Volume Special Issue, Issue 1, Page 96-102, 2024 https://doi.org/10.46810/tdfd.1423927 Research Article

BİNGÖL UNIVERSITY

Türk Doğa ve Fen Dergisi
Turkish Journal of Nature and Science
www.dergipark.gov.tr/tdfd

Some Novel Integral Inequalities on the Co-ordinates for Geometrically Exponentially Convex Functions

Sinan ASLAN[1*]

[1] Ağrı Türk Telekom Social Sciences High School, Ağrı, Türkiye
Sinan ASLAN ORCID No: 0000-0001-5970-1926

**Corresponding author: sinanaslan0407@gmail.com*

(Received: 22.01.2024, Accepted: 28.05.2024, Online Publication: 01.10.2024)

Keywords
Hermite-Hadamard, Geometrically exponentially convex functions, Inequalities in the Coordinates

Abstract: The main purpose of this study is to define geometrically exponentially convex functions, which are a more general version, by expanding geometrically convex functions and to create the relevant lemmas. Some properties of geometrically exponentially convex functions are proven using definitions and lemmas. While obtaining the main findings, in addition to basic analysis information, Young and Hölder inequalities, well known in the literature, were also used for the powers of some functions. In the new theorems obtained, some special results were obtained for $\alpha = 0$.

Geometrik Eksponansiyel Konveks Fonksiyonlar için Koordinatlarda Bazı Yeni İntegral Eşitsizlikler

Anahtar Kelimeler
Hermite-Hadamard, Geometrik Üstel Konveks, Koordinatlarda Eşitsizlikler

Öz: Bu çalışmanın temel amacı geometrik konveks fonksiyonları genişleterek daha genel bir versiyonu olan geometrik eksponansiyel konveks fonksiyonları tanımlamak ve ilgili lemmaları oluşturmaktır. Geometrik eksponansiyel konveks fonksiyonların bazı özellikleri tanım ve lemmalar kullanılarak ispatlanmıştır. Ana bulguları elde ederken temel analiz bilgilerinin yanı sıra bazı fonksiyonların kuvvetleri için literatürde iyi bilinen Young ve Hölder eşitsizliklerinden yararlanılmıştır. Elde edilen yeni teoremlerde $\alpha = 0$ için bazı özel sonuçlar elde edilmiştir.

1. INTRODUCTION

In the 21st century, we see that inequalities are not only limited to mathematics, but also have an important place in many different sciences, especially engineering. For this reason, it has attracted the attention of many researchers and has been examined from different perspectives. The concept of convexity, which has an important place in inequality theory, is widely used by many researchers working in the field of inequality theory. We begin this study by giving the definition of the concept of convexity [2].

Definition 1. Let $A \subset R$. Then $\Upsilon : A \to \mathbb{R}$ is said to be convex, if

$$\Upsilon(\varpi\mu_1 + (1-\varpi)\mu_2) \le \varpi\Upsilon(\mu_1) + (1-\varpi)\Upsilon(\mu_2) \quad (1)$$

holds for all $\mu_1, \mu_2 \in A$ and $\varpi \in [0,1]$ (Peajcariaac et al. [2]).

The main goal of studies on different types of convexity is to optimize the bounds and generalize some known classical inequalities. Based on this basic purpose, an important class of convex functions whose definition is given is exponentially convex functions and whose definition reference [1] is given as follows.

Definition 2. A function $\Upsilon : A \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponential convex function, if

$$\Upsilon((1-\varpi)\mu_1+\varpi\mu_2)\leq(1-\varpi)\frac{\Upsilon(\mu_1)}{e^{\alpha\mu_1}}+\varpi\frac{\Upsilon(\mu_2)}{e^{\alpha\mu_2}} \quad (2)$$

for all $\mu_1,\mu_2\in A, \alpha\in\mathrm{R}$ and $\varpi\in[0,1]$ (Awan et al. [1]).

The definition of the concept of geometrically convex functions, which is well known in the literature and used by many researchers in their studies, is given as follows in reference [3].

Definition 3. A function $\Upsilon: A\subseteq(0,\infty)\to(0,\infty)$ is said to be a geometrically convex function, if

$$\Upsilon(\mu_1^{\varpi}\mu_2^{1-\varpi})\leq[\Upsilon(\mu_1)]^{\varpi}[\Upsilon(\mu_2)]^{1-\varpi} \quad (3)$$

for all $\mu_1,\mu_2\in A$ and $\varpi\in[0,1]$.

There are many studies in the literature about geometrically convex functions. Some of these are available in reference [4-10].

Aslan and Akdemir gave the definition of exponentially convex functions on the coordinates, which is a more general version of convex functions on coordinates, as follows in reference [11].

Definition 4. Let us consider the bidimensional interval $\Delta=[\vartheta_1,\vartheta_2]x[\vartheta_3,\vartheta_4]$ in R^2 with $\vartheta_1<\vartheta_2$ and $\vartheta_3<\vartheta_4$. The mapping $\Upsilon:\Delta\to R$ is exponentially convex on the co-ordinates on Δ, if the following inequality holds,

$$\Upsilon(\varpi\mu_1+(1-\varpi)\mu_3,\varpi\mu_2+(1-\varpi)\mu_4)$$
$$\leq\varpi\frac{\Upsilon(\mu_1,\mu_2)}{e^{\alpha(\mu_1+\mu_2)}}+(1-\varpi)\frac{\Upsilon(\mu_3,\mu_4)}{e^{\alpha(\mu_3+\mu_4)}} \quad (4)$$

for all $(\mu_1,\mu_2),(\mu_3,\mu_4)\in\Delta, \alpha\in R$ and $\varpi\in[0,1]$ (Aslan et al. [11]).

Aslan and Akdemir gave another definition of coordinates equivalent to the exponentially convex function definition as follows:

Definition 5. A function $\Upsilon:\Delta\to\mathrm{R}$ is exponentially convex function on the co-ordinates on Δ, if the following inequality holds,

$$\Upsilon(\varpi\mu_1+(1-\varpi)\mu_2,\omega\mu_3+(1-\omega)\mu_4)$$
$$\leq\varpi\omega\frac{\Upsilon(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}}+\varpi(1-\omega)\frac{\Upsilon(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \quad (5)$$
$$+(1-\varpi)\omega\frac{\Upsilon(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}}+(1-\varpi)(1-\omega)\frac{\Upsilon(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}}$$

for all $(\mu_1,\mu_3),(\mu_1,\mu_4),(\mu_2,\mu_3),(\mu_2,\mu_4)\in\Delta,\alpha\in\mathrm{R}$ and $\varpi,\omega\in[0,1]$ (Aslan et al. [11]).

With the definition of convex functions on coordinates, it brought to mind the question that the Hermite-Hadamard inequality could also be extended to coordinates. We see the answer to this enormous question in Dragomir's article [12]. We see the equivalent of the Hermite-Hadamard inequality in coordinates in the theorem below.

Theorem1. (Dragomir [12]) Lets assume that $\Upsilon:\Delta=[\mu_1,\mu_2]\times[\mu_3,\mu_4]\to\mathrm{R}$ is convex on the co-ordinates on Δ. Then one has the inequalit

$$\Upsilon\left(\frac{\mu_1+\mu_2}{2},\frac{\mu_3+\mu_4}{2}\right)$$
$$\leq\frac{1}{(\mu_2-\mu_1)(\mu_4-\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\Upsilon(x,y)dxdy$$
$$\leq\frac{\Upsilon(\mu_1,\mu_3)+\Upsilon(\mu_1,\mu_4)+\Upsilon(\mu_2,\mu_3)+\Upsilon(\mu_2,\mu_4)}{4}. \quad (6)$$

The above inequalities are sharp.

Theorem2. (Aslan et al. [11]) Let $\Upsilon:\Delta=[\mu_1,\mu_2]\times[\mu_3,\mu_4]\to\mathrm{R}$ be partial differentiable mapping on $\Delta=[\mu_1,\mu_2]\times[\mu_3,\mu_4]$ and $\Upsilon\in L(\Delta)$, $\alpha\in\mathrm{R}$. If Υ is exponentially convex function on the co-ordinates on Δ, then the following inequality holds;

$$\frac{1}{(\mu_2-\mu_1)(\mu_4-\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\Upsilon(x,y)dxdy$$
$$\leq\frac{\frac{\Upsilon(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}}+\frac{\Upsilon(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}}+\frac{\Upsilon(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}}+\frac{\Upsilon(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}}}{4}. \quad (7)$$

Many studies have been carried out in the literature on exponentially convex functions and exponentially convex functions on coordinates. Some of these are given in reference [13-21].

Anderson et al. gave the following definition in (Anderson et al. [22])

Definition 6. A function $M: R^+\times R^+\to R^+$ is called a Mean function if

(1) $M(\mu_1,\mu_2)=M(\mu_2,\mu_1)$,

(2) $M(\mu_1,\mu_1)=\mu_1$,

(3) $\mu_1<M(\mu_1,\mu_2)<\mu_2$, whenever $\mu_1<\mu_2$,

(4) $M(a\mu_1,a\mu_2)=aM(\mu_1,\mu_2)$ for all $a>0$.

Let us recall special means (See [22,23,24])

1. Arithmetic Mean:

$$M(\mu_1,\mu_2)=A(\mu_1,\mu_2)=\frac{\mu_1+\mu_2}{2}.$$

2. Geometric Mean:

$M(\mu_1,\mu_2)=G(\mu_1,\mu_2)=\sqrt{\mu_1\mu_2}$.

3. Harmonic Mean:

$$M(\mu_1,\mu_2)=H(\mu_1,\mu_2)=1/A\left(\frac{1}{\mu_1},\frac{1}{\mu_2}\right).$$

4. Logarithmic Mean:

$$M(\mu_1,\mu_2)=L(\mu_1,\mu_2)=(\mu_1-\mu_2)/(\log\mu_1-\log\mu_2)$$

for $\mu_1\neq\mu_2$ and $L(\mu_1,\mu_1)=\mu_1$.

5. Identric Mean:

$$M(\mu_1,\mu_2)=I(\mu_1,\mu_2)=(1/e)\left(\mu_1^{\mu_1}/\mu_2^{\mu_2}\right)^{1/(\mu_1-\mu_2)}$$

for $\mu_1\neq\mu_2$ and $I(\mu_1,\mu_1)=\mu_1$.

Now we are in a position to put in order as:

$$H(\mu_1,\mu_2)\leq G(\mu_1,\mu_2)\leq L(\mu_1,\mu_2)$$
$$\leq I(\mu_1,\mu_2)\leq A(\mu_1,\mu_2)\leq K(\mu_1,\mu_2).$$

In [22], authors also gave a definition which is called MN-convexity as the following:

Definition 7. Let $\Upsilon:I\to(0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say Υ is MN-convex (concave) if

$$\Upsilon(M(\mu_1,\mu_2))\leq(\geq)N(\Upsilon(\mu_1),\Upsilon(\mu_2)) \tag{8}$$

for all $\mu_1,\mu_2\in I$.

2. MAIN RESULTS

Definition 8. Let us consider the bidimensional interval $\Delta=[\mu_1,\mu_2]\times[\mu_3,\mu_4]$ in R^2 with $\mu_1<\mu_2$ and $\mu_3<\mu_4$. The mapping $\Upsilon:\Delta\to R^+$ is geometrically-exponentially convex on the co-ordinates on Δ, if the following inequality holds,

$$\Upsilon\left(\mu_1^{\varpi}\mu_3^{(1-\varpi)},\mu_2^{\varpi}\mu_4^{(1-\varpi)}\right)\leq\frac{\Upsilon^{\varpi}(\mu_1,\mu_2)}{e^{\alpha(\mu_1+\mu_2)}}\frac{\Upsilon^{(1-\varpi)}(\mu_3,\mu_4)}{e^{\alpha(\mu_3+\mu_4)}} \tag{9}$$

for all $(\mu_1,\mu_2),(\mu_3,\mu_4)\in\Delta,\alpha\in R$ and $\varpi\in[0,1]$.

A second definition of geometrically exponentially convex functions on coordinates equivalent to the above definition can be made as follows:

Definition 9. A function $\Upsilon:\Delta\to R^+$ is geometrically exponentially convex on the co-ordinates on Δ, if the following inequality holds,

$$\Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right)$$
$$\leq\frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}}\frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}}\frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}}\frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}} \tag{10}$$

for all $(\mu_1,\mu_3),(\mu_1,\mu_4),(\mu_2,\mu_3),(\mu_2,\mu_4)\in\Delta,\alpha\in R$ and $\varpi,\omega\in[0,1]$.

Lemma 1. A function $\Upsilon:\Delta\to R^+$ will be called geometrically exponentially convex on the co-ordinates on Δ, if the partial mappings $\Upsilon_{\rho_2}:[\mu_1,\mu_2]\to R$, $\Upsilon_{\rho_2}(u)=e^{\alpha\rho_2}f(u,\rho_2)$ and $\Upsilon_{\rho_1}:[\mu_3,\mu_4]\to R$, $\Upsilon_{\rho_1}(v)=e^{\alpha\rho_1}f(\rho_1,v)$ are geometrically exponentially convex on the co-ordinates on Δ, where defined for all $\rho_2\in[\mu_3,\mu_4]$ and $\rho_1\in[\mu_1,\mu_2]$

Proof. From the definition of partial mapping Υ_{ρ_1} we can write

$$\Upsilon_{\rho_1}\left(v_1^{\varpi}v_2^{(1-\varpi)}\right)=e^{\alpha\rho_1}\Upsilon\left(\rho_1,v_1^{\varpi}v_2^{(1-\varpi)}\right)$$
$$=e^{\alpha\rho_1}\Upsilon\left(\rho_1^{\varpi}\rho_1^{(1-\varpi)},v_1^{\varpi}v_2^{(1-\varpi)}\right)$$
$$\leq e^{\alpha\rho_1}\left[\frac{\Upsilon^{\varpi}(\rho_1,v_1)}{e^{\alpha(\rho_1+v_1)}}\frac{\Upsilon^{(1-\varpi)}(\rho_1,v_2)}{e^{\alpha(\rho_1+v_2)}}\right]$$
$$=\frac{\Upsilon^{\varpi}(\rho_1,v_1)}{e^{\alpha v_1}}\frac{\Upsilon^{(1-\varpi)}(\rho_1,v_2)}{e^{\alpha v_2}}$$
$$=\frac{\Upsilon_{\rho_1}^{\varpi}(v_1)}{e^{\alpha v_1}}\frac{\Upsilon_{\rho_1}^{(1-\varpi)}(v_2)}{e^{\alpha v_2}}. \tag{11}$$

Similarly, one can easily see that

$$\Upsilon_{\rho_2}\left(u_1^{\varpi}u_2^{(1-\varpi)}\right)\leq\frac{\Upsilon_{\rho_2}^{\varpi}(u_1)}{e^{\alpha u_1}}\frac{\Upsilon_{\rho_2}^{(1-\varpi)}(u_2)}{e^{\alpha u_2}}. \tag{12}$$

The proof is completed.

Proposition 1. If $\Upsilon,\Phi:\Delta\to R$ are two geometrically exponentially convex functions on the co-ordinates on Δ, then $\Upsilon\Phi$ is geometrically exponentially convex functions on the co-ordinates on Δ.

Proof.

$$\Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right)\times\Phi\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right)$$
$$\leq\frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}}\frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}}\frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}}\frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}}$$

$$\times \frac{\Phi^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}} \frac{\Phi^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \frac{\Phi^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}} \frac{\Phi^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}}$$

$$= \frac{(\Upsilon\Phi)^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}} \frac{(\Upsilon\Phi)^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \times \frac{(\Upsilon\Phi)^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}} \frac{(\Upsilon\Phi)^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}} \tag{13}$$

Therefore $\Upsilon\Phi$ is geometrically exponentially convex functions on the co-ordinates on Δ.

Theorem 3. Let $\Upsilon : \Delta = [\mu_1,\mu_2]\times[\mu_3,\mu_4] \to R^+$ be partial differentiable mapping on $\Delta = [\mu_1,\mu_2]\times[\mu_3,\mu_4]$ and $\Upsilon \in L(\Delta)$, $\alpha \in R.$ If Υ is geometrically exponentially convex function on the co-ordinates on Δ, then the following inequality holds;

$$\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2 \le \frac{L(\Upsilon(\mu_1,\mu_3),\Upsilon(\mu_1,\mu_4))+L(\Upsilon(\mu_2,\mu_3)\Upsilon(\mu_2,\mu_4))}{2e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \tag{14}$$

where $\rho_1 \in [\mu_1,\mu_2]$ and $\rho_2 \in [\mu_3,\mu_4]$ dir.

Proof. Using inequality (10), the following expression is written

$$\Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right) \le \frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}} \frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}} \frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}} \tag{15}$$

By integrating both sides of inequality (15) with respect to ϖ,ω on $[0,1]^2$, we have

$$\int_0^1\int_0^1 \Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right)d\varpi d\omega$$
$$\le \int_0^1\int_0^1\left(\frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}} \frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \times \frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}} \frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}}\right)d\varpi d\omega$$
$$= \frac{1}{e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \times \int_0^1\int_0^1\left(\Upsilon^{\varpi\omega}(\mu_1,\mu_3)\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)\Upsilon^{(1-\varpi)\omega} \times (\mu_2,\mu_3)\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)\right)d\varpi d\omega$$

$$= \frac{1}{e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \times \int_0^1 \frac{\Upsilon^{\omega}(\mu_1,\mu_3)\Upsilon^{(1-\omega)}(\mu_1,\mu_4)-\Upsilon^{\omega}(\mu_2,\mu_3)\Upsilon^{(1-\omega)}(\mu_2,\mu_4)}{\ln\Upsilon^{\omega}(\mu_1,\mu_3)\Upsilon^{(1-\omega)}(\mu_1,\mu_4)-\ln\Upsilon^{\omega}(\mu_2,\mu_3)\Upsilon^{(1-\omega)}(\mu_2,\mu_4)}d\omega$$
$$= \frac{1}{e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \times \int_0^1 L\left(\Upsilon^{\omega}(\mu_1,\mu_3)\Upsilon^{(1-\omega)}(\mu_1,\mu_4),\Upsilon^{\omega}(\mu_2,\mu_3)\Upsilon^{(1-\omega)}(\mu_2,\mu_4)\right)d\omega \tag{16}$$

If the $\rho_1 = \mu_1^{\varpi}\mu_2^{1-\varpi}, \rho_2 = \mu_3^{\omega}\mu_4^{1-\omega}$ variable is changed and the $L(a,b) < A(a,b)$ feature is taken into account, the following result is obtained.

$$\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2$$
$$\le \frac{1}{e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \times \int_0^1 A\left(\Upsilon^{\omega}(\mu_1,\mu_3)\Upsilon^{(1-\omega)}(\mu_1,\mu_4),\Upsilon^{\omega}(\mu_2,\mu_3)\Upsilon^{(1-\omega)}(\mu_2,\mu_4)\right)d\omega$$
$$= \frac{1}{e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \times \int_0^1 \frac{\Upsilon^{\omega}(\mu_1,\mu_3)\Upsilon^{(1-\omega)}(\mu_1,\mu_4)+\Upsilon^{\omega}(\mu_2,\mu_3)\Upsilon^{(1-\omega)}(\mu_2,\mu_4)}{2}d\omega$$
$$= \frac{L(\Upsilon(\mu_1,\mu_3),\Upsilon(\mu_1,\mu_4))+L(\Upsilon(\mu_2,\mu_3)\Upsilon(\mu_2,\mu_4))}{2e^{2\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} \tag{17}$$

proof is completed.

Corollary 1. If we choose $\alpha = 0$ in Theorem 2.1, the result agrees geometrically with convexity on the coordinates.

$$\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2 \le \frac{L(\Upsilon(\mu_1,\mu_3),\Upsilon(\mu_1,\mu_4))+L(\Upsilon(\mu_2,\mu_3)\Upsilon(\mu_2,\mu_4))}{2} \tag{18}$$

where $\rho_1 \in [\mu_1,\mu_2]$ and $\rho_2 \in [\mu_3,\mu_4]$ dir.

Theorem 4. Let $\Upsilon : \Delta = [\mu_1,\mu_2]\times[\mu_3,\mu_4] \to R^+$ be partial differentiable mapping on $\Delta = [\mu_1,\mu_2]\times[\mu_3,\mu_4]$ and $\Upsilon \in L(\Delta)$, $\alpha \in R.$ If $|\Upsilon|$ is geometrically exponentially convex function on the co-ordinates on Δ, $p > 1$ then the following inequality holds;

$$\left|\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|$$
$$\leq\left(\frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{L\left(|\Upsilon(\mu_1,\mu_3)|^q|\Upsilon(\mu_1,\mu_4)|^q\right)+L\left(|\Upsilon(\mu_2,\mu_3)|^q|\Upsilon(\mu_2,\mu_4)|^q\right)}{2}\right)^{\frac{1}{q}} \quad (19)$$

where $\rho_1\in[\mu_1,\mu_2]$, $\rho_2\in[\mu_3,\mu_4]$ and $p^{-1}+q^{-1}=1$ dir.

Proof. Using inequality (10), the following expression is written

$$\Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right)$$
$$\leq\frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}}\frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}}\frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}}\frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}} \quad (20)$$

If the absolute value of both sides of inequality (20) is taken and integrated with respect to ϖ,ω on $[0,1]^2$, we can write

$$\left|\int_0^1\int_0^1\Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)},\mu_3^{\omega}\mu_4^{(1-\omega)}\right)d\varpi d\omega\right|$$
$$\leq\int_0^1\int_0^1\left|\begin{pmatrix}\frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}}\frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}}\\ \times\frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}}\frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}}\end{pmatrix}\right|d\varpi d\omega \quad (21)$$

In inequality (21), apply Hölder's inequality to the right side of the inequality and $\rho_1=\mu_1^{\varpi}\mu_2^{1-\varpi},\rho_2=\mu_3^{\omega}\mu_4^{1-\omega}$ variable is changed , we get

$$\left|\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|$$
$$\leq\left(\int_0^1\int_0^1\frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}d\varpi d\omega\right)^{\frac{1}{p}}$$
$$\times\left(\begin{matrix}\int_0^1\int_0^1\left(|\Upsilon(\mu_1,\mu_3)|^{\varpi\omega q}|\Upsilon(\mu_1,\mu_4)|^{\varpi(1-\omega)q}\right.\\ \left.\times|\Upsilon(\mu_2,\mu_3)|^{(1-\varpi)\omega q}|\Upsilon(\mu_2,\mu_4)|^{(1-\varpi)(1-\omega)q}\right)d\varpi d\omega\end{matrix}\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 L\left(|\Upsilon(\mu_1,\mu_3)|^{\omega q}|\Upsilon(\mu_1,\mu_4)|^{(1-\omega)q},|\Upsilon(\mu_2,\mu_3)|^{\omega q}|\Upsilon(\mu_2,\mu_4)|^{(1-\omega)q}\right)d\omega\right)^{\frac{1}{q}}$$

Because of the $L(a,b)<A(a,b)$ property, we can write

$$\left|\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|$$
$$\leq\left(\frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 A\left(|\Upsilon(\mu_1,\mu_3)|^{\omega q}|\Upsilon(\mu_1,\mu_4)|^{(1-\omega)q},|\Upsilon(\mu_2,\mu_3)|^{\omega q}|\Upsilon(\mu_2,\mu_4)|^{(1-\omega)q}\right)d\omega\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\frac{|\Upsilon(\mu_1,\mu_3)|^{\omega q}|\Upsilon(\mu_1,\mu_4)|^{(1-\omega)q}+|\Upsilon(\mu_2,\mu_3)|^{\omega q}|\Upsilon(\mu_2,\mu_4)|^{(1-\omega)q}}{2}d\omega\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{L\left(|\Upsilon(\mu_1,\mu_3)|^q|\Upsilon(\mu_1,\mu_4)|^q\right)+L\left(|\Upsilon(\mu_2,\mu_3)|^q|\Upsilon(\mu_2,\mu_4)|^q\right)}{2}\right)^{\frac{1}{q}}. \quad (22)$$

Proof is completed.

Corollary 2. If we choose $\alpha=0$ in Theorem 4, the result agrees geometrically with convexity on the coordinates.

$$\left|\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|$$
$$\leq\left(\frac{L\left(|\Upsilon(\mu_1,\mu_3)|^q|\Upsilon(\mu_1,\mu_4)|^q\right)+L\left(|\Upsilon(\mu_2,\mu_3)|^q|\Upsilon(\mu_2,\mu_4)|^q\right)}{2}\right)^{\frac{1}{q}} \quad (23)$$

where $\rho_1\in[\mu_1,\mu_2]$, $\rho_2\in[\mu_3,\mu_4]$ and $p^{-1}+q^{-1}=1$ dir.

Theorem 5. Let $\Upsilon:\Delta=[\mu_1,\mu_2]\times[\mu_3,\mu_4]\rightarrow R^+$ be partial differentiable mapping on $\Delta=[\mu_1,\mu_2]\times[\mu_3,\mu_4]$ and $\Upsilon\in L(\Delta)$, $\alpha\in R.$ If $|\Upsilon|$ is geometrically exponentially convex function on the co-ordinates on Δ, $p>1$ then the following inequality holds;

$$\left|\frac{1}{(\ln\mu_2-\ln\mu_1)(\ln\mu_4-\ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4}\frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|$$

$$\leq \frac{1}{pe^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} + \frac{L\left(\left|\Upsilon(\mu_1,\mu_3)\right|^q \left|\Upsilon(\mu_1,\mu_4)\right|^q\right) + L\left(\left|\Upsilon(\mu_2,\mu_3)\right|^q \left|\Upsilon(\mu_2,\mu_4)\right|^q\right)}{2q} \quad (24)$$

where $\rho_1 \in [\mu_1, \mu_2]$, $\rho_2 \in [\mu_3, \mu_4]$ and $p^{-1} + q^{-1} = 1$ dir.

Proof. Using inequality (10), the following expression is written

$$\Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)}, \mu_3^{\omega}\mu_4^{(1-\omega)}\right) \leq \frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}} \frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}} \frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}} \quad (25)$$

If the absolute value of both sides of inequality (20) is taken and integrated with respect to ϖ, ω on $[0,1]^2$, we can write

$$\left|\int_0^1\int_0^1 \Upsilon\left(\mu_1^{\varpi}\mu_2^{(1-\varpi)}, \mu_3^{\omega}\mu_4^{(1-\omega)}\right) d\varpi d\omega\right| \leq \int_0^1\int_0^1 \left|\begin{pmatrix} \frac{\Upsilon^{\varpi\omega}(\mu_1,\mu_3)}{e^{\alpha(\mu_1+\mu_3)}} \frac{\Upsilon^{\varpi(1-\omega)}(\mu_1,\mu_4)}{e^{\alpha(\mu_1+\mu_4)}} \\ \times \frac{\Upsilon^{(1-\varpi)\omega}(\mu_2,\mu_3)}{e^{\alpha(\mu_2+\mu_3)}} \frac{\Upsilon^{(1-\varpi)(1-\omega)}(\mu_2,\mu_4)}{e^{\alpha(\mu_2+\mu_4)}} \end{pmatrix}\right| d\varpi d\omega \quad (26)$$

In inequality (26), apply Young's inequality to the right side of the inequality and $\rho_1 = \mu_1^{\varpi}\mu_2^{1-\varpi}, \rho_2 = \mu_3^{\omega}\mu_4^{1-\omega}$ variable is changed , we get

$$\left|\frac{1}{(\ln\mu_2 - \ln\mu_1)(\ln\mu_4 - \ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4} \frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2} d\rho_1 d\rho_2\right|$$
$$\leq \frac{1}{p}\left(\int_0^1\int_0^1 \frac{1}{e^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} d\varpi d\omega\right)$$
$$+ \frac{1}{q}\left(\int_0^1\int_0^1 \begin{pmatrix} \left|\Upsilon(\mu_1,\mu_3)\right|^{\varpi\omega q} \left|\Upsilon(\mu_1,\mu_4)\right|^{\varpi(1-\omega)q} \\ \times \left|\Upsilon(\mu_2,\mu_3)\right|^{(1-\varpi)\omega q} \left|\Upsilon(\mu_2,\mu_4)\right|^{(1-\varpi)(1-\omega)q} \end{pmatrix} d\varpi d\omega\right)$$
$$= \frac{1}{pe^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}$$
$$+ \frac{1}{q}\left(\int_0^1 L\left(\left|\Upsilon(\mu_1,\mu_3)\right|^{\omega q}\left|\Upsilon(\mu_1,\mu_4)\right|^{(1-\omega)q}, \left|\Upsilon(\mu_2,\mu_3)\right|^{\omega q}\left|\Upsilon(\mu_2,\mu_4)\right|^{(1-\omega)q}\right) d\omega\right) \quad (27)$$

Because of the $L(a,b) < A(a,b)$ property, we can write

$$\left|\frac{1}{(\ln\mu_2 - \ln\mu_1)(\ln\mu_4 - \ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4} \frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2} d\rho_1 d\rho_2\right|$$
$$\leq \frac{1}{pe^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} +$$
$$\frac{1}{q}\left(\int_0^1 A\left(\left|\Upsilon(\mu_1,\mu_3)\right|^{\omega q}\left|\Upsilon(\mu_1,\mu_4)\right|^{(1-\omega)q}, \left|\Upsilon(\mu_2,\mu_3)\right|^{\omega q}\left|\Upsilon(\mu_2,\mu_4)\right|^{(1-\omega)q}\right) d\omega\right)$$
$$= \frac{1}{pe^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}}$$
$$+ \left(\int_0^1 \frac{\left|\Upsilon(\mu_1,\mu_3)\right|^{\omega q}\left|\Upsilon(\mu_1,\mu_4)\right|^{(1-\omega)q} + \left|\Upsilon(\mu_2,\mu_3)\right|^{\omega q}\left|\Upsilon(\mu_2,\mu_4)\right|^{(1-\omega)q}}{2q} d\omega\right)$$
$$= \frac{1}{pe^{2p\alpha(\mu_1+\mu_3+\mu_1+\mu_4)}} + \frac{L\left(\left|\Upsilon(\mu_1,\mu_3)\right|^q \left|\Upsilon(\mu_1,\mu_4)\right|^q\right) + L\left(\left|\Upsilon(\mu_2,\mu_3)\right|^q \left|\Upsilon(\mu_2,\mu_4)\right|^q\right)}{2q} \quad (28)$$

proof is completed.

Corollary 3. If we choose $\alpha = 0$ in Theorem 5, the result agrees geometrically with convexity on the co-ordinates.

$$\left|\frac{1}{(\ln\mu_2 - \ln\mu_1)(\ln\mu_4 - \ln\mu_3)}\int_{\mu_1}^{\mu_2}\int_{\mu_3}^{\mu_4} \frac{\Upsilon(\rho_1,\rho_2)}{\rho_1\rho_2} d\rho_1 d\rho_2\right|$$
$$\leq \frac{1}{p} + \frac{L\left(\left|\Upsilon(\mu_1,\mu_3)\right|^q \left|\Upsilon(\mu_1,\mu_4)\right|^q\right) + L\left(\left|\Upsilon(\mu_2,\mu_3)\right|^q \left|\Upsilon(\mu_2,\mu_4)\right|^q\right)}{2q} \quad (29)$$

where $\rho_1 \in [\mu_1, \mu_2]$, $\rho_2 \in [\mu_3, \mu_4]$ and $p^{-1} + q^{-1} = 1$ dir.

Acknowledgement

This study was presented as an oral presentation at the "6th International Conference on Life and Engineering Sciences (ICOLES 2023)" conference.

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