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Lacunary Invariant Statistical Convergence in Fuzzy Normed Spaces

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Abstract

In the study done here regarding the theory of summability, we introduce some new concepts in fuzzy normed spaces. First, at the beginning of the original part of our study, we define the lacunary invariant statistical convergence. Then, we examine some characteristic features like uniqueness, linearity of this new notion and give its important relation with pre-given concepts.

1. Introduction and Definitions

First, we note some basic information available in the literature for a better understanding of our work and to use for the definitions of new concepts that we will give in the original chapter. The convergence of sequences in real numbers was generalized to statistical convergence by Schoenberg [1] and Fast [2]. Several features have been studied like being subspace of bounded sequence space by Salat [3], statistical Cauchy sequence by Fridy [4], statistical convergence and its equivalence of strong p-Cesaro summability for bounded sequences by Connor [5] and so on.

Lacunary convergence with the relation to strong Cesàro summability was studied by Freedman et al. [6]. Also Das and Patel [7] investigated this issue comprehensively. Fridy and Orhan [8,9] contributed to the literature about lacunary statistical convergence. Additionally, Ulusu and Nuray have been studied on this issue [10–12].

Banach limit was first introduced by Banach [13]. In case all Banach limits are equal for a given bounded sequence, Lorentz [14] called that almost convergence. Later, as a generalization of Banach limit and almost convergence, the notions of invariant mean and invariant convergence were presented by Raimi et al. [15,16]. Also, it has been studied by several authors [17–21]. Especially, Savaş and Nuray [22,23] proved important theorems in their studies.

The definitions of concepts such as statistical convergence, Banach limit, invariant mean, invariant convergence, lacunary sequence and lacunary convergence are not given here, and the references are based on the studies mentioned above.

Zadeh [24] proposed fuzzy set as a new concept to study on imprecise phenomena. A fuzzy set having certain properties was described as a fuzzy number [25,26]. The literature includes studies on concepts such as fuzzy topological spaces [27–29], fuzzy metric [30,31], fuzzy norm [32,33].

Now, based on these studies, the definition of a fuzzy number, arithmetic operations on fuzzy numbers, convergence on fuzzy numbers sequence, and fuzzy norm introduced by Felbin [32] will be given.

A fuzzy number u is a fuzzy set provided that
(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;
(iii) u is upper semi-continuous;
(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.

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We will denote all fuzzy numbers by the set $\mathscr{L}(\mathbb{R})$. Every $r \in \mathbb{R}$ is also a fuzzy number denoted by $\tilde{r} = \tilde{r}(t)$ and its value is 1 when $t = r$ and 0 otherwise. So, $\mathbb{R}$ is included by $\mathscr{L}(\mathbb{R})$.
The α-level sets, partial ordering, arithmetic equations and supremum metric on $\mathscr{L}(\mathbb{R})$ are very important in fuzzy numbers and will be used in the operations performed in our study. Now let's give the definitions and features of these concepts.
The α-level set of $u \in \mathscr{L}(\mathbb{R})$ is given by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ cl\,\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{cases}$$

and is written as a non-empty interval $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ which is also bounded and closed for every $\alpha \in [0,1]$. Here, $[-\infty, \infty]$ is admissible. When $u(x) = 0$ for all $x < 0$, $u \in \mathscr{L}(\mathbb{R})$ is a non-negative fuzzy number. We will denote all non-negative fuzzy numbers by the set $\mathscr{L}^*(\mathbb{R})$. It is clearly understood that $\tilde{0} \in \mathscr{L}^*(\mathbb{R})$.
For $u, v \in \mathscr{L}(\mathbb{R})$ and all $\alpha \in [0,1]$, the partial ordering $\preceq$ in $\mathscr{L}(\mathbb{R})$ is given as following

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \& \quad u_\alpha^+ \leq v_\alpha^+.$$

On $\mathscr{L}(\mathbb{R})$, arithmetic equations are defined as follows
(i) $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}$,
(ii) $(u \odot v)(t) = \sup_{s \in \mathbb{R} s \neq 0} \{u(s) \wedge v(t/s)\}$,
(iii) $ru(t) = u(t/r)$ for $r \in \mathbb{R}^+$ and $0u(t) = \tilde{0}$,
for $u, v \in \mathscr{L}(\mathbb{R})$ and $t \in \mathbb{R}$.
Using α-level sets, arithmetic equations are given as follows
(i) $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$, for $u, v \in \mathscr{L}(\mathbb{R})$,
(ii) $[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$, for $u, v \in \mathscr{L}^*(\mathbb{R})$,
(iii) For $u \in \mathscr{L}(\mathbb{R})$,

$$[ru]_\alpha = r[u]_\alpha = \begin{cases} [ru_\alpha^-, ru_\alpha^+], & \text{if } r \geq 0, \\ [ru_\alpha^+, ru_\alpha^-], & \text{if } r < 0. \end{cases}$$

On $\mathscr{L}(\mathbb{R})$, it is described that a metric known as supremum metric;

$$\mathscr{D}(u,v) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\},$$

for $u, v \in \mathscr{L}(\mathbb{R})$. Obviously,

$$\mathscr{D}\left(u, \tilde{0}\right) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^-\right|, \left|u_\alpha^+\right|\right\} = \max\left\{\left|u_0^-\right|, \left|u_0^+\right|\right\}$$

and for $u \in \mathscr{L}^*(\mathbb{R})$, we obtain $\mathscr{D}\left(u, \tilde{0}\right) = u_0^+$.
In $\mathscr{L}(\mathbb{R})$, the sequence (u_n) is convergent to $u \in \mathscr{L}(\mathbb{R})$ if $\lim_{n \to \infty} \mathscr{D}(u_n, u) = 0$. This convergence is denoted by $\mathscr{D} - \lim_{n \to \infty} u_n = u$.
Now let's give the definition and features of fuzzy normed space.
For a vector space $\mathscr{X}$ over $\mathbb{R}$, consider $\|.\| : \mathscr{X} \to \mathscr{L}^*(\mathbb{R})$. For the symmetric and non-decreasing mappings $L, R : [0,1] \times [0,1] \to [0,1]$, let the conditions $L(0,0) = 0$ and $R(1,1) = 1$ be satisfied.
If the followings
(i) $\|x\| = \tilde{0}$ iff x is zero vector.
(ii) $\|rx\| = |r| \odot \|x\|$ for $r \in \mathbb{R}, x \in \mathscr{X}$.
(iii) For all $x, y \in \mathscr{X}$,
(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, if $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ & $s+t \leq \|x+y\|_1^-$,
(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, if $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ & $s+t \geq \|x+y\|_1^-$
hold, then $\|.\|$ is named fuzzy norm and the quadruple $(\mathscr{X}, \|.\|, L, R)$ is fuzzy normed space (FNS).
We substitute min and max for L and R in (iii), then we have

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \quad \& \quad \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+,$$

for $x, y \in \mathscr{X}$ and all $\alpha \in (0,1]$. Also, $\|.\|_\alpha^-$ and $\|.\|_\alpha^+$ satisfy the other norm conditions.
From now on, in our study, we take $(\mathscr{X}, \|.\|)$ as a FNS.
Let's take $\mathscr{X}$ as a topological structure. For any $\varepsilon > 0$ and all $\alpha \in [0,1]$, the $\varepsilon(\alpha)-$neighborhood of $x \in \mathscr{X}$ is the set

$$\mathscr{N}_{\varepsilon(\alpha)}^x = \{y \in \mathscr{X} : \|x-y\|_\alpha^+ < \varepsilon\}.$$

Recently, several convergence types have been studied on fuzzy normed spaces by a lot of authors [34–39].
If for each $\varepsilon > 0$, an $n_0 \in \mathbb{N}$ exists and satisfy

$$\mathscr{D}\left(\|x_n - x_0\|, \tilde{0}\right) = \sup_{\alpha \in [0,1]} \|x_n - x_0\|_\alpha^+ = \|x_n - x_0\|_0^+ < \varepsilon$$

for all $n > n_0$, then the sequence $(x_n) \subset \mathscr{X}$ is convergent to $x_0 \in \mathscr{X}$ and we write $x_n \overset{FN}{\to} x_0$. In other words, in terms of neighborhoods, it can be said this way: for all $\varepsilon > 0$, an $n_0 \in \mathbb{N}$ exists such that $x_n \in \mathscr{N}_{\varepsilon(0)}^{x_0}$, for $n > n_0$.
The lacunary sequence, which has been studied in many different spaces in the theory of summability in recent years, is well known in the literature, and its convergence types rather than its basic definition will be noted here.

Then after this, A lacunary sequence will be taken as $\theta = \{k_r\}$. For the sequence $(x_n) \subset \mathscr{X}$, if there is an $\ell \in \mathscr{X}$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left(\sum_{k\in I_r} \mathscr{D}\left(\|x_k - \ell\|, \tilde{0}\right) \right) = 0$$

holds, then it is lacunary summable to ℓ.
If for every $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \mathscr{D}\left(\|x_k - \ell\|, \tilde{0}\right) \geq \varepsilon\right\}\right| = 0$ holds then the sequence $(x_n) \subset \mathscr{X}$ is lacunary st-convergent to ℓ, briefly $x_n \stackrel{FS_\theta}{\longrightarrow} \ell$.
The concepts of invariant mean and invariant convergence types are studied in many different spaces in summability theory and are well-known in the literature. Here, rather than its basic definition, some types of convergence, especially in fuzzy normed spaces, will be noted.
Now, let

$$t_{mn} = \frac{x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m}.$$

The bounded sequence $(x_n) \subset \mathscr{X}$ is invariant convergent to the ℓ iff $\lim_{m\to\infty} t_{mn} = \ell$ uniformly in n, namely $(\mathscr{D}) - \lim_{m\to\infty} \|t_{mn} - \ell\| = \tilde{0}$, uniformly in n, that is, there exists an $m_0 \in \mathbb{N}$ for every $\varepsilon > 0$ such that

$$\mathscr{D}(\|t_{mn} - \ell\|, \tilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - \ell\|_\alpha^+ = \|t_{mn} - \ell\|_0^+ < \varepsilon,$$

for all $m > m_0$ and every $n \in \mathbb{N}$, in other words, in terms of neighborhoods, it can be said this way: There exists an $m_0 \in \mathbb{N}$ for every $\varepsilon > 0$ such that $t_{mn} \in \mathscr{N}^{\ell}_{\varepsilon(0)}$ for all $m > m_0$ and every $n \in \mathbb{N}$. For this convergence, we write $x_n \stackrel{\sigma-FN}{\longrightarrow} \ell$.
If for every $\varepsilon > 0$, $\lim_{m\to\infty} \frac{1}{m} \left|\left\{k \leq m : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| = 0$, uniformly in n, then the sequence $(x_n) \subset \mathscr{X}$ is invariant statistical convergent to ℓ and we write $x_n \stackrel{S_\sigma FN}{\longrightarrow} \ell$. The set of sequences that have this convergence is denoted by $S_\sigma FN$.
Any $(x_n) \subset X$ is lacunary invariant convergent to ℓ and denoted by $x_n \stackrel{\sigma-FN_\theta}{\longrightarrow} \ell$ iff

$$\lim_{r\to\infty} \mathscr{D}\left(\left\| \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(n)} - \ell \right\|, \tilde{0} \right) = \lim_{r\to\infty} \left\| \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(n)} - \ell \right\|_0^+ = 0,$$

uniformly in n. By $\sigma - FN_\theta$, we show the set of sequences have this convergence.
Any $(x_n) \subset \mathscr{X}$ is strongly lacunary invariant convergent to ℓ and denoted by $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} \ell$ iff

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \mathscr{D}\left(\left\|x_{\sigma^k(n)} - \ell\right\|, \tilde{0} \right) = \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left\|x_{\sigma^k(n)} - \ell\right\|_0^+ = 0,$$

uniformly in n. By $[\sigma - FN]_\theta$, we show the set of sequences have this convergence.

2. Main Results

First, in the beginning of the original part of our study, we want to give $S_{\sigma\theta}FN$-convergence and $S_\sigma FN_\theta$-convergence, which have not been defined in the literature before.

Definition 2.1. *For a sequence $(x_n) \subset \mathscr{X}$, if for every $\varepsilon > 0$ and uniformly in n,*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \mathscr{D}\left(\left\|x_{\sigma^k(n)} - \ell\right\|, \tilde{0} \right) \geq \varepsilon \right\}\right| = \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \left\|x_{\sigma^k(n)} - \ell\right\|_0^+ \geq \varepsilon \right\}\right| = 0,$$

then the sequence $(x_n) \subset \mathscr{X}$ is lacunary invariant statistically convergent to ℓ and we write $x_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell$. The set of sequences that have this convergence is denoted by $S_{\sigma\theta}FN$.

Definition 2.2. *Let*

$$t_{rn} = \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(n)}.$$

For a sequence $(x_n) \subset \mathscr{X}$, if for every $\varepsilon > 0$ and uniformly in n,

$$\lim_{m\to\infty} \frac{1}{m} \left|\left\{r \leq m : \mathscr{D}\left(\|t_{rn} - \ell\|, \tilde{0}\right) \geq \varepsilon\right\}\right| = \lim_{m\to\infty} \frac{1}{m} \left|\left\{r \leq m : \|t_{rn} - \ell\|_0^+ \geq \varepsilon\right\}\right| = 0,$$

therefore, the sequence $(x_n) \subset \mathscr{X}$ is statistically lacunary invariant convergent to ℓ and we write $x_n \stackrel{S_\sigma FN_\theta}{\longrightarrow} \ell$. The set of sequences that have this convergence is denoted by $S_\sigma FN_\theta$.

Now we will give the theorem examining the relations between $[\sigma - FN]_\theta$ and $S_{\sigma\theta}FN$ with its proof.

Theorem 2.3. *For $0 < q < \infty$ and a sequence $(x_n) \subset \mathscr{X}$, the followings hold:*
(i) *If $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} \ell$, then $x_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell$.*
(ii) *If (x_n) is bounded sequence and $x_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell$, then $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} \ell$.*

Proof. **(i)** According to our assumption, uniformly in n, we get

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=0.$$

For every $\varepsilon>0$ and n, from the following inequality

$$\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geq\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_{\sigma^k(n)}-\ell\|_0^+\geq\varepsilon}}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+$$

$$\geq\frac{1}{h_r}\varepsilon\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|,$$

we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|=0,$$

uniformly in n. Thus, (x_n) is lacunary invariant statistically convergent to ℓ.
(ii) Let's presume that the bounded sequence $(x_n)\subset\mathscr{X}$ is lacunary invariant statistically convergent to ℓ. So, an $M>0$ exists such that

$$\|x_{\sigma^k(n)}-\ell\|_0^+<M$$

for every $k\in\mathbb{N}$ and all $n\in\mathbb{N}$. Also we have for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|=0,$$

uniformly in n. We know

$$\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_{\sigma^k(n)}-\ell\|_0^+\geq\varepsilon}}\left\|x_{\sigma^k(n)}-\ell\right\|_0^++\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_{\sigma^k(n)}-\ell\|_0^+<\varepsilon}}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+$$

$$\leq\frac{1}{h_r}M\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|+\varepsilon,$$

for every n. Therefore, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=0,$$

uniformly in n. Hence, (x_n) is strongly lacunary invariant convergent to ℓ. □

Now, we will prove the theorem about the uniqueness of the limit.We will now prove the theorem about the uniqueness of the limit, which has an important place in summability theory.

Theorem 2.4. *Let $(x_n)\subset\mathscr{X}$ be a sequence. If $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell$, in this case ℓ is unique.*

Proof. Let's presume that $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell_1$, $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell_2$ and $\ell_1\neq\ell_2$. Then for any given $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_1\right\|_0^+\geqslant\frac{\varepsilon}{2}\right\}\right|=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_2\right\|_0^+\geqslant\frac{\varepsilon}{2}\right\}\right|=0,$$

uniformly in n. Put

$$N_1^r=\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_1\right\|_0^+<\frac{\varepsilon}{2}\right\}\ \text{ and }\ N_2^r=\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_2\right\|_0^+<\frac{\varepsilon}{2}\right\}.$$

We know as $r\to\infty$,

$$\frac{\left|N_1^r\cap I_r\right|}{h_r}\to 1\text{ and }\frac{\left|N_2^r\cap I_r\right|}{h_r}\to 1. \tag{2.1}$$

Since $\ell_1\neq\ell_2$, $\|\ell_1-\ell_2\|_0^+\geq\varepsilon$ for some $\varepsilon>0$.
Obviously,

$$N_1^r\cap N_2^r=\emptyset\ \text{ and }\ N_1^r\cup N_2^r\subseteq I_r. \tag{2.2}$$

We can write

$$(N_1^r \cap I_r) \cup (N_2^r \cap I_r) = (N_1^r \cup N_2^r) \cap I_r \subseteq I_r$$

and

$$\frac{|N_1^r \cap I_r|}{h_r} + \frac{|N_2^r \cap I_r|}{h_r} \leq \frac{|I_r|}{h_r}, \text{ from (2.2)}.$$

Because of (2.1) we obtain

$$1 + 1 \leq 1 \text{ as } r \to \infty$$

which is the contradiction. Therefore, $\ell_1 = \ell_2$. □

Now we will give the theorems examining the linearity properties of lacunary invariant statistical convergence and their proofs. We will give these properties in two parts in the following theorem.

Theorem 2.5. *Let $x = (x_n)$, $y = (y_n)$ be sequences in $\mathscr{X}$ and assume that $x_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell_1$ and $y_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell_2$. In this case, we obtain the following hypotheses:*
(i) $x_n + y_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell_1 + \ell_2$,
(ii) $(cx_n) \stackrel{S_{\sigma\theta}FN}{\longrightarrow} c\ell_1$ *where c is a scaler.*

Proof. **(i)** Let's presume that $x_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell_1$ and $y_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell_2$. Then, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| y_{\sigma^k(n)} - \ell_2 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| = 0,$$

uniformly in n. From the triangle inequality,

$$\left\| \left(x_{\sigma^k(n)} + y_{\sigma^k(n)} \right) - (\ell_1 + \ell_2) \right\|_0^+ \leq \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ + \left\| y_{\sigma^k(n)} - \ell_2 \right\|_0^+,$$

for any given $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \left(x_{\sigma^k(n)} + y_{\sigma^k(n)} \right) - (\ell_1 + \ell_2) \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| \\ &\leq \frac{1}{h_r} \left| \left\{ k \in I_r : \| x_{\sigma^k(n)} - \ell_1 \|_0^+ + \| y_{\sigma^k(n)} - \ell_2 \|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| \\ &\leq \frac{1}{h_r} \left| \left\{ k \in I_r : \| x_{\sigma^k(n)} - \ell_1 \|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| + \frac{1}{h_r} \left| \left\{ k \in I_r : \| y_{\sigma^k(n)} - \ell_2 \|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

So, we concluded that

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \left(x_{\sigma^k(n)} + y_{\sigma^k(n)} \right) - (\ell_1 + \ell_2) \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| = 0,$$

that is,

$$x_n + y_n \stackrel{S_{\sigma\theta}FN}{\longrightarrow} \ell_1 + \ell_2.$$

(ii) Let c be a scaler. From the inequality

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \left\| c x_{\sigma^k(n)} - c\ell_1 \right\|_0^+ \geqslant \varepsilon \right\} \right| \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ \geqslant \frac{\varepsilon}{|c|} \right\} \right|,$$

we obtain

$$(cx_n) \stackrel{S_{\sigma\theta}FN}{\longrightarrow} cL.$$

□

We give the following lemma without the proof. It can be proved like in [23].

Lemma 2.6. *Let $(x_n) \subset \mathscr{X}$ be a sequence. Presume for given $\varepsilon_1 > 0$ and for all $\varepsilon > 0$, n_0 and m_0 exist such that for all $n \geq n_0$ and $m \geq m_0$,*

$$\frac{1}{m} \left| \left\{ k \leq m : \| x_{\sigma^k(n)} - \ell \|_0^+ \geq \varepsilon \right\} \right| < \varepsilon_1,$$

then (x_n) is invariant statistical convergent to ℓ.

Finally, we will show the relation between invariant statistical convergence and lacunary invariant statistical convergence with the following theorem.

Theorem 2.7. *$S_{\sigma\theta}FN = S_\sigma FN$ for every lacunary sequence θ.*

Proof. Let the sequence $(x_n) \in S_{\sigma\theta}FN$. According to definition, for all $\varepsilon > 0$ and for any $\varepsilon_1 > 0$, r_0 and ℓ exist such that

$$\frac{1}{h_r}\left|\left\{0 < k \leq h_r : \left\|x_{\sigma^k(n)} - \ell\right\|_0^+ \geq \varepsilon\right\}\right| < \varepsilon_1,$$

for $r \geq r_0$ and $n = \sigma^{k_r - 1}(n')$ and $n' \geq 0$. Let $m \geq h_r$, write $m = th_r + s$ where $0 \leq s \leq h_r$ and t is a integer. Since $m \geq h_r$, $t \geq 1$. Now

$$\begin{aligned}
&\frac{1}{m}\left|\left\{0 < k \leq m : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{m}\left|\left\{0 < k \leq (t+1)h_r : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{m}\sum_{i=1}^{t}\left|\left\{ih_r < k \leq (i+1)h_r : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{m}(t+1)\, h_r \varepsilon_1 \\
&\leq \frac{2th_r\varepsilon_1}{m}, \quad (t \geq 1)
\end{aligned}$$

for $\frac{h_r}{m} \leq 1$ and since $\frac{th_r}{m} \leq 1$,

$$\frac{1}{m}\left|\left\{0 < k \leq m : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \leq 2\varepsilon_1.$$

Then by Lemma, $S_{\sigma\theta}FN \subseteq S_\sigma FN$. Also, obviously $S_\sigma FN \subseteq S_{\sigma\theta}FN$. We concluded that $S_{\sigma\theta}FN = S_\sigma FN$. □

3. Conclusion

In the Fuzzy normed spaces, using the lacunary sequence, we introduce some new concepts in summability. In this sense, firstly, we define the lacunary invariant statistical convergence. Then, we examine some characteristic features like uniqueness, linearity of this new notion and give its important relation with pre-given concepts. In the future, these studies are also debatable in terms of regularly convergence for double sequences.

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References

[1] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361-375.
[2] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2**(3-4) (1951), 241-244.
[3] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**(2) (1980), 139-150.
[4] J. A. Fridy, *On statistical convergence*, Analysis, **5**(4) (1985), 301-314.
[5] J. S. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis, **8**(1-2) (1988), 47-64.
[6] A. R. Freedman, J. J. Sember, M. Raphael, *Some Cesàro-type summability spaces*, Proc. Lond. Math. Soc., **3**(3) (1978), 508-520.
[7] G. Das, B. K. Patel, *Lacunary distribution of sequences*, Indian J. Pure Appl. Math., **20**(1) (1989), 64-74.
[8] J. A. Fridy, C. Orhan, *Lacunary statistical convergence*, Pacific. J. Math., **160** (1993), 43-51.
[9] J. A. Fridy, C. Orhan *Lacunary statistical summability*, J. Math. Anal. Appl., **173**(2) (1993), 497-504.
[10] U. Ulusu, F. Nuray, *Lacunary statistical convergence of sequences of sets*, Progr. Appl. Math., **4**(2) (2012), 99-109.
[11] U. Ulusu, F. Nuray, *Statistical lacunary summability of sequences of sets*, AKU J. Sci. Eng., **13** (2013), 9-14.
[12] U. Ulusu, F. Nuray, *On strongly lacunary summability of sequences of sets*, J. Appl. Math. Bioinform., **3**(3) (2013), 75-88.
[13] S. Banach, *Théorie des Opérations Linéaires*, Warszawa, 1932.
[14] G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math., **80** (1948), 167-190.
[15] D. Dean, R. A. Raimi, *Permutations with comparable sets of invariant means*, Duke Math. J., **27** (1960), 467-479.
[16] R. A. Raimi, *Invariant means and invariant matrix methods of summability*, Duke Math. J., **30** (1963), 81-94.

[17] M. Mursaleen, O. H. H. Edely, *On the invariant mean and statistical convergence*, Appl. Math. Lett., **22**(11) (2009), 1700-1704.
[18] M. Mursaleen, *On some new invariant matrix methods of summability*, Q. J. Math., **34**(1) (1983), 77-86.
[19] E. Savaş, *Some sequence spaces involving invariant means*, Indian J. Math., **31** (1989), 1-8.
[20] E. Savaş, *Strong σ-convergent sequences*, Bull. Calcutta Math., **81** (1989), 295-300.
[21] P. Schaefer, *Infinite matrices and invariant means*, Proc. Mer. Math. Soc., **36** (1972), 104-110.
[22] E. Savaş, *On lacunary strong $\sigma-$convergence*, Indian J. Pure Appl. Math., **21** (1990), 359-365.
[23] E. Savaş, F. Nuray, *On σ-statistically convergence and lacunary σ-statistically convergence*, Math. Slovaca, **43**(3) (1993), 309-315.
[24] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338-353.
[25] D. Dubois, H. Prade, *Operations on fuzzy numbers*, Internat. J. Systems Sci., **9**(6) (1978), 613-626.
[26] D. Dubois, H. Prade, *Fuzzy real algebra: some results*, Fuzzy Sets and Systems, **2**(4) (1979), 327-348.
[27] C. L. Chang, *Fuzzy topolojical spaces*, J. Math. Anal. Appl., **24**(1) (1968), 182-190.
[28] C. K. Wong, *Covering properties of fuzzy topological spaces*. J. Math. Anal. Appl., **43**(3) (1973), 697-704.
[29] C. K. Wong, *Fuzzy topology: product and quotient theorems*. J. Math. Anal. Appl., **45**(2) (1974), 512-521.
[30] O. Kaleva, S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Sytems, **12**(3) (1984), 215-229.
[31] I. Kramosil, J. Michálek, *Fuzzy metrics and statstical metric spaces*, Kybernetika, **11**(5) (1975), 336-344.
[32] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48**(2) (1992), 293-248.
[33] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems, **12**(2) (1984), 143-154.
[34] C. Şençimen, S. Pehlivan, *Statistical convergence in fuzzy normed linear spaces*, Fuzzy Sets and Systems, **159** (2008), 361-370.
[35] M. R. Türkmen, M. Çınar, *Lacunary statistical convergence in fuzzy normed linear spaces*, Appl. Comput. Math., **6**(5) (2017), 233-237.
[36] M. R. Türkmen, M. Çınar, *λ-statistical convergence in fuzzy normed linear spaces*, J. Intell. Fuzzy Syst., **34**(6) (2018), 4023-4030.
[37] M. R. Türkmen, E. Dündar, *On lacunary statistical convergence of double sequences and some properties in fuzzy normed spaces*, J. Intell. Fuzzy Syst., **36**(2) (2019), 1683-1690.
[38] Ş. Yalvaç, E. Dündar, *Invariant convergence in fuzzy normed spaces*, Honam Math. J., **43**(3) (2021), 433-440.
[39] Ş. Yalvaç, E. Dündar, *Lacunary strongly invariant convergence in fuzzy normed spaces*, Math. Sci. Appl. E-Notes, **11**(2) (2023), 89-96.