popular generalizations of the metric function, which was depicted by Bakhtin [6] and mainly, Czerwik [7, 8] in 1993 and 1998, as noted below.

Definition 1.1. [7] *A function $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ is a b−metric with $\tau \geq 1$ on a non−empty set $\mathcal{U}$ provided that the following axioms hold, for all $\lambda, \zeta, z \in \mathcal{U}$:*

(ρ_1) $\rho(\lambda, \zeta) = 0 \Leftrightarrow \lambda = \zeta$,

(ρ_2) $\rho(\lambda, \zeta) = \rho(\zeta, \lambda)$,

(ρ_3) $\rho(\lambda, \zeta) \leq \tau[\rho(\lambda, z) + \rho(z, \zeta)]$.

Thereupon, we say that the pair $(\mathcal{U}, \rho)$ is a b−metric space, and, by choosing $\tau = 1$, b−metric is reduced to ordinary metric.

Also, except for the continuity, other topological features of b−metric can be defined as in metric ones. For continuity, the subsequent lemma can be a guide in b−metric.

Lemma 1.2. [9] *Let $(\mathcal{U}, \rho)$ be a b−metric space with $\tau \geq 1$ and $\{\lambda_\eta\}$ and $\{\zeta_\eta\}$ be convergent to λ and ζ, respectively. Then*

$$\frac{1}{\tau^2}\rho(\lambda, \zeta) \leq \liminf_{\eta \to \infty} \rho(\lambda_\eta, \zeta_\eta) \leq \limsup_{\eta \to \infty} \rho(\lambda_\eta, \zeta_\eta) \leq \tau^2 \rho(\lambda, \zeta).$$

Especially, if $\lambda = \zeta$, then $\lim_{\eta \to \infty} \rho(\lambda_\eta, \zeta_\eta) = 0$. Also, for $z \in \mathcal{U}$, we have

$$\frac{1}{\tau}\rho(\lambda, z) \leq \liminf_{\eta \to \infty} \rho(\lambda_\eta, z) \leq \limsup_{\eta \to \infty} \rho(\lambda_\eta, z) \leq \tau \rho(\lambda, z).$$

On the other hand, in 2010, Chistyakov [10, 11] put forth a novel concept which is known as modular metric space.

Definition 1.3. [10, 11] *A function $\mu : (0, \infty) \times \mathcal{U} \times \mathcal{U} \to [0, \infty]$, defined by $\mu(\sigma, \lambda, \zeta) = \mu_\sigma(\lambda, \zeta)$, is called a modular metric on a non-void set $\mathcal{U}$ if it satisfies the below statements for all $\lambda, \zeta, z \in \mathcal{U}$:*

(μ_1) $\mu_\sigma(\lambda, \zeta) = 0$ *for all* $\sigma > 0 \Leftrightarrow \lambda = \zeta$,

(μ_2) $\mu_\sigma(\lambda, \zeta) = \mu_\sigma(\zeta, \lambda)$ *for all* $\sigma > 0$,

(μ_3) $\mu_{\sigma+\chi}(\lambda, \zeta) \leq \mu_\sigma(\lambda, z) + \mu_\chi(z, \zeta)$ *for all* $\sigma, \chi > 0$.

If instead of (μ_1), the condition

(μ_1') $\mu_\sigma(\lambda, \lambda) = 0$ for all $\sigma > 0$

is fulfilled, then μ is said to be a (metric) pseudomodular on $\mathcal{U}$.

By using the constant $\tau \geq 1$, the axiom (μ_3) is revised with the following one by M. E. Ege and C. Alaca [12], and in this case, the function μ is entitled as modular b−metric:

(μ_3') $\mu_{\sigma+\chi}(\lambda, \zeta) \leq \tau\left[\mu_\sigma(\lambda, z) + \mu_\chi(z, \zeta)\right]$ for all $\sigma, \chi > 0$.

Consequently, the pair $(\mathcal{U}, \mu)$ is a modular b−metric space, which denotes $\mathcal{M}_b\mathcal{MS}$.

Note that the notation of modular b−metric and modular metric coincide when $\tau = 1$. Also, considering modular b−metric μ on $\mathcal{U}$, a modular set is specified by

$$\mathcal{U}_\mu = \left\{\zeta \in \mathcal{U} : \zeta \overset{\mu}{\sim} \lambda\right\},$$

where $\overset{\mu}{\sim}$ is a binary relation on $\mathcal{U}$ identified by $\lambda \sim \zeta \Leftrightarrow \lim_{\sigma \to \infty} \mu_\sigma(\lambda, \zeta) = 0$ for $\lambda, \zeta \in \mathcal{U}$. Moreover, the set

$$\mathcal{U}_\mu^* = \{\lambda \in \mathcal{U} : \exists \sigma = \sigma(\lambda) > 0 \text{ such that } \mu_\sigma(\lambda, \lambda_0) < \infty\} \ (\lambda_0 \in \mathcal{U})$$

is mentioned as $\mathcal{M}_b\mathcal{MS}$ (around λ_0).

Example 1.4. [12] *Consider the space*

$$\ell_p = \left\{ (\lambda_\eta) \subset \mathbb{R} : \sum_{j=1}^{\infty} |\lambda_\eta|^p < \infty \right\} \quad 0 < p < 1,$$

$\sigma \in (0,\infty)$ *and* $\mu_\sigma(\lambda,\zeta) = \frac{d(\lambda,\zeta)}{\sigma}$ *such that*

$$d(\lambda,\zeta) = \left(\sum_{j=1}^{\infty} |\lambda_\eta - \zeta_\eta|^p \right)^{\frac{1}{p}}, \quad \lambda = \lambda_\eta,\ \zeta = \zeta_\eta \in \ell_p.$$

Eventually, one can conclude that $(\mathcal{U},\mu)$ *is an* $\mathcal{M}_b\mathcal{MS}$.

Example 1.5. [13] *Consider the equality* $\mu_\sigma(\lambda,\zeta) = (\omega_\sigma(\lambda,\zeta))^s$, *where* $(\mathcal{U},\omega)$ *is a modular metric space and* $s \geq 1$. *Thereupon, take into Jensen inequality account, together with the convexity of the function* $\mathcal{P}(\lambda) = \lambda^s$ *for* $\lambda \geq 0$, *we get*

$$(a+b)^s \leq 2^{s-1}(a^s + b^s)$$

for $a, b \in \mathbb{R}^+$. *Hence,* $(\mathcal{U},\mu)$ *is an* $\mathcal{M}_b\mathcal{MS}$ *with* $\tau = 2^{s-1}$.

Definition 1.6. [12] *Let* $\mathcal{U}_\mu^*$ *be an* $\mathcal{M}_b\mathcal{MS}$ *and* $\{\lambda_\eta\}_{\eta\in\mathbb{N}} \in \mathcal{U}_\mu^*$ *be a sequence.*

(c_1) *The sequence* $\{\lambda_\eta\}_{\eta\in\mathbb{N}}$ *is* $\mu-$*convergent to* $\lambda \in \mathcal{U}_\mu^* \Leftrightarrow \mu_\sigma(\lambda_\eta,\lambda) \to 0$, *as* $\eta \to \infty$ *for all* $\sigma > 0$.

(c_2) *The* $\{\lambda_\eta\}_{\eta\in\mathbb{N}}$ *in* $\mathcal{U}_\mu^*$ *is a* $\mu-$*Cauchy sequence if* $\lim_{\eta,m\to\infty} \mu_\sigma(\lambda_\eta,\lambda_m) = 0$ *for all* $\sigma > 0$.

(c_3) *The space* $\mathcal{U}_\mu^*$ *is called* $\mu-$*complete provided that any* $\mu-$*Cauchy sequence in* $\mathcal{U}_\mu^*$ *is* $\mu-$*convergent to the point of* $\mathcal{U}_\mu^*$.

(c_4) $\mathcal{P} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ *is a* $\mu-$*continuous mapping if* $\mu_\sigma(\lambda_\eta,\lambda) \to 0$, *provided to* $\mu_\sigma(\mathcal{P}\lambda_\eta,\mathcal{P}\lambda) \to 0$ *as* $\eta \to \infty$.

Further, for more detail on modular $b-$metric, see [14]-[17].

As an auxiliary function, the class of simulation functions (briefly, $\mathcal{SF}$) was identified by Khojasteh et al. [18] in 2015, as noted below.

Definition 1.7. [18] *Let* $\Xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *be a mapping. If the axioms*

(Ξ_1) $\Xi(0,0) = 0$,

(Ξ_2) $\Xi(\ell,k) < k - \ell$ *for all* $\ell, k > 0$,

(Ξ_3) *if* $\{\ell_\eta\}$, $\{k_\eta\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{\eta\to\infty} \ell_\eta = \lim_{\eta\to\infty} k_\eta > 0$, *then* $\limsup_{\eta\to\infty} \Xi(\ell_\eta,k_\eta) < 0$

are fulfilled, then, Ξ is an $\mathcal{SF}$, and $\mathcal{Z}$ represents the set of all $\mathcal{SF}$. Also, note that, from (Ξ_2), we have $\Xi(\ell,\ell) < 0$ for all $\ell > 0$.

Definition 1.8. [18] *A self-mapping* $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ *on a metric space* $(\mathcal{U},d)$ *is called* $\mathcal{Z}$*-contraction with respect to* $\Xi \in \mathcal{Z}$ *provided that, for all* $\lambda,\zeta \in \mathcal{U}$, *the subsequent inequality hold:*

$$\Xi(d(\mathcal{P}\lambda,\mathcal{P}\zeta), d(\lambda,\zeta)) \geq 0.$$

Moreover, Banach contraction mapping can be expressed via $\mathcal{SF}$ $\Xi \in \mathcal{Z}$ for which $\Xi(\ell,k) = \gamma k - \ell$ for all $\ell, k \in [0,\infty)$ and $\gamma \in [0,1)$.

The following expression was used for the first time by Fulga and Proca [19] in 2017 and subsequently referred to as $\mathcal{E}-$contraction or $\mathcal{E}$ type contraction:

$$\mathcal{E}(\lambda,\zeta) = d(\lambda,\zeta) + |d(\lambda,\mathcal{P}\lambda) - d(\zeta,\mathcal{P}\zeta)|, \tag{1.1}$$

whenever $(\mathcal{U},d)$ is a complete metric space and $\lambda,\zeta \in \mathcal{U}$. Also, some studies involve such contraction; see [20]-[22]. One of them was presented by A. Fulga and E. Karapınar [23] via $\mathcal{SF}$ in 2018, as indicated below:

Theorem 1.9. [23] *Let $\mathcal{P}$ be a self-mapping on a complete metric space $(\mathcal{U}, d)$. If there exists $\Xi \in \mathcal{Z}$ satisfying, for all $\lambda, \zeta \in \mathcal{U}$,*

$$\Xi\left(d\left(\mathcal{P}\lambda, \mathcal{P}\zeta\right), \mathcal{E}\left(\lambda, \zeta\right)\right) \geq 0,$$

where $\mathcal{E}(\lambda, \zeta)$ is defined as in (1.1), then $\mathcal{P}$ owns a fixed point.

In 2014, A.H. Ansari [24] proposed $C-$class functions as characterized in the subsequent definition.

Definition 1.10. [24] *A continuous function $\mathscr{A} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is entitled $C-$class function if, for all $\ell, k \in [0,\infty)$, the below statements hold:*

$(\mathscr{A}_1)$ $\mathscr{A}(\ell, k) \leq \ell$;

$(\mathscr{A}_2)$ $\mathscr{A}(\ell, k) = \ell$ *implies that either* $\ell = 0$ *or* $k = 0$.

Let $C-$class functions symbolize as $\mathscr{C}$.

In 2018, Radenovic et al. [25] identified the idea of $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ by means of the $C-$class functions and $\mathcal{SF}$.

Definition 1.11. [25] *A mapping $\Omega : [0,\infty)^2 \to \mathbb{R}$ is referred to as $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ if the conditions*

(Ω_1) $\Omega(\ell, k) \leq \mathscr{A}(k, \ell)$ *for all* $\ell, k > 0$, *where* $\mathscr{A} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is a $C-$class functions,*

(Ω_2) *if* $\{\ell_\mathfrak{y}\}, \{k_\mathfrak{y}\} \in (0,\infty)$ *are sequences such that* $\lim_{\mathfrak{y}\to\infty} \ell_\mathfrak{y} = \lim_{\mathfrak{y}\to\infty} k_\mathfrak{y} > 0$ *and* $k_\mathfrak{y} < \ell_\mathfrak{y}$, *then* $\limsup_{\mathfrak{y}\to\infty} \Omega(\ell_\mathfrak{y}, k_\mathfrak{y}) < \mathscr{C}_{\mathscr{A}}$

are provided.

Presume that $\mathscr{Z}^*$ symbolizes the family of all $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$.

Definition 1.12. *[25] A mapping $\mathscr{A} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ has the property $\mathscr{C}_{\mathscr{A}}$, if $\mathscr{C}_{\mathscr{A}} \geq 0$ exists such that*

(1) $\mathscr{A}(\ell, k) > \mathscr{C}_{\mathscr{A}}$ *implies* $\ell > k$,

(2) $\mathscr{A}(\ell, \ell) \leq \mathscr{C}_{\mathscr{A}}$ *for all* $\ell \in [0,\infty)$.

The following theorem has a new precondition added to a contractive mapping and was proved by Suzuki [26] in 2009. Herewith, many authors have mentioned this notation as a Suzuki-type contraction.

Theorem 1.13. [26] *Let $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ be a self-mapping on a compact metric space $(\mathcal{U}, d)$. If, for all distinc $\lambda, \zeta \in \mathcal{U}$, the statement*

$$\frac{1}{2} d\left(\lambda, \mathcal{P}\lambda\right) < d\left(\lambda, \zeta\right) \Rightarrow d\left(\mathcal{P}\lambda, \mathcal{P}\zeta\right) < d\left(\lambda, \zeta\right)$$

is hold, then, $\mathcal{P}$ owns a unique fixed point.

Very recently, Proinov [27] demonstrated a novel fixed point theorem by introducing some auxiliary functions, and subsequently, via this theorem, many significant results were obtained.

Definition 1.14. [27] *Let $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ be a self mapping on a metric space $(\mathcal{U}, d)$ and $\mathcal{F}, \mathcal{Q} : (0,\infty) \to \mathbb{R}$ are two functions that provide the following features:*

(i) $\mathcal{F}$ *is non-decreasing,*

(ii) $\mathcal{Q}(s) < \mathcal{F}(s)$ *for all* $s > 0$,

(iii) $\limsup_{s\to s_0+} \mathcal{Q}(s) < \mathcal{F}(s_0+)$ *for any* $s_0 > 0$.

If, for all $\lambda, \zeta \in \mathcal{U}$ and $d(\mathcal{P}\lambda, \mathcal{P}\zeta) > 0$, the inequality

$$\mathcal{F}\left(d\left(\mathcal{P}\lambda, \mathcal{P}\zeta\right)\right) \leq \mathcal{Q}\left(d\left(\lambda, \zeta\right)\right)$$

is fulfilled, then $\mathcal{P}$ is called Proinov type contraction.

Theorem 1.15. [27] *Let $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ be a Proinov type contraction on a complete metric space $(\mathcal{U}, m)$. Then, $\mathcal{P}$ admits a unique fixed point.*

Various fixed point results involving Proinov type contraction appear in the literature. Some examples are in [28]-[35].

In 2012, Samet et al. [36] introduced the class of $\alpha-$admissible mappings, and subsequently, many new notations appear via this mapping.

Definition 1.16. *Let $\mathcal{P}, \mathcal{S} : \mathcal{U} \to \mathcal{U}$ be two mappings and $\alpha : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be a function. Then, we have the following ideas.*

(α_1) [36] *If* $\alpha(\lambda, \zeta) \geq 1$ implies $\alpha(\mathcal{P}\lambda, \mathcal{P}\zeta) \geq 1$, *then $\mathcal{P}$ is $\alpha-$admissible,*

(α_2) [37] *if* $\alpha(\lambda, \mathcal{P}\lambda) \geq 1$ implies $\alpha\left(\mathcal{P}\lambda, \mathcal{P}^2\zeta\right) \geq 1$, *then, $\mathcal{P}$ is $\alpha-$orbital admissible,*

(α_3) [37] *together with (α_2), if* $\alpha(\lambda, \zeta) \geq 1$ and $\alpha(\zeta, \mathcal{P}\zeta) \geq 1$ imply $\alpha(\lambda, \mathcal{P}\zeta) \geq 1$, *then, $\mathcal{P}$ is triangular $\alpha-$orbital admissible,*

(α_4) [38] *together with (α_1), if* $\alpha(\lambda, z) \geq 1$ and $\alpha(z, \zeta) \geq 1$ imply $\alpha(\lambda, \zeta) \geq 1$, *then $\mathcal{P}$ is triangular $\alpha-$ admissible,*

(α_5) [39] *together with (α_4), if* $\alpha(\lambda, \zeta) \geq 1$ implies $\alpha(\mathcal{P}\lambda, \mathcal{S}\zeta) \geq 1$ and $\alpha(\mathcal{S}\mathcal{P}\lambda, \mathcal{P}\mathcal{S}\zeta) \geq 1$, *then, the pair $(\mathcal{P}, \mathcal{S})$ is triangular $\alpha-$admissible.*

Lemma 1.17. [37] *Let $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ be a triangular $\alpha-$orbital admissible mapping. Assume that a $\lambda_0 \in \mathcal{U}$ exists such that $\alpha(\lambda_0, \mathcal{P}\lambda_0) \geq 1$. Construct a sequence $\{\lambda_\mathfrak{y}\}$ by $\lambda_{\mathfrak{y}+1} = \mathcal{P}\lambda_\mathfrak{y}$. Then we have $\alpha(\lambda_\mathfrak{y}, \lambda_m) \geq 1$ for all $\mathfrak{y}, m \in \mathbb{N}$ with $\mathfrak{y} < m$.*

2. Main Results

Primarily, it is necessary to mention the below conditions to guarantee the existence and uniqueness of fixed points in $\mathcal{M}_\flat\mathcal{MS}$ owing to not having to be finite.

$(\mathfrak{C}_1)$ $\mu_\sigma(\lambda, \mathcal{P}\lambda) < \infty$ for all $\sigma > 0$ and $\lambda \in \mathcal{U}^*_\mu$,

$(\mathfrak{C}_2)$ $\mu_\sigma(\lambda, \zeta) < \infty$ for all $\sigma > 0$ and $\lambda, \zeta \in \mathcal{U}^*_\mu$.

Next, we establish a new contraction mapping by defining Suzuki−Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction w.r.t Ω in the sense of $\mathcal{M}_\flat\mathcal{MS}$, as follows.

Definition 2.1. *Let $\mathcal{U}^*_\mu$ be an $\mathcal{M}_\flat\mathcal{MS}$ with constant $\tau \geq 1$ and let $\mathcal{P}, \mathcal{S} : \mathcal{U}^*_\mu \to \mathcal{U}^*_\mu$ and $\alpha : \mathcal{U}^*_\mu \times \mathcal{U}^*_\mu \to \mathbb{R}$ be mappings. Then, we say that $\mathcal{P}$ and $\mathcal{S}$ are Suzuki−Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction if there exists a $\mathscr{C}_\mathscr{A}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that*

$$\frac{1}{2\tau}\min\{\mu_\sigma(\lambda, \mathcal{P}\lambda), \mu_\sigma(\zeta, \mathcal{S}\zeta)\} \leq \mu_\sigma(\lambda, \zeta)$$

implies

$$\Omega\left(\alpha(\lambda, \zeta)\,\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda, \mathcal{S}\zeta)^2\right), \mathcal{Q}(\mathcal{E}^*(\lambda, \zeta)\,\mathcal{R}(\lambda, \zeta))\right) \geq \mathscr{C}_\mathscr{A}, \tag{2.1}$$

where the functions $\mathcal{F}, \mathcal{Q} : (0, \infty) \to \mathbb{R}$ are hold the below requirement:

(c_1) *$\mathcal{F}$ is lower semi-continuous and non-decreasing;*

(c_2) *$\mathcal{Q}(s) < \mathcal{F}(s)$ for all $s > 0$;*

(c_3) *$\limsup\limits_{s \to s_0+} \mathcal{Q}(s) < \mathcal{F}(s_0+)$ for any $s_0 > 0$,*

and also,

$$\mathcal{E}^*(\lambda, \zeta) = \mu_\sigma(\lambda, \zeta) + |\mu_\sigma(\lambda, \mathcal{P}\lambda) - \mu_\sigma(\zeta, \mathcal{S}\zeta)|$$

and

$$\mathcal{R}(\lambda, \zeta) = \frac{\mu_\sigma(\lambda, \mathcal{P}\lambda)\mu_\sigma(\lambda, \mathcal{S}\zeta) + [\mu_\sigma(\lambda, \zeta)]^2 + \mu_\sigma(\lambda, \mathcal{P}\lambda)\mu_\sigma(\lambda, \zeta)}{\mu_\sigma(\lambda, \mathcal{P}\lambda) + \mu_\sigma(\lambda, \zeta) + \mu_\sigma(\lambda, \mathcal{S}\zeta)}$$

for all distinct $\lambda, \zeta \in \mathcal{U}^*_\mu$, $\mu_\sigma(\mathcal{P}\lambda, \mathcal{S}\zeta) > 0$ and for all $\sigma > 0$.

Theorem 2.2. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$ with constant $\tau \geq 1$ and $\mathcal{P}$ and $\mathcal{S}$ be a Suzuki−Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction w.r.t. Ω. Assume that the following conditions hold:*

(i) the pair $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible,

(ii) there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0, \mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P},\mathcal{S}$ are $\mu-$continuous,

(iv) there exists $\lambda,\zeta \in C_{Fix}(\mathcal{P},\mathcal{S})$, where $C_{Fix}(\mathcal{P},\mathcal{S})$ represents set of common fixed points of $\mathcal{P}$ and $\mathcal{S}$, such that $\alpha(\lambda,\zeta) \geq 1$.

In case of satisfying $(\mathfrak{C}_1)$, there there exists $\lambda^* \in \mathcal{U}_\mu^*$ such that $\lambda^* \in C_{Fix}(\mathcal{P},\mathcal{S})$. Also, additionally, if $(\mathfrak{C}_2)$ is hold, then $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$.

Proof. Let $\lambda_0 \in \mathcal{U}_\mu^*$ be a specified point such that $\alpha(\lambda_0, \mathcal{P}\lambda_0) \geq 1$. Construct an iterative sequence $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ in $\mathcal{U}_\mu{}^*$ such that

$$\lambda_{2\mathfrak{y}+1} = \mathcal{P}\lambda_{2\mathfrak{y}} \quad \text{and} \quad \lambda_{2\mathfrak{y}+2} = \mathcal{S}\lambda_{2\mathfrak{y}+1}, \quad \text{for all } \mathfrak{y} \in \mathbb{N}.$$

On the other hand, regarding that $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible, we derive

$$\alpha(\lambda_0,\lambda_1) = \alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1 \ \Rightarrow \begin{cases} \alpha(\mathcal{P}\lambda_0, \mathcal{S}\lambda_1) = \alpha(\lambda_1,\lambda_2) \geq 1 \\ \text{and} \\ \alpha(\mathcal{S}\mathcal{P}\lambda_0, \mathcal{P}\mathcal{S}\lambda_1) = \alpha(\mathcal{S}\lambda_1, \mathcal{P}\lambda_2) = \alpha(\lambda_2,\lambda_3) \geq 1. \end{cases}$$

Likewise, we get

$$\alpha(\lambda_2,\lambda_3) \geq 1 \ \Rightarrow \begin{cases} \alpha(\mathcal{P}\lambda_2, \mathcal{S}\lambda_3) = \alpha(\lambda_3,\lambda_4) \geq 1 \\ \text{and} \\ \alpha(\mathcal{S}\mathcal{P}\lambda_2, \mathcal{P}\mathcal{S}\lambda_3) = \alpha(\mathcal{S}\lambda_3, \mathcal{P}\lambda_4) = \alpha(\lambda_2,\lambda_3) \geq 1. \end{cases}$$

Thereby, recursively, we conclude that

$$\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) \geq 1. \tag{2.2}$$

Also, if there is some $\mathfrak{y}_0 \in \mathbb{N}$ such that $\lambda_{\mathfrak{y}_0} = \lambda_{\mathfrak{y}_0+1}$, then $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\mathfrak{y}_0\}$. Thereupon, we presume that $\lambda_k \neq \lambda_{k+1}$ for all $k \in \mathbb{N}$, which indicates that $\mu_\sigma(\lambda_k,\lambda_{k+1}) > 0$ for all $\sigma > 0$. Next, we assume that $k = 2\mathfrak{y}$ for some $\mathfrak{y} \in \mathbb{N}$. Because

$$\begin{aligned} \tfrac{1}{2\tau}\min\{\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}}), \mu_\sigma(\lambda_{2\mathfrak{y}+1},\mathcal{S}\lambda_{2\mathfrak{y}+1})\} &= \tfrac{1}{2\tau}\min\{\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}), \mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})\} \\ &\leq \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}), \end{aligned}$$

from (2.1) and (Θ_1), we have

$$\begin{aligned} \mathscr{C}_\mathscr{A} &\leq \Omega\left(\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda_{2\mathfrak{y}},\mathcal{S}\lambda_{2\mathfrak{y}+1})^2\right), \mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}))\right) \\ &= \Omega\left(\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right), \mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}))\right) \\ &< \mathscr{A}\left(\mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})), \alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right)\right), \end{aligned}$$

and by (c_2), (2.2) and the properties $\mathscr{C}_\mathscr{A}$, we yield

$$\begin{aligned} \mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right) \leq \alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right) &< \mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})) \\ &< \mathcal{F}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) &= \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) + |\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}}) - \mu_\sigma(\lambda_{2\mathfrak{y}+1},\mathcal{S}\lambda_{2\mathfrak{y}+1})| \\ &= \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) + |\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) - \mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})| \end{aligned}$$

and

$$\begin{aligned}\mathcal{R}\left(\lambda_{2\eta},\lambda_{2\eta+1}\right) &= \frac{\mu_\sigma\left(\lambda_{2\eta},\mathcal{P}\lambda_{2\eta}\right)\mu_\sigma\left(\lambda_{2\eta},\mathcal{S}\lambda_{2\eta+1}\right)+\left[\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)\right]^2+\mu_\sigma\left(\lambda_{2\eta},\mathcal{P}\lambda_{2\eta}\right)\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)}{\mu_\sigma\left(\lambda_{2\eta},\mathcal{P}\lambda_{2\eta}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)+\mu_\sigma\left(\lambda_{2\eta},\mathcal{S}\lambda_{2\eta+1}\right)}\\
&= \frac{\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+2}\right)+\left[\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)\right]^2+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)}{\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+2}\right)}\\
&= \frac{\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)\left[\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+2}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)\right]}{\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)+\mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+2}\right)}\\
&= \mu_\sigma\left(\lambda_{2\eta},\lambda_{2\eta+1}\right).\end{aligned}$$

Denote $\mu_\sigma\left(\lambda_\eta,\lambda_{\eta+1}\right)$ by κ_η. Now, if $\max\{\kappa_{2\eta},\kappa_{2\eta+1}\}=\kappa_{2\eta+1}$, then, we get $\mathcal{E}^*\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)=\kappa_{2\eta+1}$ and $\mathcal{R}\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)=\kappa_{2\eta}$. Thereupon, (2.3) turns into

$$\mathcal{F}\left(\kappa^2_{2\eta+1}\right)\le\mathcal{F}\left(\tau^6\kappa^2_{2\eta+1}\right)<\mathcal{Q}\left(\kappa_{2\eta+1}.\kappa_{2\eta}\right)<\mathcal{F}\left(\kappa_{2\eta+1}.\kappa_{2\eta}\right),$$

such that, by utilizing the function $\mathcal{F}$'s characteristics, we conclude that $\kappa_{2\eta+1}<\kappa_{2\eta}$. However, this contradicts our assumptions. Thereby, we achieve $\max\{\kappa_{2\eta},\kappa_{2\eta+1}\}=\kappa_{2\eta}$, which implies that $\mathcal{E}^*\left(\lambda_{2\eta},\lambda_{2\eta+1}\right)=2\kappa_{2\eta}-\kappa_{2\eta+1}$. Then, (2.3) becomes

$$\mathcal{F}\left(\kappa^2_{2\eta+1}\right)\le\mathcal{F}\left(\tau^6\kappa^2_{2\eta+1}\right)<\mathcal{Q}\left(\left(2\kappa_{2\eta}-\kappa_{2\eta+1}.\right)\kappa_{2\eta}\right)<\mathcal{F}\left(\left(2\kappa_{2\eta}-\kappa_{2\eta+1}.\right)\kappa_{2\eta}\right), \tag{2.4}$$

by (c_1), we obtain that

$$\begin{aligned}\kappa^2_{2\eta+1}<\left(2\kappa_{2\eta}-\kappa_{2\eta+1}.\right)\kappa_{2\eta} &\Leftrightarrow \kappa^2_{2\eta+1}<2\kappa^2_{2\eta}-\kappa_{2\eta}\kappa_{2\eta+1}<2\kappa^2_{2\eta}-\kappa^2_{2\eta+1}\\
&\Leftrightarrow 2\kappa^2_{2\eta+1}<2\kappa^2_{2\eta}\\
&\Leftrightarrow \kappa_{2\eta+1}<\kappa_{2\eta}.\end{aligned}$$

Likewise, one concludes that $\kappa_{2\eta}<\kappa_{2\eta-1}$. So, we say that that $\{\kappa_\eta\}_{\eta\in\mathbb{N}}=\{\mu_\sigma\left(\lambda_\eta,\lambda_{\eta+1}\right)\}_{\eta\in\mathbb{N}}$ is a non-increasing sequence of non-negative real numbers. Also, a similar consequence can be obtained when k is an odd number. Then, there exists $p\ge 0$ such that $\lim_{\eta\to\infty}\kappa_\eta=p$. Assume on the contrary, we aim to show that $p>0$. Then, by (2.4), we have

$$\mathcal{F}\left(p^2\right)\le\lim_{\eta\to\infty}\mathcal{F}\left(\kappa^2_{2\eta+1}\right)<\lim_{\eta\to\infty}\mathcal{Q}\left(\left(2\kappa_{2\eta}-\kappa_{2\eta+1}.\right)\kappa_{2\eta}\right)<\lim_{\eta\to\infty}\mathcal{F}\left(\left(2\kappa_{2\eta}-\kappa_{2\eta+1}.\right)\kappa_{2\eta}\right)=\mathcal{F}\left(p^2\right),$$

which emerges a contradiction, which means that, for all $\sigma>0$,

$$\lim_{\eta\to\infty}\mu_\sigma\left(\lambda_\eta,\lambda_{\eta+1}\right)=0. \tag{2.5}$$

Now, it is required to indicate $\{\lambda_\eta\}_{\eta\in\mathbb{N}}$ is a $\mu-$Cauchy sequence. Rather, presume that $\{\lambda_\eta\}_{\eta\in\mathbb{N}}$ is not a $\mu-$Cauchy sequence. Then, for at least a $\varepsilon>0$ and $\eta_\hbar>m_\hbar>\hbar$ whenever $\hbar\in\mathbb{N}\cup\{0\}$ and let $\eta_\hbar$ be the smallest index such that the following expressions are provided:

$$\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\eta_\hbar}\right)\ge\varepsilon \quad\text{and}\quad \mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\eta_\hbar-2}\right)<\varepsilon,\quad \text{for all } \sigma>0. \tag{2.6}$$

By using (2.5), (2.6) and (μ_3'), we yield

$$\begin{aligned}\varepsilon\le\mu_{4\sigma}\left(\lambda_{2m_\hbar},\lambda_{2\eta_\hbar}\right)&\le\tau\mu_{2\sigma}\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)+\tau^2\mu_\sigma\left(\lambda_{2m_\hbar+1},\lambda_{2\eta_\hbar+2}\right)\\
&+\tau^3\mu_{\sigma/2}\left(\lambda_{2\eta_\hbar+2},\lambda_{2\eta_\hbar+1}\right)+\tau^3\mu_{\sigma/2}\left(\lambda_{2\eta_\hbar+1},\lambda_{2\eta_\hbar}\right)\end{aligned}$$

such that

$$\limsup_{\hbar\to\infty}\mu_\sigma\left(\lambda_{2m_\hbar+1},\lambda_{2\eta_\hbar+2}\right)\ge\frac{\varepsilon}{\tau^2}. \tag{2.7}$$

Also, we get

$$\begin{aligned}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) \le \tau\mu_{\sigma/2}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar-2}\right)+\tau^2\mu_{\sigma/4}\left(\lambda_{2\mathfrak{y}_\hbar-2},\lambda_{2\mathfrak{y}_\hbar-1}\right)\\ +\tau^3\mu_{\sigma/8}\left(\lambda_{2\mathfrak{y}_\hbar-1},\lambda_{2\mathfrak{y}_\hbar}\right)+\tau^3\mu_{\sigma/8}\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right).\end{aligned} \tag{2.8}$$

Thereby, by taking the limit superior in (2.8) and using (2.5), we obtain that

$$\limsup_{\hbar\to\infty}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) \le \tau\varepsilon. \tag{2.9}$$

Also, from the (2.5) and (2.6), we achieve that

$$\begin{aligned}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right) \le \tau\mu_{\sigma/2}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar-2}\right)+\tau^2\mu_{\sigma/4}\left(\lambda_{2\mathfrak{y}_\hbar-2},\lambda_{2\mathfrak{y}_\hbar-1}\right)\\ +\tau^3\mu_{\sigma/8}\left(\lambda_{2\mathfrak{y}_\hbar-1},\lambda_{2\mathfrak{y}_\hbar}\right)+\tau^4\mu_{\sigma/16}\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\tau^4\mu_{\sigma/16}\left(\lambda_{2\mathfrak{y}_\hbar+1},\lambda_{2\mathfrak{y}_\hbar+2}\right)\end{aligned}$$

and letting $\hbar\to\infty$, we attain

$$\limsup_{\hbar\to\infty}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right) \le \tau\varepsilon. \tag{2.10}$$

Furthermore, if $\mathfrak{y}_\hbar > m_\hbar > \hbar$ for sufficiently large $\hbar\in\mathbb{N}$, we assert

$$\frac{1}{2\tau}\min\left\{\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right),\mu_\sigma\left(\lambda_{2m_\hbar-1},\mathcal{S}\lambda_{2m_\hbar-1}\right)\right\} \le \mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right). \tag{2.11}$$

Given the fact that, $\mathfrak{y}_\hbar > m_\hbar$ and $\{\mu_\sigma(\lambda_\mathfrak{y},\lambda_{\mathfrak{y}+1})\}_{\mathfrak{y}\ge1}$ is non-decreasing, we acquire

$$\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right)=\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\le\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)\le\mu_\sigma\left(\lambda_{2m_\hbar-1},\lambda_{2m_\hbar}\right)=\mu_\sigma\left(\lambda_{2m_\hbar-1},\mathcal{S}\lambda_{2m_\hbar-1}\right).$$

Hence,

$$\frac{1}{2\tau}\min\left\{\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right),\mu_\sigma\left(\lambda_{2m_\hbar-1},\mathcal{S}\lambda_{2m_\hbar-1}\right)\right\}=\frac{1}{2\tau}\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right)=\frac{1}{2\tau}\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right).$$

According to (2.5), there exists $\hbar_1\in\mathbb{N}$ such that for any $\hbar>\hbar_1$,

$$\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)<\frac{\varepsilon}{2\tau}.$$

Also, there exists $\hbar_2\in\mathbb{N}$ such that for any $\hbar>\hbar_2$,

$$\mu_\sigma\left(\lambda_{2m_\hbar-1},\lambda_{2m_\hbar}\right)<\frac{\varepsilon}{2\tau}.$$

Hence, for any $\hbar>\max\{\hbar_1,\hbar_2\}$ and $\mathfrak{y}_\hbar>m_\hbar>\hbar$, we get

$$\varepsilon\le\mu_{2\sigma}\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar}\right)\le\tau\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right)+\tau\mu_\sigma\left(\lambda_{2m_\hbar-1},\lambda_{2m_\hbar}\right)\le\tau\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right)+\tau\frac{\varepsilon}{2\tau}.$$

So, one can conclude that

$$\frac{\varepsilon}{2\tau}\le\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right).$$

Thus, we deduce that for any $\hbar>\max\{\hbar_1,\hbar_2\}$ and $\mathfrak{y}_\hbar>m_\hbar>\hbar$,

$$\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)<\frac{\varepsilon}{2\tau}\le\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right)$$

which implies that (2.11) is hold. Also, by using that $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible pair, we can write $\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\ge1$. Therefore, from (2.1), we conclude that

$$\begin{aligned}\mathscr{C}_\mathscr{A} &\le \Omega\left(\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\right)\\ &=\Omega\left(\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\right)\\ &<\mathscr{A}\left(\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right),\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right)\right),\end{aligned}$$

and by the properties of $\mathscr{C}_{\mathscr{A}}$ and (c_2), we deduce that

$$\begin{aligned}\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right) &\leq \alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right)\\ &< \mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\\ &< \mathcal{F}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right),\end{aligned}\tag{2.12}$$

where

$$\begin{aligned}\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) &= \mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\left|\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)-\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar+1},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)\right|\\ &= \mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\left|\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)-\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar+1},\lambda_{2\mathfrak{y}_\hbar+2}\right)\right|\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) &= \frac{\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)+\left[\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right]^2+\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)}{\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)}\\ &= \frac{\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right)+\left[\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right]^2+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)}{\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right)}.\end{aligned}\tag{2.14}$$

Next, letting $\hbar\to\infty$ in (2.12), (2.13) and (2.14), and also, by using (2.5), (2.7), (2.9) and (2.10), we acquire that

$$\begin{aligned}\mathcal{F}\left(\tau^2\varepsilon^2\right)=\mathcal{F}\left(\tau^6\left(\frac{\varepsilon}{\tau^2}\right)^2\right) &\leq \limsup_{\mathfrak{y}\to\infty}\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right)\\ &< \limsup_{\mathfrak{y}\to\infty}\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\\ &< \limsup_{\mathfrak{y}\to\infty}\mathcal{F}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\\ &\leq \mathcal{F}\left(\tau\varepsilon.\frac{(\tau\varepsilon)^2}{\tau\varepsilon+\tau\varepsilon}\right)=\mathcal{F}\left(\frac{\tau^2\varepsilon^2}{2}\right).\end{aligned}$$

This causes a contradictory, that is, $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ is a $\mu-$Cauchy sequence on a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$. Thereby, a point λ^* exists in $\mathcal{U}_\mu^*$ such that

$$\lim_{\mathfrak{y}\to\infty}\lambda_\mathfrak{y}=\lambda^*.\tag{2.15}$$

Considering the continuity of the mappings and (2.15), we get

$$\begin{aligned}\mathcal{P}\lambda^* &= \mathcal{P}\left(\lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}}\right)=\lim_{\mathfrak{y}\to\infty}\mathcal{P}\lambda_{2\mathfrak{y}}=\lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}+1}=\lambda^*\\ &= \lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}+2}=\lim_{\mathfrak{y}\to\infty}\mathcal{S}\lambda_{2\mathfrak{y}+1}\\ &= \mathcal{S}\left(\lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}+1}\right)=\mathcal{S}\lambda^*.\end{aligned}$$

Thereupon, we conclude that λ^* is a common fixed point of $\mathcal{P}$ and $\mathcal{S}$. Finally, we prove that the point λ^* is unique. For this, there is $\hat{\lambda}$, which is another common fixed point, such that $\lambda^*\neq\hat{\lambda}$. So, from the condition (iv), we deduce that $\alpha\left(\lambda^*,\hat{\lambda}\right)\geq 1$. Hence, since

$$0=\frac{1}{2\tau}\min\left\{\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right),\mu_\sigma\left(\hat{\lambda},\mathcal{S}\hat{\lambda}\right)\right\}\leq\mu_\sigma\left(\lambda^*,\hat{\lambda}\right),$$

by using (2.1) and (Θ_1), we gain

$$\begin{aligned}\mathscr{C}_{\mathscr{A}} &\leq \Omega\left(\alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda^*,\mathcal{S}\hat{\lambda}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right)\right)\\ &= \Omega\left(\alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right)\right)\\ &< \mathscr{A}\left(\mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right),\alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right)\right)\end{aligned}$$

and by Definition 1.12 and (c_2), we get

$$\begin{aligned}\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right) \leq \alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right) &< \mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right)\\ &< \mathcal{F}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right),\end{aligned} \tag{2.16}$$

where

$$\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right) = \mu_\sigma\left(\lambda^*,\hat{\lambda}\right) + \left|\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right) - \mu_\sigma\left(\hat{\lambda},\mathcal{S}\hat{\lambda}\right)\right| = \mu_\sigma\left(\lambda^*,\hat{\lambda}\right)$$

and

$$\mathcal{R}\left(\lambda^*,\hat{\lambda}\right) = \frac{\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right)\mu_\sigma\left(\lambda^*,\mathcal{S}\hat{\lambda}\right) + \left[\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)\right]^2 + \mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right)\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)}{\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right) + \mu_\sigma\left(\lambda^*,\hat{\lambda}\right) + \mu_\sigma\left(\lambda^*,\mathcal{S}\hat{\lambda}\right)} = \frac{\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)}{2}.$$

Consequently, considering the above equalities, the inequality (2.16) turns into

$$\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right) < \mathcal{F}\left(\mu_\sigma\left(\lambda^*,\hat{\lambda}\right).\frac{\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)}{2}\right) = \mathcal{F}\left(\frac{\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2}{2}\right),$$

which causes a contradiction. In turn, we achieve that $\lambda^* = \hat{\lambda}$, which means that $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$. This ends the proof. □

3. Consequences

In this part of the study, we discuss some of the implications of the fundamental observation. Primarily, if the restriction

$$\frac{1}{2\tau}\min\{\mu_\sigma(\lambda,\mathcal{P}\lambda),\mu_\sigma(\zeta,\mathcal{S}\zeta)\} \leq \mu_\sigma(\lambda,\zeta)$$

is ignored, Theorem 2.2 yields the subsequent consequence.

Corollary 3.1. *Let $\mathcal{U}_\mu^*$ be a μ−complete $\mathcal{M}_b\mathcal{MS}$ with $\tau \geq 1$, $\alpha : \mathcal{U}_\mu^* \times \mathcal{U}_\mu^* \to \mathbb{R}$ be a function and $\mathcal{P},\mathcal{S} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be two self-mappings. Assume that the following assertions are true:*

(i) there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that

$$\Omega\left(\alpha(\lambda,\zeta)\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right),\mathcal{Q}(\mathcal{E}^*(\lambda,\zeta)\mathcal{R}(\lambda,\zeta))\right) \geq \mathscr{C}_{\mathscr{A}},$$

where $\mathcal{F}$, $\mathcal{Q}$, $\mathcal{E}(\lambda,\zeta)$ and $\mathcal{R}(\lambda,\zeta)$ are defined as in Definition 2.1 for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta) > 0$ and for all $\sigma > 0$,

(ii) the pair $(\mathcal{P},\mathcal{S})$ is triangular α−admissible and there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P},\mathcal{S}$ are μ−continuous,

(iv) there exist $\lambda,\zeta \in C_{Fix}(\mathcal{P},\mathcal{S})$ such that $\alpha(\lambda,\zeta) \geq 1$.

Under the conditions $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$, $\lambda^* \in \mathcal{U}_\mu^*$ exists such that $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$.

Moreover, take into $\alpha(\lambda,\zeta) = 1$ account in Corollary 3.1, the next result is determined.

Corollary 3.2. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_\flat\mathcal{MS}$ with $\tau \geq 1$ and $\mathcal{P},\mathcal{S} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be two self-mappings there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that*

$$\Omega\left(\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right),\mathcal{Q}\left(\mathcal{E}^*(\lambda,\zeta)\mathcal{R}(\lambda,\zeta)\right)\right) \geq \mathscr{C}_{\mathscr{A}},$$

where $\mathcal{F},\mathcal{Q}$, $\mathcal{E}(\lambda,\zeta)$ and $\mathcal{R}(\lambda,\zeta)$ are defined as in Definition 2.1 for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta) > 0$ and for all $\sigma > 0$. Thereupon, together with $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$, we conclude that $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$.

Corollary 3.3. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_\flat\mathcal{MS}$ with a constant $\tau \geq 1$, $\alpha : \mathcal{U}_\mu^* \times \mathcal{U}_\mu^* \to \mathbb{R}$ be a function and $\mathcal{P} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be a self-mapping. Assume that the below requirements are met:*

(i) there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that

$$\frac{1}{2\tau}\mu_\sigma(\lambda,\mathcal{P}\lambda) \leq \mu_\sigma(\lambda,\zeta)$$

implies

$$\Omega\left(\alpha(\lambda,\zeta)\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{P}\zeta)^2\right),\mathcal{Q}\left(\mathcal{E}^*(\lambda,\zeta)\mathcal{R}(\lambda,\zeta)\right)\right) \geq \mathscr{C}_{\mathscr{A}},$$

where the functions $\mathcal{F},\mathcal{Q}$ are as indicated in Definition 2.1 and also, $\mathcal{E}(\lambda,\zeta)$ as in (1.1) and

$$\mathcal{R}(\lambda,\zeta) = \frac{\mu_\sigma(\lambda,\mathcal{P}\lambda)\mu_\sigma(\lambda,\mathcal{P}\zeta) + [\mu_\sigma(\lambda,\zeta)]^2 + \mu_\sigma(\lambda,\mathcal{P}\lambda)\mu_\sigma(\lambda,\zeta)}{\mu_\sigma(\lambda,\mathcal{P}\lambda) + \mu_\sigma(\lambda,\zeta) + \mu_\sigma(\lambda,\mathcal{P}\zeta)}$$

for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{P}\zeta) > 0$ and for all $\sigma > 0$,

(ii) $\mathcal{P}$ is a triangular $\alpha-$orbital admissible mapping and there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P}$ is $\mu-$continuous,

(iv) there exist $\lambda,\zeta \in Fix(\mathcal{P})$ such that $\alpha(\lambda,\zeta) \geq 1$.

So, under the conditions $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$, $\mathcal{P}$ has a unique fixed point.

Proof. Letting $\mathcal{P} = \mathcal{S}$ in Theorem 2.2, and by Lemma 1.17, we achieve the desired results. □

Corollary 3.4. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_\flat\mathcal{MS}$ with a constant $\tau \geq 1$, $\alpha : \mathcal{U}_\mu^* \times \mathcal{U}_\mu^* \to \mathbb{R}$ be a function and $\mathcal{P},\mathcal{S} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be two self-mappings. Assume that the following assertions are true:*

(i) there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that

$$\frac{1}{2\tau}\min\{\mu_\sigma(\lambda,\mathcal{P}\lambda),\mu_\sigma(\zeta,\mathcal{S}\zeta)\} \leq \mu_\sigma(\lambda,\zeta)$$

implies

$$\alpha(\lambda,\zeta)\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right) \leq \mathcal{Q}\left(\mathcal{E}^*(\lambda,\zeta)\mathcal{R}(\lambda,\zeta)\right),$$

where $\mathcal{F},\mathcal{Q}$, $\mathcal{E}(\lambda,\zeta)$ and $\mathcal{R}(\lambda,\zeta)$ are defined as in Definition 2.1 for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta) > 0$ and for all $\sigma > 0$;

(ii) the pair $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible and there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P},\mathcal{S}$ are $\mu-$continuous,

(iv) there exists $\lambda,\zeta \in C_{Fix}(\mathcal{P},\mathcal{S})$ such that $\alpha(\lambda,\zeta) \geq 1$.

Thereupon, $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$ provided that $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$ are met.

Proof. Letting $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ with the properties $\mathscr{C}_{\mathscr{A}}$ in Definition 1.12. □

Remark 3.5. *Note that all of the results can be again evaluated with respect to $\Xi \in \mathcal{Z}$ in place of $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$. Besides, as in Corollary 3.3, different results can be obtained when $\mathcal{P} = \mathcal{S}$.*

4. An Application to Dynamic Programming

We assume that Λ and Φ are Banach spaces, $\Sigma \subseteq \Lambda$ and $\Upsilon \subseteq \Phi$ such that Σ and Υ are state space and decision space, respectively. Consider the system of functional equations:

$$q(\lambda) = \max_{\zeta \in \Upsilon} \{f(\lambda,\zeta) + G(\lambda,\zeta,q(\xi(\lambda,\zeta)))\}, \ \lambda \in \Sigma$$

where $f : \Sigma \times \Upsilon \to \mathbb{R}$ and $G : \Sigma \times \Upsilon \times \mathbb{R} \to \mathbb{R}$ are bounded, $\xi : \Sigma \times \Upsilon \to \Sigma$. Let $\mathcal{U}_\mu = B(\Sigma)$ denotes the space of all bounded real-valued functions on Σ. Consider the metric defined by

$$\mu_\sigma(\varsigma,\varpi) = \frac{1}{\sigma} \max_{\lambda \in \Sigma} |\varsigma(\lambda) - \varpi(\lambda)|^2, \text{ for all } \varsigma,\varpi \in \Lambda \text{ and } \sigma > 0.$$

Then, $\mathcal{U}_\mu$ is a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$ with $\tau = 2$. Moreover, let $\mathcal{P} : \mathcal{U}_\mu \to \mathcal{U}_\mu$ be given by

$$\mathcal{P}\varsigma(\lambda) = \sup_{\zeta \in \Upsilon} \{f(\lambda,\zeta) + G(\lambda,\zeta,\varsigma(\xi(\lambda,\zeta)))\}, \tag{4.1}$$

where $\lambda \in \Sigma$ and $\varsigma \in \mathcal{U}_\mu$. If the functions f and G are bounded, then Λ and Φ are well-defined.

Theorem 4.1. *Let $\mathcal{P} : \mathcal{U}_\mu \to \mathcal{U}_\mu$ be an operator defined by (4.1) and suppose that the following conditions are hold:*

(i) f and G are bounded;

(ii) for $\forall \varsigma,\varpi \in \mathcal{U}_\mu$, $\forall \lambda \in \Sigma$, $\forall \zeta \in \Upsilon$, there exists $\delta \in (0,1)$ such that

$$|G(\lambda,\zeta,\varsigma(\lambda)) - G(\lambda,\zeta,\varpi(\lambda))| < \delta^{1/4} |\varsigma(\lambda) - \varpi(\lambda)|.$$

Then, the function equation (4.1) has a bounded solution; that is, $\mathcal{P}$ has a fixed point.

Proof. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary, $\lambda \in \Sigma$ and $\varsigma \in \mathcal{U}_\mu$. Assume that $\mathcal{P}\varsigma \neq \varsigma$. Then, $\zeta_1,\zeta_2 \in \Upsilon$ exist such that

$$\mathcal{P}\varsigma(\lambda) < f(\lambda,\zeta_1) + G(\lambda,\zeta_1,\varsigma(\xi(\lambda,\zeta_1))) + \varepsilon, \tag{4.2}$$

$$\varpi(\lambda) < f(\lambda,\zeta_2) + G(\lambda,\zeta_2,\varpi(\xi(\lambda,\zeta_1))) + \varepsilon, \tag{4.3}$$

$$\mathcal{P}\varsigma(\lambda) \geq f(\lambda,\zeta_2) + G(\lambda,\zeta_2,\varsigma(\xi(\lambda,\zeta_2))), \tag{4.4}$$

$$\varpi(\lambda) \geq f(\lambda,\zeta_1) + G(\lambda,\zeta_1,\varpi(\xi(\lambda,\zeta_1))). \tag{4.5}$$

Then, from (4.2) and (4.5), we yield that

$$\begin{aligned}
\mathcal{P}\varsigma(\lambda) - \varpi(\lambda) &< G(\lambda,\zeta_1,\varsigma(\xi(\lambda,\zeta_1))) - G(\lambda,\zeta_1,\varpi(\xi(\lambda,\zeta_1))) + \varepsilon \\
&\leq |G(\lambda,\zeta_1,\varsigma(\xi(\lambda,\zeta_1))) - G(\lambda,\zeta_1,\varpi(\xi(\lambda,\zeta_1)))| + \varepsilon \\
&< \delta^{1/4} |\varsigma(\lambda) - \varpi(\lambda)| + \varepsilon.
\end{aligned}$$

Likewise, from (4.3) and (4.4), we get

$$\begin{aligned}
\varpi(\lambda) - \mathcal{P}\varsigma(\lambda) &< G(\lambda,\zeta_2,\varpi(\xi(\lambda,\zeta_2))) - G(\lambda,\zeta_2,\varsigma(\xi(\lambda,\zeta_2))) + \varepsilon \\
&\leq |G(\lambda,\zeta_2,\varpi(\xi(\lambda,\zeta_2))) - G(\lambda,\zeta_2,\varsigma(\xi(\lambda,\zeta_2)))| + \varepsilon \\
&< \delta^{1/4} |\varsigma(\lambda) - \varpi(\lambda)| + \varepsilon.
\end{aligned}$$

Hence, by considering the above inequalities, we conclude that

$$|\mathcal{P}\varsigma(\lambda) - \varpi(\lambda)| < \delta^{1/4}|\varsigma(\lambda) - \varpi(\lambda)| + \varepsilon,$$

and, for an arbitrary ε

$$|\mathcal{P}\varsigma(\lambda) - \varpi(\lambda)| \leq \delta^{1/4}|\varsigma(\lambda) - \varpi(\lambda)|.$$

So, we have

$$\mu_\sigma(\mathcal{P}\varsigma(\lambda), \varpi(\lambda)) = \frac{1}{\sigma}|\mathcal{P}\varsigma(\lambda) - \varpi(\lambda)|^2 \leq \frac{1}{\sigma}\delta^{1/2}|\varsigma(\lambda) - \varpi(\lambda)|^2 = \delta^{1/2}\mu_\sigma(\varsigma(\lambda), \varpi(\lambda)). \tag{4.6}$$

Now, in Theorem 2.2, we take $\Omega(\ell, k) = \gamma k - \ell$ with $\gamma \in (0,1)$, $\mathscr{C}_{\mathscr{A}} = 0$ and $\mathscr{A}(\ell, k) = \ell - k$, and also, $\alpha(\lambda, \zeta) = 1$, $\mathcal{F}(s) = s$, $\mathcal{Q}(s) = \frac{s}{2}$ and lastly $\mathcal{S} = I$, which means that

$$\mathcal{E}^*(\varsigma(\lambda), \varpi(\lambda)) = \mu_\sigma(\varsigma(\lambda), \varpi(\lambda)) + \mu_\sigma(\varsigma(\lambda), \mathcal{P}\varsigma(\lambda))$$

and

$$\mathcal{R}(\varsigma(\lambda), \varpi(\lambda)) = \frac{\mu_\sigma(\varsigma(\lambda), \varpi(\lambda))[2\mu_\sigma(\varsigma(\lambda), \mathcal{P}\varsigma(\lambda)) + \mu_\sigma(\varsigma(\lambda), \varpi(\lambda))]}{\mu_\sigma(\varsigma(\lambda), \mathcal{P}\varsigma(\lambda)) + 2\mu_\sigma(\varsigma(\lambda), \varpi(\lambda))}.$$

Thereby, by a simple calculation, Theorem 2.2 turns into

$$\begin{aligned}\mu_\sigma(\mathcal{P}\varsigma(\lambda), \varpi(\lambda))^2 &\leq \tfrac{\gamma}{128}\mathcal{E}^*(\varsigma(\lambda), \varpi(\lambda))\mathcal{R}(\varsigma(\lambda), \varpi(\lambda)) \\ &\leq \tfrac{\gamma}{128}[\mu_\sigma(\varsigma(\lambda), \varpi(\lambda)) + \mu_\sigma(\varsigma(\lambda), \mathcal{P}\varsigma(\lambda))\mu_\sigma(\varsigma(\lambda), \varpi(\lambda))].\end{aligned} \tag{4.7}$$

Consequently, from the inequality (4.6), we deduce that

$$\begin{aligned}\mu_\sigma(\mathcal{P}\varsigma(\lambda), \varpi(\lambda))^2 &\leq \delta\mu_\sigma(\varsigma(\lambda), \varpi(\lambda))^2 \\ &\leq \delta[\mu_\sigma(\varsigma(\lambda), \varpi(\lambda)) + \mu_\sigma(\varsigma(\lambda), \mathcal{P}\varsigma(\lambda))\mu_\sigma(\varsigma(\lambda), \varpi(\lambda))],\end{aligned}$$

which means that, by taking $\delta = \frac{\gamma}{128} \in (0,1)$, (4.7) is satisfied, that is, all the conditions of Theorem 2.2 are met. Thus, we gain that $\mathcal{P}$ has a fixed point, i.e., the functional equation (4.1) has a bounded solution.

□

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