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The Essential Gronwall Inequality Demands the $(\rho,\varphi)-$Fractional Operator with Applications in Economic Studies

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Abstract

Gronwall's inequalities are important in the study of differential equations and integral inequalities. Gronwall inequalities are a valuable mathematical technique with several applications. They are especially useful in differential equation analysis, stability research, and dynamic systems modeling in domains spanning from science and math to biology and economics. In this paper, we present new generalizations of Gronwall inequalities of integral versions. The proposed results involve $(\rho,\varphi)-$Riemann-Liouville fractional integral with respect to another function. Some applications on differential equations involving $(\rho,\varphi)-$Riemann-Liouville fractional integrals and derivatives are established.

1. Introduction

In recent years, fractional calculus has been applied to real and complex domains, like physics and engineering, see for instance [1–3], also in chemistry and biology, see for example [4]. It has also been used in relaxation-oscillation phenomena and diffusion vibrations, see the research papers [5–8]. Fractional calculus has some investigations in mechanical systems, we invite the reader to consult the articles [9–12]. Such applications have motivated researchers to work on new fractional calculus theories(see [13–15]. The generalized Riemann-Liouville fractional operators are a class of integral operators that extend the classical Riemann-Liouville fractional integral and derivative operators [16–20]. They are defined for functions that are not necessarily differentiable, but satisfy certain integrability conditions [21–23]. The generalized Riemann-Liouville fractional operators have many properties and applications in mathematics and physics, including fractional differential equations, fractional calculus, and signal processing. They also have connections to other areas of mathematics, such as complex analysis and number theory [24–30].

In the same way, in the present paper, we shall discuss some fractional integral variants of the well-known Gronwall integral inequality and some applications on differential equations that involve "new fractional order derivatives". The Gronwall inequality is a fundamental result in the theory of differential equations. It provides a bound on the growth of a function that satisfies a certain type of differential inequality [31–35]. The Gronwall inequality is often used to study the existence, uniqueness, and stability of solutions to differential equations. It is named after the Swedish mathematician T.H. Gronwall, who first proved it in 1919. Before presenting our results, we need to present to the reader some motivated papers for the present paper. We recall the paper [36] where a generalized Gronwall-type inequality involving Riemann-Liouville derivatives has been considered. Then, in [18], the Gronwall inequality has been proved and some applications for differential equations of the hybrid type, that involve Hadamard derivatives, have been established. Other types of inequalities have also been considered in [37–40].

The fractional Gronwall inequality is a generalization of the classical Gronwall inequality, which is a fundamental result in the theory of ordinary differential equations. It provides an estimate on the growth of a function in terms of an integral involving the function and

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its derivatives. The fractional Gronwall inequality has important applications in the study of fractional differential equations, which are differential equations involving fractional derivatives. It can be used to establish existence, uniqueness, and stability results for solutions of such equations. The main aim of this work is to establish generalizations for Gronwall inequality by applying fractional integrals with respect to another function. Also, our aim is to establish sufficient guaranteeing conditions on the boundedness of solutions for some classes of differential equations involving "$(\rho,\varphi)-$Riemann-Liouville derivatives".

The structure of this research paper is given as follows: In Section 2, we give some used preliminaries. In Section 3, our integral results are proved. In Section 4, we continue with the main results; two classes of differential equations, in the sense of "$(\rho,\varphi)-$Riemann-Liouville", are studied. At the end, a conclusion follows.

2. Preliminaries

We introduce some used preliminaries [6,9,12]. In particular, the generalized fractional derivatives in the sense of $(\rho,\varphi)-$Riemann-Liouville involving (or with respect to) the function φ are introduced, for the first time, in this section.
We begin by noticing that the classical form of the inequality of Gronwall says that if a positive function u over $I:=[t_0,T);T\leq\infty$, that satisfies the inequality

$$u(z)\leq f(z)+\int_{t_0}^{z}L(t)\,u(t)\,dt,\ \ z\in[t_0,T),$$

where f is a continuous function on $[t_0,T)$, and $L(t)\geq 0$ over the same interval,
then, one has the following result:

Lemma 2.1. *We have*

$$u(z)\leq f(z)+\int_{t_0}^{z}f(t)\,L(t)\exp\left(\int_{t}^{z}L(\tau)\,d\tau\right)dt,\ \ t_0\leq z<T.$$

The following Lemma is also needed in the present work.

Lemma 2.2. *(Jensen) If we take $n\in\mathbb{N}^*$, and also the nonnegative numbers $r_1,...,r_n$, then, for $m>1$,*

$$\left(\sum_{i=1}^{n}r_i\right)^m\leq n^{m-1}\sum_{i=1}^{n}r_i^m.$$

We are also concerned with the following auxiliary result:

Lemma 2.3. *Let $T\leq\infty, I=[t_0,T)\subset\mathbb{R},\ \ f,g,q\in C(I,\mathbb{R}_+)$. We suppose also that for $u\in C(I,\mathbb{R}_+)$, the inequality holds*

$$u(x)\leq f(x)+\int_{t_0}^{x}g(t)\,u(t)\,dt+\int_{t_0}^{x}q(t)\,u^{\gamma}(t)\,dt,\ \ x\in I,$$

with $0\leq\gamma<1$.
Hence, for any $x\in I$, the inequality

$$u(x)\leq\left[F^{1-\gamma}(x)+(1-\gamma)\int_{t_0}^{x}p(t)\exp\left((\gamma-1)\int_{t_0}^{t}g(\tau)\,d\tau\right)dt\right]^{\frac{1}{1-\gamma}}\times\exp\left(\int_{t_0}^{x}g(t)\,dt\right),$$

is valid, such that $F(x)=\max\limits_{t_0\leq t\leq x}f(t)$.

Under the same interval I, the following estimate of u holds.

Theorem 2.4. *Let consider the nonnegative continuous functions $u,f,g,k_i,i\in\{1,...,n\}$ and suppose there are some positive real numbers $r_1,r_2,...,r_n$. If u satisfies the estimate:*

$$u^r(x)\leq g(x)+f(x)\int_0^x\sum_{i=1}^{n}k_i(t)\,u^{r_i}(t)\,dt,\ x\in I,$$

then, we have

$$u(x)\leq\left[f(x)+g(x)\int_0^x\sum_{i=1}^{n}\%k_i(t)\left(\frac{r_i}{r}f(t)+\frac{r-r_i}{r}\right)\times\exp\left(\int_{\sigma}^{x}g(\tau)\sum_{\%i=1}^{n}\frac{r_i}{r}k_i(\tau)\,d\tau\right)dt\right]^{\frac{1}{r}},$$

for $r\geq\max\{r_i,i=1,...,n\}$.

Now, we recall the following $(\rho,\varphi)-$Riemann-Liouville fractional integrals of a function f on $[a,b]$ with respect to φ, see the paper of M Bezziou et al. [6]:

$${}_{\rho}I^{\alpha}_{a+,\varphi}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,f(t)dt,$$

and

$${}_{\rho}I^{\alpha}_{b-,\varphi}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\varphi^{\rho}(t)-\varphi^{\rho}(x)\right)^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,f(t)dt,$$

where $\alpha,\rho>0$.
Let us now pass to introduce, for the first time, new generalized fractional derivatives in the sense of $(\rho,\varphi)-$Riemann-Liouville with respect to the function φ. We define the proposed derivatives as follows:

Definition 2.5. *Consider $\alpha>0$ and take f as an integrable function over $[a,b]$. If $\varphi\in C^1([a,b],\mathbb{R})$ is an increasing function, such that $\varphi'(x)\varphi^{\rho-1}(x)\neq 0, \forall x\in[a,b], \rho>0$,*
then the left-sided (respectively), the right-sided $(\rho,\varphi)-$ Riemann-Liouville fractional derivative of order α is, respectively, defined by:

$$\rho D^{\alpha}_{a+,\varphi}f(x)=\left(\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\%\right)^n \rho I^{n-\alpha}_{a+,\varphi}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\right)^n\int_a^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{n-\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,f(t)dt,$$

and

$$\rho D^{\alpha}_{b-,\varphi}f(x)=\left(-\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\%\right)^n \rho I^{n-\alpha}_{b-,\varphi}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\right)^n\int_x^b(\varphi^{\rho}(t)-\varphi^{\rho}(x))^{n-\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,f(t)dt,$$

where $n=[\alpha]+1$.

Remark 2.6. *Several existing operators can be derived from Definition 2.5 as follows:*

(i) *Letting $\rho=1$ and $\varphi(x)=x$, thus we can obtain the definition of Riemann-Liouville derivative [22, 34].*

(ii) *Letting $\rho=1$ and $\varphi(x)=\ln(x)$, hence, we can get the definition of Hadamard derivative [22, 34].*

(iii) *Letting $\rho=1$ and $\varphi(x)=\dfrac{x^{\nu+1}}{\upsilon+1}$, where $\upsilon\neq-1$ is a real number, hence, we can obtain the definition of the Katugampola derivative proposed in [10, 21].*

We pass to prove the following important two properties

Theorem 2.7. *Consider $0<\alpha<1$, and take f as an integrable function over $[a,b]$. If $\varphi\in C^1([a,b],\mathbb{R})$ is increasing, such that $\varphi'(x)\varphi^{\rho-1}(x)\neq 0, \forall x\in[a,b], \rho>0$, then, the following two properties:*

$$\left(_{\rho}D^{\alpha}_{a+,\varphi}\ _{\rho}I^{\alpha}_{a+,\varphi}\right)f(x)-f(x)=0 \tag{2.1}$$

and

$$\left(_{\rho}D^{\alpha}_{b-,\varphi}\ _{\rho}I^{\alpha}_{b-,\varphi}\right)f(x)-f(x)=0 \tag{2.2}$$

hold.

Proof. We begin by proving the left-sided fractional operator (2.1)
Thanks to the Fubini theorem, we can write

$$\begin{aligned}
&\left(_{\rho}D^{\alpha}_{a+,\varphi}\ _{\rho}I^{\alpha}_{a+,\varphi}\right)f(x)=_{\rho}D^{\alpha}_{a+,\varphi}\left(_{\rho}I^{\alpha}_{a+,\varphi}f(x)\right)=\tfrac{1}{\Gamma(1-\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\dfrac{\varphi\prime(t)\,\varphi^{\rho-1}(t)}{(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha}}\left(_{\rho}I^{\alpha}_{a+,\varphi}f(t)\right)dt\\
&=\tfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\left[\tfrac{\varphi\prime(t)\varphi^{\rho-1}(t)}{(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha}}\times\int_a^t(\varphi^{\rho}(t)-\varphi^{\rho}(s))^{\alpha-1}\varphi\prime(s)\,\varphi^{\rho-1}(s)\,f(s)ds\right]dt\\
&=\tfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\left[\varphi\prime(s)\,\varphi^{\rho-1}(s)\,f(s)\times\int_s^x\tfrac{(\varphi^{\rho}(t)-\varphi^{\rho}(s))^{\alpha-1}}{(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha}}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,dt\right]ds\\
&=\tfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\varphi\prime(s)\,\varphi^{\rho-1}(s)\,f(s)ds\times\Gamma(1-\alpha)\Gamma(\alpha)\\
&=f(x).
\end{aligned}$$

Notice here that to achieve the proof of (2.1), we can use the transformation:

$$u:=\frac{\varphi^{\rho}(t)-\varphi^{\rho}(s)}{\varphi^{\rho}(x)-\varphi^{\rho}(s)}.$$

The proof of (2.2) can be achieved by using the same arguments as in the proof of (2.1). □

3. Results

We have first to present the following estimate for the continuous positive function u.

Theorem 3.1. *Consider $\alpha>0$ and γ in $]0,1[$. Then, take f,g and p in $C(I,\mathbb{R}_+)$. If $u\in C(I,\mathbb{R}_+)$ satisfies*

$$u(x)\leq f(x)+\int_{t_0}^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,g(t)u(t)dt+\int_{t_0}^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,p(t)u^{\gamma}(t)dt, t_0<x, \tag{3.1}$$

then the following two inequalities are valid: (i) If $\alpha-\dfrac{1}{2}>0$, then, we have

$$\begin{aligned}
u(x)\leq&\left[F_1^{1-\gamma}(x)+(1-\gamma)B_1\textstyle\int_{t_0}^x\exp\left((\gamma-1)B_1\int_{t_0}^t g^2(z)\,dz\right)\times p^2(t)\,\varphi\prime(t)\,\varphi^{\rho-1}(t)\exp\left((2\gamma-2)\,\varphi^{\rho}(t)\right)dt\right]^{\frac{1}{2(1-\gamma)}}\\
&\times\exp\left(\left(\varphi^{\rho}(x)+\tfrac{B_1}{2}\%\right)\textstyle\int_{t_0}^x g^2(t)\,\varphi\prime(t)\,\varphi^{\rho-1}(t)\,dt\right),\ \mathrm{t}\in I,
\end{aligned}\tag{3.2}$$

where $F_1(x) = \max_{t_0 \le t \le x} 3e^{-2\varphi^\rho(t)} f^2(t)$, *and* $B_1 = \dfrac{6\Gamma(2\alpha-1)}{\rho 4^\alpha}$.

(ii) *In the case where* α *is in* $\left(0, \frac{1}{2}\right]$. *If* $0 = q - \dfrac{1+\alpha}{\alpha}, 0 = -p+1+\alpha$, *then one has*

$$u(x) \le \left[F_2^{1-\gamma}(x) + (1-\gamma)B_2 \int_{t_0}^x \exp\left((\gamma-1)B_2 \int_{t_0}^t b^q(\tau)\,d\tau\right) \times p^q(t)\,\varphi\prime(t)\,\varphi^{\rho-1}(t) \exp\left(q(\gamma-1)\varphi^\rho(t)\right) dt\right]^{\frac{1}{q(1-\gamma)}} \times \exp\left(\left(\varphi^\rho(x) + \tfrac{B_2}{q}\%\right) \int_{t_0}^x g^q(t)\,\varphi\prime(t)\,\varphi^{\rho-1}(t)\,dt\right), \tag{3.3}$$

where $F_2(x) = \max_{t_0 \le t \le x} 3^{q-1} e^{-q\varphi^\rho(t)} f^q(t)$, *and* $B_2 := 3^{-1}3^q \left(\dfrac{\Gamma((p\alpha-p)+1)}{\rho p^{(\alpha p-p)+1}}\right)^{\frac{q}{p}}$.

Proof. Taking $x \in I$, we obtain:

$$u(x) - f(x) \le \int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{\alpha-1} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,g(t)u(t)dt + \int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{\alpha-1} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,p(t)u^\gamma(t)dt. \tag{3.4}$$

(i) Using Cauchy-Schwarz inequality to (3.4), we get

$$\begin{aligned}
u(x) - f(x) &\le \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{(2\alpha-2)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{2\varphi^\rho(t)}dt\right)^{1/2} \\
&\times \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,(gu)^2 dt\right)^{1/2} \\
&+ \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{(2\alpha-2)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{2\varphi^\rho(t)}dt\right)^{1/2} \\
&\times \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,p^2(t)u^{2\gamma}(t)dt\right)^{1/2} \\
&\le \left(\tfrac{2\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right)^{1/2} \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,(gu)^2 dt\right)^{1/2} \\
&+ \left(\tfrac{2\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right)^{1/2} \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,(pu^\gamma)^2(t)dt\right)^{1/2},
\end{aligned} \tag{3.5}$$

where, $\alpha > \dfrac{1}{2}$.

Thanks to Lemma 2.2, with $m = 2, n = 3$, we observe that (3.5) is equivalent to:

$$\begin{aligned}
u^2(x) - 3f^2(x) &\le \left(\tfrac{6\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right) \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,g^2(t)u^2(t)dt\right) \\
&+ \left(\tfrac{6\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right) \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,u^{2\gamma}(t)p^2(t)dt\right).
\end{aligned} \tag{3.6}$$

We shall now consider $R(x)$ the quantity $\left(u(x)e^{-\varphi^\rho(x)}\right)^2$. Then, (3.6) can be transformed into:

$$R(x) \le F_1(x) + B_1 \left(\int_{t_0}^x \varphi^{\rho-1}(t)\,\varphi'(t)\,g^2(t)R(t)dt\right) + B_1 \left(\int_{t_0}^x e^{2(\gamma-1)\varphi^\rho(t)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,p^2(t)R^\gamma(t)dt\right).$$

As $F_1(x)$ is nondecreasing, then by Lemma 2.3, we observe that

$$R(x) \le \left[F_1^{1-\gamma}(x) + (1-\gamma)B_1 \left(\int_{t_0}^x e^{2(\gamma-1)\varphi^\rho(t)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,p^2(t) \times \exp\left((1-\gamma)B_1 \int_{t_0}^t \varphi^{\rho-1}(\tau)\,\varphi\prime(\tau)\,g^2(\tau)d\tau\right) dt\right]^{\frac{1}{1-\gamma}} \times \exp\left(B_1 \int_{t_0}^x \varphi^{\rho-1}(t)\,\varphi\prime(t)\,g^2(t)dt\right). \tag{3.7}$$

So, from (3.7), we obtain (3.2).

(ii) Now, for $\alpha \in \left(0, \frac{1}{2}\right]$, $q = \dfrac{\alpha+1}{\alpha}, p = \alpha+1$, and using Holder inequality on the two integrals of (3.1), we get:

$$\begin{aligned}
u(x) &\le f(x) + \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{p(\alpha-1)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,e^{p\varphi^\rho(t)}dt\right)^{\frac{1}{p}} \\
&\times \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,g^q(t)u^q(t)dt\right)^{\frac{1}{q}} \\
&+ \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{p(\alpha-1)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,e^{p\varphi^\rho(t)}dt\right)^{\frac{1}{p}} \\
&\times \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,p^q(t)u^{q\gamma}(t)dt\right)^{\frac{1}{q}} \\
&\le f(x) + \left(\tfrac{\Gamma(p(\alpha-1)+1)}{p^{p(\alpha-1)+1}} e^{p\varphi^\rho(x)}\right)^{\frac{1}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi^{\rho-1}(t)\,\varphi\prime(t)\,g^q(t)u^q(t)dt\right)^{\frac{1}{q}} \\
&+ \left(\tfrac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}} e^{p\varphi^\rho(x)}\right)^{\frac{1}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,p^q(t)u^{q\gamma}(t)dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.8}$$

In view of Lemma 2.2, (with $m = q$ and $n = 3$), and thanks to (3.8), we obtain

$$\begin{aligned}
u^q(x) &\le 3^{q-1} f^q(x) + 3^{q-1} \left(\tfrac{e^{p\varphi^\rho(x)}\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}\right)^{\frac{q}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,(gu)^q(t)dt\right) \\
&+ 3^{q-1} \left(\tfrac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}} e^{p\varphi^\rho(x)}\right)^{\frac{q}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\,\varphi^{\rho-1}(t)\,p^q(t)u^{q\gamma}(t)dt\right).
\end{aligned} \tag{3.9}$$

Let us now take $R(x)$ equal to the quantity $\left(u(x)e^{-\varphi^{\rho}(x)}\right)^{q}$. Then (3.9) implies that

$$R(x) \leq F_2(x) + B_2\left(\int_{t_0}^{x} \varphi'(t)\,\varphi^{\rho-1}(t)\,g^q(t)R(t)dt\right) + B_2\left(\int_{t_0}^{x} e^{q(\gamma-1)\varphi^{\rho}(t)}\varphi^{\rho-1}(t)\,p^q(t)\varphi\prime(t)\,R^{\gamma}(t)dt\right).$$

The function R is nondecreasing on $[t_0,T)$, from Lemma 2.3.
Thus, the reader can see that

$$\begin{aligned} R(x) \leq & \Big[F_2^{1-\gamma}(x) + (1-\gamma)B_2\left(\textstyle\int_{t_0}^{x} e^{q(\gamma-1)\varphi^{\rho}(t)}\varphi^{\rho-1}(t)\,\varphi\prime(t)\,p^q(t)\right. \\ & \times \exp\left((1-\gamma)B_2\left(\textstyle\int_{t_0}^{t}\varphi^{\rho-1}(\tau)\,g^q(\tau)\varphi\prime(\tau)\,d\tau\right)dt\right]^{\frac{1}{1-\gamma}} \\ & \times \exp\left(B_2\textstyle\int_{t_0}^{x}\varphi^{\rho-1}(t)\,g^q(t)\varphi\prime(t)\,dt\right). \end{aligned}$$

By the relations of $u(x)$ and $R(x)$, we conclude that (3.3) holds. □

Remark 3.2. (1) *When $\rho=1$ and $\varphi(x)=x$ on $[t_0,T)\subset\mathbb{R}$, the inequalities established in Theorem 3.1 can be transformed into the inequalities established in Theorem* 4 *given in [36].*

(2) *Taking $\rho=1$ and $\varphi(x)=\ln(x)$ on $[t_0,T), t_0\geq 1$, then the inequalities established in Theorem 3.1 become the inequalities established in Theorem* 3.1 *given in [18].*

The second main result to be presented to the reader is given by.

Theorem 3.3. *Let us take over the interval I the nonnegative and continuous functions u,f and $g_i, i\in\{1,2,..,n\}$.*
If

$$u(x)-f(x) \leq \textstyle\int_{t_0}^{x}\left(\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\right)\sum\limits_{i=1}^{n} g_i(t)\,u^{\gamma_i}(t)\,dt, \tag{3.10}$$

then we have the following cases: (i) If $\alpha>\dfrac{1}{2}$, then

$$\begin{aligned} u(x) \leq & \Bigg[2f^2(x) + 2\tfrac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\textstyle\int_{t_0}^{x}\sum\limits_{i=1}^{n} n\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-2\varphi^{\rho}(t)}g_i^2(t)\left[\gamma_i\left(2f^2(t)-1\right)+1\right] \\ & \times \exp\left(\textstyle\int_{t}^{x} 2\tfrac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(\tau)}\sum\limits_{i=1}^{n} n\varphi\prime(\tau)\,\varphi^{\rho-1}(\tau)\,e^{-2\varphi^{\rho}(\tau)}\gamma_i g_i^2(\tau)\,d\tau\right)dt\Bigg]^{\frac{1}{2}}. \end{aligned}$$

(ii) *If $\alpha\in\left(0,\frac{1}{2}\right]$, $0=-q+\dfrac{1+\alpha}{\alpha}, p-1-\alpha=0$, then we have*

$$\begin{aligned} u(x) \leq & \Bigg[2^{q-1}f^q(x) + 2^{q-1}\left(\frac{\Gamma\left(p(\alpha-1)+1\right)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{q}{p}} \\ & \times \textstyle\int_{t_0}^{x}\sum\limits_{i=1}^{n} n^{q-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-q\varphi(t)}g_i^q(t)\left[\gamma_i\left(2^{q-1}f^q(t)-1\right)+1\right] \\ & \times \exp\left(\textstyle\int_{t}^{x} 2^{q-1}\left(\tfrac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(\tau)}\right)^{\frac{q}{p}}\sum\limits_{i=1}^{n} n^{q-1}\varphi\prime(\tau)\,\varphi^{\rho-1}(\tau)\,e^{-q\varphi^{\rho}(\tau)}\gamma_i g_i^q(\tau)\,d\tau\right)dt\Bigg]^{\frac{1}{q}}. \end{aligned}$$

Proof. As $x\in[t_0,T)$, we get

$$u(x)-f(x) \leq \textstyle\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\varphi^{\rho-1}(t)\,\varphi\prime(t)\,e^{\varphi^{\rho}(t)}\sum\limits_{i=1}^{n} g_i(t)\,u^{\gamma_i}(t)\,e^{-\varphi^{\rho}(t)}dt.$$

(i) By employing Cauchy-Schwarz inequality and Lemma 2.2, we obtain:

$$\begin{aligned} u(x) \leq & f(x) + \left(\textstyle\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{2(\alpha-1)}\varphi^{\rho-1}(t)\,\varphi\prime(t)\,e^{2\varphi^{\rho}(t)}dt\right)^{\frac{1}{2}} \\ & \times\left(\textstyle\int_{t_0}^{x}\sum\limits_{i=1}^{n} n\varphi^{\rho-1}(t)\,\varphi\prime(t)\,e^{-2\varphi^{\rho}(t)}g_i^2(t)\,u^{2\gamma_i}(t)\,dt\right)^{\frac{1}{2}} \\ \leq & f(x) + \left(\tfrac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\right)^{\frac{1}{2}}\times\left(\textstyle\int_{t_0}^{x}\sum\limits_{i=1}^{n} n\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-2\varphi^{\rho}(t)}g_i^2(t)\,u^{2\gamma_i}(t)\,dt\right)^{\frac{1}{2}}. \end{aligned}$$

And using Lemma 2.3 for $m=2$, the above inequality becomes

$$u^2(x)-2f^2(x) \leq \left(2\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\right)\times\left(\int_{t_0}^{x}\sum_{i=1}^{n} n\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-2\varphi^{\rho}(t)}g_i^2(t)\,u^{2\gamma_i}(t)\,dt\right).$$

So,

$$u^{\tilde{p}}(x)-\tilde{f}(x) \leq \tilde{g}(x)\left(\textstyle\int_{0}^{x}\sum\limits_{i=1}^{n}\tilde{h}_i(t)\,u^{\tilde{p}_i}(t)\,dt\right),$$

where

$$\tilde{p}=2,\tilde{p}_i=2\gamma_i,\tilde{h}_i(x)=ng_i^2(x)\,\varphi\prime(x)\,\varphi^{\rho-1}(x)\,e^{-2\varphi^{\rho}(x)},\ \tilde{f}(x)=2f^2(x)$$

and

$$\tilde{g}(x)=2\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}.$$

Theorem 2.4 permits us to write

$$u(x)\leq\left[\tilde{f}(t)+\tilde{g}(t)\int_0^x\sum_{i=1}^n\tilde{h}_i(t)\left[\gamma_i\left(\tilde{f}\%(t)-1\right)+1\right]\times\exp\left(\int_t^x\tilde{g}(\tau)\sum_{i=1}^n\gamma_i\tilde{h}_i(\tau)\,d\tau\right)dt\right]^{\frac{1}{2}}.$$

(ii) Using Holder inequality, 3.10 allows us to write

$$\begin{aligned}
u(x)&\leq f(x)+\left(\int_{t_0}^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{p(\alpha-1)}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{p\varphi^{\rho}(t)}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_{t_0}^x\sum_{i=1}^n n^{q-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-q\varphi^{\rho}(t)}g_i^q(t)\,u^{q\gamma_i}(t)\,dt\right)^{\frac{1}{q}}\\
&\leq f(x)+\left(\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{1}{p}}\times\left(\int_{t_0}^x\sum_{i=1}^n n^{q-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-q\varphi^{\rho}(t)}g_i^q(t)\,u^{q\gamma_i}(t)\,dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{3.11}$$

The inequality (3.11) and Lemma 2.3 give us

$$u^q(x)\leq 2^{q-1}f^q(x)+2^{q-1}\left(\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{q\varphi^{\rho}(x)}\right)^{\frac{q}{p}}\times\left(\int_{t_0}^x\sum_{i=1}^n\%n^{q-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-q\varphi^{\rho}(t)}g_i^q(t)\,u^{q\gamma_i}(t)\,dt\right).$$

We consider

$$\tilde{p}=q,\ \tilde{p}_i=q\gamma_i,\ \tilde{h}_i(x)=n^{q-1}g_i^q(x)\,\varphi\prime(x)\,\varphi^{\rho-1}(x)\,e^{-q\varphi^{\rho}(x)},\tilde{f}(x)=2^{q-1}f^2(x)$$

and

$$\tilde{g}(x)=2^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{\rho\, p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{q}{p}},$$

then, we can write

$$u^{\tilde{p}}(x)\leq\tilde{f}(x)+\tilde{g}(x)\left(\int_0^x\sum_{i=1}^n\tilde{h}_i(t)\,u^{\tilde{p}_i}(t)\,dt\right).\tag{3.12}$$

Thus, from Theorem 2.4, we can conclude that

$$u(x)\leq\left[\tilde{f}(x)+\tilde{g}(x)\int_0^x\sum_{i=1}^{\%n}\tilde{h}_i(t)\left[\gamma_i\left(\tilde{f}\%(t)-1\right)+1\right]\times\exp\left(\int_t^x\tilde{g}(\tau)\overset{\%}{n}\sum_{i=1}\gamma_i\tilde{h}_i(\tau)\,d\tau\right)dt\%\right]^{\frac{1}{q}}.\tag{3.13}$$

□

4. Applications

Differential equations are commonly used in economics to represent the change of economic variables across time. These equations may define connections between variables like production, consumption, and investment. Gronwall's inequality may be used to investigate the behavior and stability of certain differential equation solutions. Assume we have a differential equation that defines the rate of variation of an identified economic variable and you want to assess the solution's long-term pattern or stability. Gronwall's inequality might be implemented to constrain the solution based on initial or boundary circumstances.

In this section, we will use the above "$(\rho,\varphi)-$ theorems" related to Gronwall inequality to investigate bounded solutions for two classes of fractional differential equations that involve $(\rho,\varphi)-$generalized derivatives with initial conditions.
Class 1: Suppose that we have:

$$\begin{cases}{}_{\rho}D_{t_0,\varphi}^{\alpha}u(x)=\psi(x,u(x))+h(x)u(x),\ \ t_0\leq x<T\leq\infty,\\ {}_{\rho}I_{t_0,\varphi}^{1-\alpha}u(x)\Big|_{x=t_0}=u_0,\end{cases}\tag{4.1}$$

where ${}_{\rho}D_{t_0,\varphi}^{\alpha}$, ${}_{\rho}I_{t_0,\varphi}^{1-\alpha}$ are respectively the $(\rho,\varphi)-$ Riemann-Liouville derivative of fractional order and $(\rho,\varphi)-$ Riemann-Liouville integral, $\rho>0,u_0\in\mathbb{R}$, with respect to $\varphi\in C^1([t_0,T),\mathbb{R}_+)$, $\varphi'(x)\,\varphi^{\rho-1}(x)\neq 0$, $\psi\in C([t_0,T)\times\mathbb{R},\mathbb{R})$ and $h\in C([t_0,T),\mathbb{R}_+)$.
From [37], we know that $u(x)$ satisfies (4.1) if $u(x)$ satisfies the equation:

$$u(x)=\frac{u_0}{\Gamma(\alpha)}(\varphi^{\rho}(x)-\varphi^{\rho}(t_0))^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha-1}\times\varphi'(t)\,\varphi^{\rho-1}(t)\,[\psi(t,u(t))+h(t)u(t)]\,dt.\tag{4.2}$$

We consider the following hypothesis:
(H_1) : There exist $g,p\in C([t_0,T),\mathbb{R}^+),0<\gamma<1$, and $|\psi(t,u(x))+h(x)u(x)|\leq g(x)\,|u(x)|+p(x)\,|u^{\gamma}(x)|$
is valid.
Under (H_1), we prove the following integral inequalities for the solution of the above differential problem of Class 1.

Theorem 4.1. *Assume that* (H_1) *is valid. If* u *is the solution of (4.1), then the following two inequalities are true:*
(i) *If* $\alpha > \frac{1}{2}$, *then, we have:*

$$\begin{aligned}|u(x)| \leq &\Big[\tilde{F}_1^{1-\gamma}(x) + (1-\gamma) B_1 \textstyle\int_{t_0}^{x} \exp\left((\gamma-1) B_1 \int_{t_0}^{t} g^2(\tau)\, d\tau\right) \\ &\times p^2(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t) \exp\left(2(\gamma-1)\varphi^{\rho}(t)\right) dt\Big]^{\frac{1}{2(1-\gamma)}} \\ &\times \exp\left(\left(\varphi^{\rho}(x) + \frac{B_1}{2}\right) \textstyle\int_{t_0}^{x} g^2(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t)\, dt\right), \quad x \in I,\end{aligned} \tag{4.3}$$

where $\tilde{F}_1(x) = \max\limits_{t_0 \leq t \leq x} 3e^{-2\varphi^{\rho}(t)} \left(\frac{|u_0|}{\Gamma(\alpha)}\right)^2 |\varphi^{\rho}(t) - \varphi^{\rho}(t_0)|^{2(\alpha-1)}$, *and* $B_1 = \dfrac{6\Gamma(2\alpha-1)}{\rho 4^{\alpha}}$.
(ii) *For* $\alpha \in \left(0, \frac{1}{2}\right]$, $q = \dfrac{1+\alpha}{\alpha}$, *and* $p = 1+\alpha$, *we have*

$$\begin{aligned}|u(x)| \leq &\Big[\tilde{F}_2^{1-\gamma}(x) + (1-\gamma) B_2 \textstyle\int_{t_0}^{x} \exp\left((\gamma-1) B_2 \int_{t_0}^{t} b^q(\tau)\, d\tau\right) \\ &\times p^q(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t) \exp\left(q(\gamma-1)\varphi^{\rho}(t)\right) dt\Big]^{\frac{1}{q(1-\gamma)}} \\ &\times \exp\left(\left(\varphi^{\rho}(x) + \frac{B_2}{q}\right) \textstyle\int_{t_0}^{x} g^q(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t)\, dt\right),\end{aligned} \tag{4.4}$$

where $\tilde{F}_2(x) = \max\limits_{t_0 \leq t \leq x} 3^{q-1} e^{-q\varphi^{\rho}(t)} \left(\frac{|u_0|}{\Gamma(\alpha)}\right)^q |\varphi^{\rho}(t) - \varphi^{\rho}(t_0)|^{q(\alpha-1)}$, *and* $B_2 = 3^{q-1}\left(\dfrac{\Gamma(p(\alpha-1)+1)}{\rho\, p^{p(\alpha-1)+1}}\right)^{\frac{q}{p}}$.

Proof. Let $x \in [t_0, T)$, then thanks to (H_1), we have

$$|u(x)| \leq \left|\frac{u_0}{\Gamma(\alpha)}\left(\varphi^{\rho}(x) - \varphi^{\rho}(t_0)\right)^{\alpha-1}\right| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{x} \left(\varphi^{\rho}(x) - \varphi^{\rho}(t)\right)^{\alpha-1} \times \varphi'(t)\, \varphi^{\rho-1}(t) \left(g(t)\,|u(t)| + p(t)\,|u^{\gamma}(t)|\right) dt.$$

Applying Theorem 3.1, we deduce the desired result. □

Let us now consider another class of differential equations.

Example 4.2. *Assume the following data:*

- *let* $\varphi(x) = x, u_0 = 0, (\psi + h)(x) = c, c \in \mathbb{R}$. *Then the solution* u *is given by the form, using Mathematica 13.3, as follows (see Figs.4.1 and 4.2)*

$$u(x) = \frac{c}{\Gamma(\alpha)} \frac{x^{\alpha\rho}}{\alpha\rho}, \quad x > 0, \alpha > 0, \rho > 0, c \in \mathbb{R}.$$

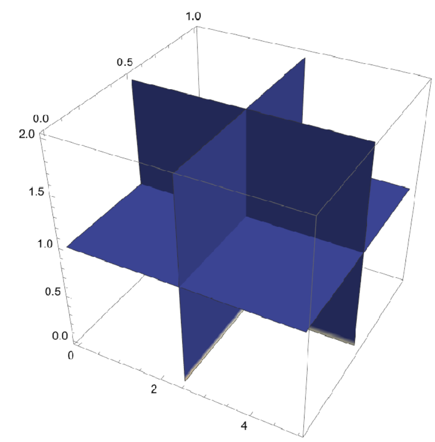

Figure 4.1: The SliceDensityPlot3D of $u(x) = \frac{c}{\Gamma(\alpha)} \frac{x^{\alpha\rho}}{\alpha\rho}$ for $(x, \alpha, \rho), c = 1$.

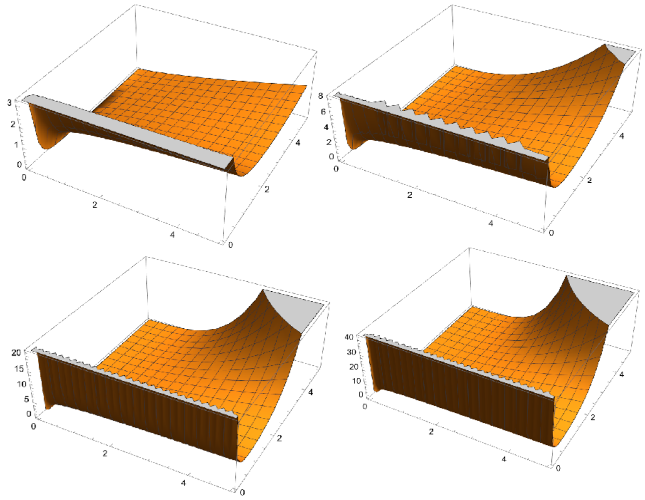

Figure 4.2: The Plot of $u(x)=\dfrac{c}{\Gamma(\alpha)}\dfrac{x^{\alpha\rho}}{\alpha\rho}$ for $(x,\rho), c=1$ and $\alpha=0.25,0.5,0.75,0.95$.

- *let* $\varphi(x)=exp(x), u_0=0, (\psi+h)(x)=c, c\in\mathbb{R}$. *Then the solution* u *is given by the form (see Fig.4.3 and 4.4),*

$$u(x)=\frac{c}{\Gamma(\alpha)}\frac{\left((e^x)^{\rho}-1\right)^{\alpha}}{\alpha\rho},\quad x>0,\alpha>0,\rho>0,c\in\mathbb{R}.$$

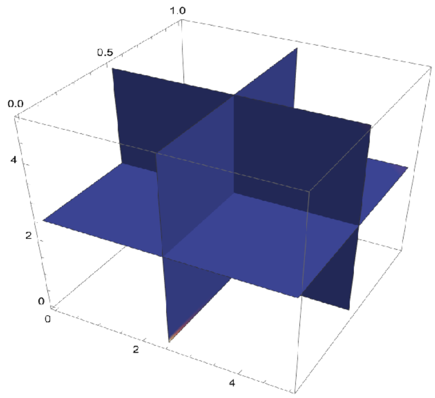

Figure 4.3: The SliceDensityPlot3D of $u(x)=\dfrac{c}{\Gamma(\alpha)}\dfrac{\left((e^x)^{\rho}-1\right)^{\alpha}}{\alpha\rho}$ for $(x,\alpha,\rho), c=1$.

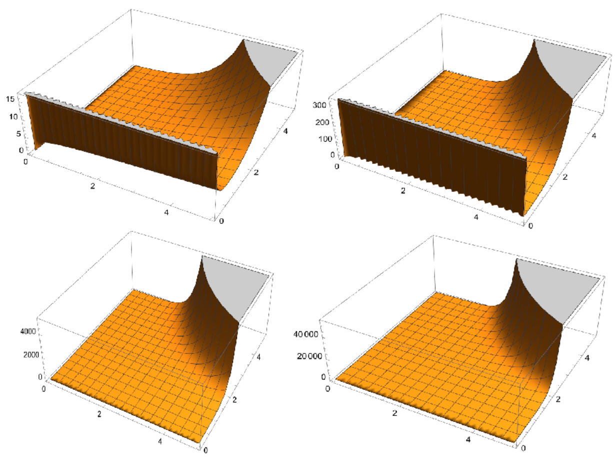

Figure 4.4: The Plot of $u(x)=\dfrac{c}{\Gamma(\alpha)}\dfrac{\left((e^x)^{\rho}-1\right)^{\alpha}}{\alpha\rho}$ for $(x,\rho), c=1$ and $\alpha=0.25,0.5,0.75,0.95$.

Class 2: We take the following differential problem:

$$\begin{cases} {}_{\rho}D^{\alpha}_{t_0,\varphi}u(x)=\sum\limits_{i=1}^{n}g_i\left(x,u(x)\right), \;\; t_0\leq x<T\leq\infty, \\ {}_{\rho}I^{1-\alpha}_{t_0,\varphi}u(x)\Big|_{x=t_0}=u_0, \end{cases} \tag{4.5}$$

where ${}_{\rho}D^{\alpha}_{t_0,\varphi}$, ${}_{\rho}I^{1-\alpha}_{t_0,\varphi}$ are respectively the $(\rho,\varphi)-$ Riemann-Liouville fractional derivative and $(\rho,\varphi)-$ Riemann-Liouville fractional integral, $\rho>0, u_0\in\mathbb{R}$, with respect to $\varphi\in C^1\left([t_0,T),\mathbb{R}_+\right)$, $\varphi'(t)\,\varphi^{\rho-1}(x)\neq0$ and $g_i\in C\left([t_0,T)\times\mathbb{R},\mathbb{R}\right), i\in\{1,2,...,n\}$.
The fractional integral solution of (4.5) is given by:

$$u(x)=\frac{u_0}{\Gamma(\alpha)}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}$$
$$\times\varphi'(t)\,\varphi^{\rho-1}(t)\sum_{i=1}^{n}g_i\left(t,u(t)\right)dt.$$

We consider the following hypothesis.
(H_2) : Assume that there are some $\psi_i\in C([t_0,T),\mathbb{R}^+), i\in\{1,2,...,n\}$ and $0<\gamma_i<1$, such that $\sum\limits_{i=1}^{n}|g_i\left(x,u(x)\right)|\leq\sum\limits_{i=1}^{n}\psi_i(x)\,|u^{\gamma_i}(x)|$.

Based on (H_2), we prove the following estimates for the solution of Class 2.

Theorem 4.3. *If (H_2) holds, then the following two inequalities are valid:*

(i) For $\alpha>\frac{1}{2}$, we have

$$|u(x)|\leq\Bigg[2Q^2(x)+2\frac{\Gamma(2\alpha-1)}{\rho4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\int_{t_0}^{x}\sum_{i=1}^{n}n\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-2\varphi^{\rho}(t)}g_i^2(t)\left[\gamma_i\left(2Q^2(t)-1\right)+1\right]$$
$$\times\exp\left(\int_t^x2\frac{\Gamma(2\alpha-1)}{\rho4^{\alpha-1}}e^{2\varphi^{\rho}(\tau)}\sum_{i=1}^{n}n\varphi\prime(\tau)\,\varphi^{\rho-1}(\tau)\,e^{-2\varphi^{\rho}(\tau)}\gamma_ig_i^2(\tau)\,d\tau\right)dt\Bigg]^{\frac{1}{2}},$$

where, $Q(x)=\dfrac{|u_0|}{\Gamma(\alpha)}\left|\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right|^{\alpha-1}$;

(ii) For $\alpha\in\left(0,\frac{1}{2}\right]$, $q=\dfrac{1+\alpha}{\alpha}$, and $p=1+\alpha$, we have

$$|u(x)|\leq\Bigg[2^{q-1}Q^q(x)+2^{q-1}\left(\frac{\Gamma\left(p\left(\alpha-1\right)+1\right)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{q}{p}}$$
$$\times\int_{t_0}^{x}\sum_{i=1}^{n}n^{q-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-q\varphi(t)}g_i^q(t)\left[\gamma_i\left(2^{q-1}Q^q(t)-1\right)+1\right]$$
$$\times\exp\left(\int_t^x2^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(\tau)}\right)^{\frac{q}{p}}\sum_{i=1}^{n}n^{q-1}\varphi\prime(\tau)\,\varphi^{\rho-1}(\tau)\,e^{-q\varphi^{\rho}(\tau)}\gamma_ig_i^q(\tau)\,d\tau\right)dt\Bigg]^{\frac{1}{q}}.$$

Proof. Let us take $x\in[t_0,T)$. So, we have

$$|u(x)|\leq\frac{|u_0|}{\Gamma(\alpha)}\left|\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right|^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\times\varphi'(t)\,\varphi^{\rho-1}(t)\sum_{i=1}^{n}|g_i\left(t,u(t)\right)|dt.$$

Using (H_2), we get

$$|u(x)|\leq\frac{|u_0|}{\Gamma(\alpha)}\left|\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right|^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\times\varphi'(t)\,\varphi^{\rho-1}(t)\sum_{i=1}^{n}\psi_i(t)\,|u^{\gamma_i}(t)|\,dt.$$

Thanks to Theorem 3.3, the proof is achieved. □

Example 4.4. *Assume the following data: let $\varphi(x)=x, u_0=0, (\psi+h)(x)=c, c\in\mathbb{R}$ and $g_(x)=x, g_2(x)=x^2$. Then the solution u is given by the form, using Mathematica 13.3, as follows (see Fig.4.5 and 4.6)*

$$u(x)=\frac{c}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-2/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{2\Gamma\left(\frac{2}{\rho}\right)}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^2}\right), \quad x>0,\alpha>0,\rho>0,c\in\mathbb{R}.$$

And for $g_(x)=x, g_2(x)=x^2$ and $g_3(x)=x^3$ the becomes (see Figs. 4.7 and 4.8)

$$u(x)=\frac{c}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-3/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{3\Gamma\left(\frac{3}{\rho}\right)}{\Gamma\left(\alpha+\frac{3}{\rho}+1\right)}+\frac{2\Gamma\left(\frac{2}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{2/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^2}$$

The solution $u(x)$ could symbolize an economic variable in a dynamic economic framework, and the inequality may be utilized for establishing conditions in which the variable stays limited or comes together to a stable equilibrium over time. The parameters α and ρ are complex and self-replicating patterns that may be discovered at various sizes. Some economists have proposed that some patterns found in economic and financial data display fractal-fractional like features. Financial time series data, for instance stock prices or currency rates, may, for example, show self-similar patterns at multiple time scales. This indicates that short-term and long-term movements exhibit comparable patterns or tendencies. The examination of these shapes is known as fractional finance.

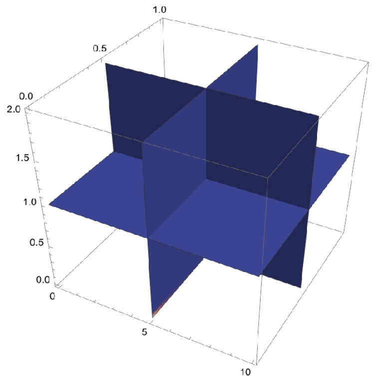

Figure 4.5: The SliceDensityPlot3D of $u(x)=\dfrac{c}{\Gamma(\alpha)}\left(\dfrac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-2/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{2\Gamma\left(\frac{2}{\rho}\right)}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}\right)$ for $(x,\alpha,\rho), c=1$.

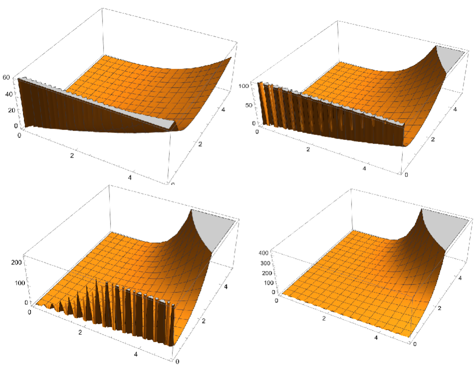

Figure 4.6: The Plot of $u(x)=\dfrac{c}{\Gamma(\alpha)}\left(\dfrac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-2/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{2\Gamma\left(\frac{2}{\rho}\right)}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}\right)$ for $(x,\rho), c=1$ and $\alpha=0.25,0.5,0.75,0.95$.

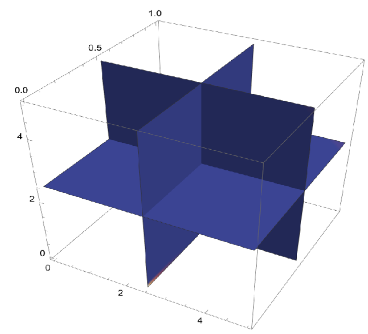

Figure 4.7: The SliceDensityPlot3D of $u(x)=\dfrac{c}{\Gamma(\alpha)}\dfrac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-3/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{3\Gamma\left(\frac{3}{\rho}\right)}{\Gamma\left(\alpha+\frac{3}{\rho}+1\right)}+\frac{2\Gamma\left(\frac{2}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{2/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}$ for $(x,\alpha,\rho), c=1$.

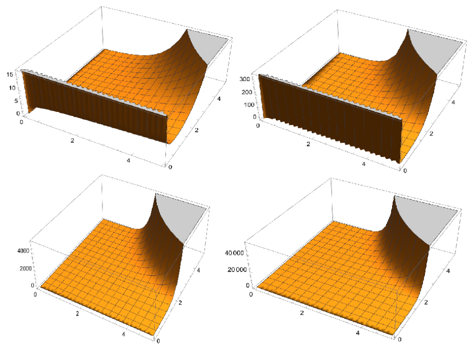

Figure 4.8: The Plot of $u(x)=\frac{c}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-3/\rho}(x^{\rho})^{\alpha}\left(\frac{3\Gamma\left(\frac{3}{\rho}\right)}{\Gamma\left(\alpha+\frac{3}{\rho}+1\right)}+\frac{2\Gamma\left(\frac{2}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{2/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^2}$ for $(x,\rho), c=1$ and $\alpha=0.25,0.5,0.75,0.95$.

5. Conclusion

The use of Gronwall's inequality in economics is part of a larger subject that is referred to as mathematical economics, which use mathematical strategies and instruments to understand economic events. It is crucial to note that the exact application of Gronwall's inequality in economics would be dependent on the specifics of the economic model under consideration. We have used one of our recent papers on $(\rho,\varphi)-$Riemann Liouville integrals to prove new results on Gronwall integral inequalities. Then, we have introduced, for the first time, the so-called $(\rho,\varphi)-$Riemann Liouville derivatives with respect to another function. We have presented some of their properties (Theorem 2.7). At the end, we have discussed two classes of differential equations that involve such derivatives. The boundedness of the solutions of these two classes has been established. We invite the interested reader to work on this "new" derivative approach since it has been shown in the study of the above two classes that the introduced derivatives are important to study differential equations. It has also been proved that they generalize several existing derivatives, see Remark 2.6.

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