Journal of Mathematical Sciences and Modelling, 7(2) (2024) 75-81
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: : https://doi.org/10.33187/jmsm.1426590

N-order Solutions to the Gardner Equation in terms of Wronskians

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Article Info

Keywords: *Gardner equation, Rational Solutions, Wronskians*
2010 AMS: *33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd*
Received: *27 January 2024*
Accepted: *23 June 2024*
Available online: *16 July 2024*

Abstract

N-order solutions to the Gardner equation (G) are given in terms of Wronskians of order N depending on $2N$ real parameters. We get solutions expressed with trigonometric or hyperbolic functions.
When one of the parameters goes to 0, we succeed to get for each positive integer N, rational solutions as a quotient of polynomials in x and t depending on $2N$ real parameters. We construct explicit expressions of these rational solutions for the first orders.

1. Introduction

Gardner equation is considered in the form

$$u_t + 6u(u-2)u_x + u_{xxx} + 6u_x = 0; \tag{1.1}$$

the subscripts x and t mean partial derivatives.
Gardner [1] considered this equation in 1968 in a slightly different form. This equation describes nonlinear wave effects for example, in plasma physics [2, 3], fluid flows [4], quantum fluid dynamics [5], in dusty plasmas [6], in ocean and atmosphere [7]. It can be used to describe large-amplitude internal waves [4, 8, 9].
We can quote many methods to get solutions to this equation, as the Hirota method [10], the series expansion method [11], the mapping method [12] or the method of leading-order analysis [13].
We can also cite more recent work on this equation realised by Lin et al. [14], Ghanbari et al. [15], Cao et al., Bokaeyan et al. [16], Ankiwicz et al. [17].

Here, we obtain general solutions in terms of a quotient of two Wronskians of order N. These solutions will be called solutions of order N. They depend on $2N$ real parameters and can be expressed in terms of trigonometric or hyperbolic functions.
We construct rational solutions by considering a passage to the limit when one of the parameters goes to 0. We get rational solutions which depend on $2N$ real parameters.
We construct solutions for the first orders.

2. Solutions to the Gardner Equation of Order N in terms of Wronskians

2.1. Solutions of order N in terms of Wronskians of hyperbolic sin functions

In the following, we use the Wronskian of order N of the functions $f_1, \dots, f_N$. We recall that is the determinant denoted $W(f_1, \dots, f_N)$, defined by $\det(\partial_x^{i-1} f_j)_{1\le i\le N, 1\le j\le N}$; ∂_x^i defines the partial derivative of order i with respect to x and $\partial_x^0 f_j$ being the function f_j.
a_j, b_j are arbitrary real numbers $1 \le j \le N$. We get the result :

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Cite as "Pierre Gaillard, *N-order Solutions to the Gardner Equation in terms of Wronskians*, J. Math. Sci. Model., 7(2) (2024), 75-81"

Theorem 2.1. *Let f_j be the functions defined by*

$$f_j(X,T) = \sinh\left(\frac{1}{2}a_jX - \frac{1}{2}a_j^3T + b_j\right), \qquad 1 \leq i \leq N,$$

then the function u expressed as

$$u(x,t) = 1 - \partial_X \ln\left(\frac{W(\partial_X(f_1),\ldots,\partial_X(f_N))}{W(f_1,\ldots,f_N)}\right)_{|X=ix,T=-it}$$

represents a solution to the Gardner equation (1.1) which depends on $2N$ real parameters a_j, b_j, $1 \leq j \leq N$.

Proof. We have proven in [18] that the function v defined by

$$v(X,T) = \partial_X \ln\left(\frac{W(\partial_X(f_1),\ldots,\partial_x(f_N))}{W(f_1,\ldots,f_N)}\right)$$

is a solution to the (mKdV) equation (2.1)

$$u_T - 6u^2u_X + u_{XXX} = 0. \tag{2.1}$$

We deduce then that the function w expresssed as

$$w(x,t) = \partial_X \ln\left(\frac{W(\partial_X(f_1),\ldots,\partial_X(f_N))}{W(f_1,\ldots,f_N)}\right)_{|X=ix,T=-it}$$

is solution to the following equation

$$u_t + 6u^2u_x + u_{xxx} = 0.$$

It can be easily checked then that the function u expressed as

$$u(x,t) = 1 - \partial_X \ln\left(\frac{W(\partial_X(f_1),\ldots,\partial_X(f_N))}{W(f_1,\ldots,f_N)}\right)_{|X=ix,T=-it}$$

is a solution to the equation (1.1). □

2.2. Different examples of solutions to the Gardner equation using sin hyperbolic generating functions

The presence of singularities in solutions makes the representation of modules of the solutions in space (x,t) inappropriate. Thus, we will not build any solution in space (x,t).
We only give the solutions of order 1, 2 and 3 in the case of generating hyperbolic sinus functions.
Solution of order 1

Proposition 2.2. *The function u expressed as*

$$\begin{aligned} u(x,t) &= \frac{n(x,t)}{d(x,t)}, \\ n(x,t) &= 2\sinh\left(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1\right)\cosh\left(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1\right) + a_1, \\ d(x,t) &= 2\sinh\left(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1\right)\cosh\left(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1\right) \end{aligned}$$

is a solution to the Gardner equation (1.1) depending on a_1, b_1 arbitrary real parameters.

Solution of order 2

Proposition 2.3. *The function u expressed as*

$$u(x,t) = \frac{n(x,t)}{d(x,t)},$$

$n(x,t) = -2\sinh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)a_2{}^2\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)\sinh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2) + 4(\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2))^2a_2a_1(\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1))^2 - 2(\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2))^2a_2a_1 - 2(\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1))^2a_1a_2 - 2\sinh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)a_1{}^2\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)\sinh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1) + a_2{}^3\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)\sinh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1) - \sinh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)a_2{}^2a_1\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2) - \sinh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)a_1{}^2a_2\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1) + a_1{}^3\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)\sinh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)$

$d(x,t) = 2(\sinh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)\cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)a_2 - \sinh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)a_1)(-\cosh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1)\sinh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)a_2 + \cosh(1/2\,ia_2x + 1/2\,ia_2{}^3t + b_2)a_1\sinh(1/2\,ia_1x + 1/2\,ia_1{}^3t + b_1))$

is a solution to the Gardner equation (1.1) with a_1, a_2, b_1, b_2 arbitrarily real parameters.

Solution of order 3

In the case of order 3, we present solution in the particular case where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$.

Proposition 2.4. *The function u expressed as*

$$u(x,t) = \frac{n(x,t)}{d(x,t)},$$

$$\begin{aligned} n(x,t) = {} & -150\,(\cosh(1/2\,ix+1/2\,it))^2 - 54\,(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2 - 384\,(\cosh(ix+4\,it))^2 - 180\,(\cosh(1/2\,ix+1/2\,it))^2(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2 \\ & + 480\,(\cosh(1/2\,ix+1/2\,it))^2(\cosh(ix+4\,it))^2 + 288\,(\cosh(ix+4\,it))^2(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2 \\ & + 320\,(\cosh(1/2\,ix+1/2\,it))^2\sinh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it)(\cosh(ix+4\,it))^2 \\ & + 576\sinh(1/2\,ix+1/2\,it)(\cosh(ix+4\,it))^2\cosh(1/2\,ix+1/2\,it)(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2 \\ & - 180\,(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2\sinh(ix+4\,it)\cosh(ix+4\,it)(\cosh(1/2\,ix+1/2\,it))^2 \\ & + 300\sinh(1/2\,ix+1/2\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(1/2\,ix+1/2\,it)\sinh(3/2\,ix+\tfrac{27}{2}\,it) \\ & - 600\sinh(1/2\,ix+1/2\,it)\cosh(ix+4\,it)\cosh(1/2\,ix+1/2\,it)\sinh(ix+4\,it) \\ & - 312\cosh(3/2\,ix+\tfrac{27}{2}\,it)\sinh(ix+4\,it)\sinh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(ix+4\,it) + 90\,(\cosh(1/2\,ix+1/2\,it))^2\sinh(ix+4\,it)\cosh(ix+4\,it) \\ & + 90\,(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2\sinh(ix+4\,it)\cosh(ix+4\,it) - 160\,(\cosh(ix+4\,it))^2\sinh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it) \\ & - 288\sinh(1/2\,ix+1/2\,it)(\cosh(3/2\,ix+\tfrac{27}{2}\,it))^2\cosh(1/2\,ix+1/2\,it) - 288\sinh(1/2\,ix+1/2\,it)(\cosh(ix+4\,it))^2\cosh(1/2\,ix+1/2\,it) \\ & - 160\,(\cosh(1/2\,ix+1/2\,it))^2\sinh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it) \\ & - 724\sinh(1/2\,ix+1/2\,it)\cosh(ix+4\,it)\sinh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(1/2\,ix+1/2\,it)\sinh(ix+4\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it) \end{aligned}$$

$$\begin{aligned} d(x,t) = {} & 2\,(16\sinh(1/2\,ix+1/2\,it)\cosh(ix+4\,it)\sinh(3/2\,ix+\tfrac{27}{2}\,it) - 9\sinh(1/2\,ix+1/2\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it)\sinh(ix+4\,it) \\ & - 5\cosh(1/2\,ix+1/2\,it)\sinh(ix+4\,it)\sinh(3/2\,ix+\tfrac{27}{2}\,it))(16\cosh(1/2\,ix+1/2\,it)\sinh(ix+4\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it) \\ & - 9\cosh(1/2\,ix+1/2\,it)\sinh(3/2\,ix+\tfrac{27}{2}\,it)\cosh(ix+4\,it) - 5\sinh(1/2\,ix+1/2\,it)\cosh(ix+4\,it)\cosh(3/2\,ix+\tfrac{27}{2}\,it)) \end{aligned}$$

is a solution to the Gardner equation (1.1).

2.3. Another solutions

We can consider other types of generating functions and get other types of solutions.

2.3.1. Solutions using hyperbolic cosine generating functions

Theorem 2.5. *Let h_j, h be the following functions*

$$h_j(X,T) = \cosh\left(\frac{1}{2}a_jX - \frac{1}{2}a_j^3T + b_j\right), \qquad \text{for } 1 \le i \le N,$$

then the function u expressed as

$$u(x,t) = 1 - \partial_X \ln\left(\frac{W(\partial_X(h_1),\ldots,\partial_X(h_N))}{W(h_1,\ldots,h_N)}\right)_{|X=ix,T=-it}$$

is a solution to the Gardner equation (1.1) with a_j, b_j $1 \le j \le N$ arbitrarily real parameters.

2.3.2. Solutions using trigonometric generating functions

Theorem 2.6. *Let g_j be the functions*

$$g_j(X,T) = \cos\left(\frac{1}{2}a_jX + \frac{1}{2}a_j^3T + b_j\right), \qquad \text{for } 1 \le i \le N,$$

then the function u expressed as

$$u(x,t) = 1 - \partial_X \ln\left(\frac{W(\partial_X(g_1),\ldots,\partial_x(g_N))}{W(g_1,\ldots,g_N)}\right)_{|X=ix;T=-it}$$

is a solution to the Gardner equation (1.1), a_j, b_j $1 \le j \le N$ being arbitrarily real numbers.

Theorem 2.7. *Let k_j be the functions*

$$k_j(X,T) = \sin\left(\frac{1}{2}a_jX + \frac{1}{2}a_j^3T + b_j\right), \qquad \text{for } 1 \le i \le N,$$

then the function u expressed as

$$u(x,t) = 1 - \partial_X \ln\left(\frac{W(\partial_X(k_1),\ldots,\partial_x(k_N))}{W(k_1,\ldots,k_N)}\right)_{|X=ix,T=-it}$$

is a solution to the Gardner equation (1.1), a_j, b_j $1 \le j \le N$ being arbitrarily real numbers.

3. Rational Solutions to the Gardner Equation

In the following, we replace all parameters a_j and b_j, $1 \leq j \leq N$ by $\hat{a}_j = \sum_{k=1}^{N} a_k (je)^{2k-1}$ and $\hat{b}_j = \sum_{k=1}^{N} b_k (je)^{2k-1}$ with e an arbitrary real parameter. We use this change of parameter to obtain rational solutions to the equation (2.1); it is enough to tend e to 0.
We get the statement:

Theorem 3.1. *Let ψ_j be the functions*

$$\psi_j(X,T,e) = \sinh\left(\frac{1}{2}\sum_{k=1}^{N} a_k (je)^{2k-1} X - \frac{1}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^3 T + \sum_{k=1}^{N} b_k (je)^{2k-1}\right), \quad \text{for } 1 \leq j \leq N$$

then the function u expressed as

$$u(x,t) = 1 - \lim_{e\to 0} \partial_X \ln\left(\frac{W(\partial_X(\psi_1),\ldots,\partial_x(\psi_N))}{W(\psi_1,\ldots,\psi_N)}\right)_{|X=ix,T=-it}$$

is a rational solution to the Gardner equation (1.1).

Proof. This can be easily deduced from the previous result. □

We can consider other generating function as trigonometric functions. The rational solutions of the Garner equation can also be expressed without the presence of a limit. We can give the statement:

Theorem 3.2. *Let ψ, φ_j be the functions*

$$\begin{aligned} \psi(X,T,e) &= \sinh\left(\frac{1}{2}\left(\sum_{k=1}^{N} a_k e^{2k-1}\right) X - \frac{1}{2}\left(\sum_{k=1}^{N} a_k e^{2k-1}\right)^3 T + \sum_{k=1}^{N} b_k e^{2k-1}\right), \\ \varphi_j(X,T) &= \frac{\partial^{2j-1}\psi(X,T,0)}{\partial_{2j-1} e}, \text{ for } 1 \leq j \leq N, \end{aligned}$$

then the function v expressed as

$$v(x,t) = 1 - \partial_X \ln\left(\frac{W(\partial_X(\varphi_1),\ldots,\partial_x(\varphi_N))}{W(\varphi_1,\ldots,\varphi_N)}\right)_{|X=ix,T=-it}$$

is a rational solution to the equation (1.1) which depend on $2N$ real numbers a_j, b_j, $1 \leq j \leq N$.

Proof. It is a consequence of the combination of the columns of the determinant of the previous theorem and the passage to the limit when e tends to 0 for each column. □

Some examples of these rational solutions are given in the rest of the text. We get rational solutions but which are singular.

3.1. Rational solutions of order 1

Proposition 3.3. *The function v expressed as*

$$v(x,t) = \frac{ia_1x + 2b_1 + a_1}{ia_1x + 2b_1},$$

is a rational solution to the Gardner equation (1.1), a_1, b_1 being arbitrarily real numbers.

3.2. Rational solutions of order 2

Proposition 3.4. *The function v expressed as*

$$v(x,t) = \frac{n(x,t)}{d(x,t)},$$

$n(x,t) = -{a_1}^5x^4 + 8i{a_1}^4b_1x^3 + 2i{a_1}^5x^3 + 24{a_1}^3{b_1}^2x^2 + 12{a_1}^4b_1x^2 - 12{a_1}^5tx + 24i{a_1}^2b_2x - 32i{a_1}^2x{b_1}^3 - 24i{a_1}^3x{b_1}^2 - 24ia_1xa_2b_1 - 12i{a_1}^5t - 16a_1{b_1}^4 + 48b_1b_2a_1 - 48a_2{b_1}^2 - 24b_2{a_1}^2 + 24a_1a_2b_1 + 24ib_1{a_1}^4t - 16{a_1}^2{b_1}^3$,

$d(x,t) = (ia_1x + 2b_1)(i{a_1}^4x^3 + 6{a_1}^3b_1x^2 - 12i{a_1}^2x{b_1}^2 + 12i{a_1}^4t + 24b_2a_1 - 24a_2b_1 - 8a_1{b_1}^3)$

is a rational solution to the Gardner equation (1.1) depending a_1, a_2, b_1, b_2 arbitrary real numbers.

3.3. Third order rational solutions

Proposition 3.5. *The function v expressed as*

$$v(x,t)=\frac{n(x,t)}{d(x,t)},$$

$$n(x,t)=-ia_1{}^{12}x^9-18a_1{}^{11}b_1x^8-3a_1{}^{12}x^8+48ia_1{}^{11}b_1x^7+144ia_1{}^{10}b_1{}^2x^7+336a_1{}^{10}b_1{}^2x^6-144b_2a_1{}^9x^6-72ia_1{}^{12}tx^6+144a_2b_1a_1{}^8x^6+
672a_1{}^9b_1{}^3x^6+144ia_1{}^9b_2x^5-144ia_1{}^8x^5b_1a_2-72a_1{}^{12}tx^5-1344ia_1{}^9x^5b_1{}^3-864a_1{}^{11}b_1tx^5+1728ib_2a_1{}^8b_1x^5-2016ia_1{}^8x^5b_1{}^4-
1728ia_2b_1{}^2a_1{}^7x^5+8640b_2a_1{}^7b_1{}^2x^4-1440a_1{}^7b_3x^4-3360a_1{}^8b_1{}^4x^4-4320a_1{}^5a_2{}^2b_1x^4+4320a_1{}^6a_2b_2x^4+720ia_1{}^{11}tb_1x^4-
8640a_2b_1{}^3a_1{}^6x^4+1440a_1{}^8b_1b_2x^4-4032a_1{}^7b_1{}^5x^4+4320ia_1{}^{10}tb_1{}^2x^4-1440a_1{}^7b_1{}^2a_2x^4+1440a_1{}^6a_3b_1x^4+5376ia_1{}^6b_1{}^6x^3+
11520ia_1{}^6b_3b_1x^3+23040ia_2b_1{}^4a_1{}^5x^3-23040ib_2a_1{}^6x^3b_1{}^3+11520a_1{}^9b_1{}^3tx^3-2880ia_1{}^7x^3b_3+8640ia_1{}^6a_2b_2x^3-11520ia_1{}^5x^3a_3b_1{}^2-
5760ia_1{}^7x^3b_1{}^2b_2-34560ib_2a_1{}^5x^3b_1a_2-8640ia_1{}^5x^3a_2{}^2b_1+2880a_1{}^{10}tb_1{}^2x^3+2880ia_1{}^6a_3b_1x^3+34560ia_2{}^2b_1{}^2a_1{}^4x^3+
5760ia_1{}^6b_1{}^3a_2x^3+5376ia_1{}^7b_1{}^5x^3+34560a_2b_1{}^5a_1{}^4x^2-11520a_1{}^6b_2b_1{}^3x^2-34560a_1{}^4b_1{}^3a_3x^2+17280a_1{}^5a_3b_1{}^2x^2+
17280a_1{}^5b_2a_2b_1x^2+17280a_1{}^6b_2{}^2x^2-34560b_2a_1{}^5b_1{}^4x^2-17280a_1{}^6b_3b_1x^2+103680a_2{}^2b_1{}^3a_1{}^3x^2+11520a_1{}^5a_2b_1{}^4x^2+
5376a_1{}^6b_1{}^6x^2-5760ia_1{}^9x^2tb_1{}^3-4320a_1{}^{12}t^2x^2-103680b_2a_1{}^4b_1{}^2a_2x^2+34560a_1{}^5b_1{}^2b_3x^2-17280ia_1{}^8x^2ta_2b_1-34560a_1{}^4a_2{}^2b_1{}^2x^2-
17280ia_1{}^8tx^2b_1{}^4+4608a_1{}^5b_1{}^7x^2+17280ia_1{}^9tb_2x^2+69120a_1{}^8b_1tb_2x-27648ia_2b_1{}^6a_1{}^3x+207360ib_2a_1{}^2a_2{}^2b_1x-3072ib_1{}^7a_1{}^5x-
138240ia_2{}^2b_1{}^4a_1{}^2x+51840a_1{}^6ta_2b_2x-2304ia_1{}^4b_1{}^8x+46080ia_1{}^3b_1{}^4a_3x-51840a_1{}^5ta_2{}^2b_1x+17280ia_1{}^{11}b_1t^2x-69120a_1{}^7b_1{}^2ta_2x+
17280a_1{}^6ta_3b_1x+138240ib_2a_1{}^3b_1{}^3a_2x+11520ib_1{}^4a_1{}^5b_2x+34560ib_1{}^3a_2{}^2a_1{}^3x-69120ia_1{}^5xb_1b_2{}^2+34560ia_2b_1{}^2a_3a_1{}^2x+
34560ib_1{}^2a_2b_2a_1{}^4x-13824a_1{}^7b_1{}^5tx+27648ib_2a_1{}^4b_1{}^5x-103680ib_2{}^2a_1{}^3a_2x-11520ib_1{}^5a_1{}^4xa_2-34560ib_1{}^3a_3xa_1{}^4-17280a_1{}^7tb_3x-
34560ib_2a_1{}^3a_3xb_1+34560ib_1{}^2b_3a_1{}^5x-34560ia_2b_1b_3a_1{}^3x-46080ia_1{}^4b_1{}^3b_3x-103680ia_2{}^3b_1{}^2xa_1+34560ib_2a_1{}^4b_3x-
5760b_1{}^4a_1{}^8tx-103680b_2{}^2a_1{}^3a_2+17280b_1{}^2a_1{}^{10}t^2-23040b_1{}^4a_3a_1{}^3-69120ib_1{}^2a_1{}^7tb_2+34560b_2a_1{}^4b_3+17280ia_1{}^7tb_3-
103680ia_1{}^6tb_2{}^2-69120b_1{}^2b_2{}^2a_1{}^4+2304ib_1{}^5a_1{}^7t+69120ib_1{}^3a_1{}^6ta_2-51840ia_1{}^6ta_2b_2+4608ia_1{}^6tb_1{}^6-51840a_2b_1a_1{}^8t^2+
51840b_2a_1{}^9t^2+103680ia_1{}^5tb_2a_2b_1+8640ia_1{}^{12}t^3-69120b_2{}^3a_1{}^3-138240a_2{}^3b_1{}^3-512a_1{}^3b_1{}^9-768b_1{}^8a_1{}^4+34560ia_1{}^6tb_3b_1-
34560ia_1{}^5ta_3b_1{}^2+51840ia_1{}^5ta_2{}^2b_1-17280ia_1{}^6ta_3b_1+69120b_1{}^3b_2a_1{}^3a_2+207360b_2a_1{}^2a_2{}^2b_1-34560b_2a_1{}^3a_3b_1+
34560a_2b_1{}^2a_1{}^2a_3-34560a_2b_1b_3a_1{}^3+69120b_2a_1{}^2a_2b_1{}^4+69120b_2a_1{}^3b_3b_1-69120b_2a_1{}^2a_3b_1{}^2+207360b_2a_1a_2{}^2b_1{}^2-
69120a_2b_1{}^2b_3a_1{}^2+69120a_2b_1{}^3a_3a_1+9216b_2a_1{}^3b_1{}^6-69120a_2{}^2b_1{}^5a_1-9216a_2b_1{}^7a_1{}^2-23040a_1{}^3b_1{}^4b_3+23040a_1{}^2b_1{}^5a_3-
103680a_1a_2{}^3b_1{}^2+4608b_1{}^5b_2a_1{}^4-4608b_1{}^6a_2a_1{}^3+23040b_1{}^3b_3a_1{}^4$$

$$d(x,t)=(ia_1{}^4x^3+6a_1{}^3b_1x^2-12ia_1{}^2b_1{}^2x+12ia_1{}^4t-8a_1b_1{}^3+24b_2a_1-24a_2b_1)(-a_1{}^8x^6+12ia_1{}^7b_1x^5+60a_1{}^6b_1{}^2x^4-60a_1{}^8tx^3+
120ib_2a_1{}^5x^3-160ia_1{}^5x^3b_1{}^3-120ia_1{}^4x^3b_1a_2-720a_1{}^3b_1{}^2a_2x^2+720a_1{}^4b_1b_2x^2+360ia_1{}^7tb_1x^2-240a_1{}^4b_1{}^4x^2-1440ia_3xa_1{}^2b_1-
1440ia_1{}^3xb_1{}^2b_2+720a_1{}^6tb_1{}^2x-4320ib_2a_1{}^2a_2x+4320ia_2{}^2a_1b_1x+1440ib_3a_1{}^3x+1440ia_1{}^2b_1{}^3a_2x+192ia_1{}^3b_1{}^5x+960b_1{}^4a_1a_2-
2880b_2a_1a_2b_1+2880b_3a_1{}^2b_1-480ia_1{}^5tb_1{}^3+720a_1{}^8t^2-2880b_2{}^2a_1{}^2-2880ia_1{}^5tb_2+64b_1{}^6a_1{}^2+5760a_2{}^2b_1{}^2-2880a_3a_1b_1{}^2+
2880ia_1{}^4ta_2b_1-960b_2a_1{}^2b_1{}^3)$$

is a rational solution to the Gardner equation (1.1).

3.4. Fourth order rational solutions

By choosing $a_j=b_j=j$ for $1\leq j\leq 4$, we obtain the following rational solution:

Proposition 3.6. *The function v expressed as*

$$v(x,t)=\frac{n(x,t)}{d(x,t)},$$

$$n(x,t)=x^{16}-36ix^{15}-600x^{14}+6160ix^{13}+240tx^{13}+43680x^{12}-6840itx^{12}-89280tx^{11}-227136ix^{11}+707520itx^{10}-896896x^{10}+
10080t^2x^{10}+3801600tx^9-237600it^2x^9+2745600ix^9+6589440x^8-2462400t^2x^8-14636160itx^8-12446720ix^7-41564160tx^7+
14860800it^2x^7+172800t^3x^7+58060800t^2x^6+88197120itx^6-18450432x^6+21245952ix^5+14515200t^3x^5+139898880tx^5-
153861120it^2x^5-96768000it^3x^4+18636800x^4-163891200itx^4+18144000t^4x^4-280627200t^2x^4-108864000it^4x^3+
348364800it^2x^3-12042240ix^3-290304000t^3x^3-137871360tx^3+464486400it^3x^2-5406720x^2+282009600t^2x^2+78888960itx^2-
217728000t^4x^2+387072000t^3x+1507328ix-217728000t^5x+145152000it^4x+27525120tx-134553600it^2x-4423680it-
28753920t^2+653184000it^5+196608-132710400it^3$$

$$d(x,t)=(-x^6+12ix^5+60x^4-60tx^3-160ix^3+360itx^2-240x^2+720tx+192ix+720t^2+64-480it)(-x^{10}+20ix^9+180x^8-
180tx^7-960ix^7+2520itx^6-3360x^6+15120tx^5+8064ix^5+13440x^4-50400itx^4-15360ix^3-100800tx^3-11520x^2+120960itx^2+
5120ix-302400t^3x+80640tx+1024-23040it+604800it^3)$$

is a rational solution to the Gardner equation (1.1).

3.5. Fifth order rational solutions

By choosing $a_j=j$ and $b_j=0$, for $1\leq j\leq N$, we get the following rational solution:

Proposition 3.7. *The function v expressed as*

$$v(x,t)=\frac{n(x,t)}{d(x,t)},$$

with

$n(x,t) = ix^{25} + 55x^{24} - 1440ix^{23} - 23920x^{22} + 600itx^{22} + 283360ix^{21} + 28680tx^{21} + 2550240x^{20} - 650160itx^{20} - 18135040ix^{19} +$
$100800it^2x^{19} - 9307200tx^{19} + 4183200t^2x^{18} + 94483200itx^{18} - 104600320x^{18} + 498389760ix^{17} + 723945600tx^{17} -$
$81648000it^2x^{17} + 1987406080x^{16} + 6955200it^3x^{16} - 997315200t^2x^{16} - 4349802240itx^{16} - 6694420480ix^{15} - 21015982080tx^{15} +$
$227404800t^3x^{15} + 8554291200it^2x^{15} + 83051827200itx^{14} - 19170385920x^{14} - 3483648000it^3x^{14} + 54780364800t^2x^{14} +$
$271580467200tx^{13} - 33191424000t^3x^{13} - 271763251200it^2x^{13} + 254016000it^4x^{13} + 46860943360ix^{13} + 220147200000it^3x^{12} -$
$740421427200itx^{12} + 97981972480x^{12} - 1069286400000t^2x^{12} + 9398592000t^4x^{12} + 3387499315200it^2x^{11} - 1690327941120tx^{11} -$
$175272099840ix^{11} - 146313216000it^4x^{11} + 1077840691200t^3x^{11} - 4029574348800it^3x^{10} - 39626496000it^5x^{10} - 267776819200x^{10} +$
$3236183408640itx^{10} + 8719674163200t^2x^{10} - 1318851072000t^4x^{10} - 11734474752000t^3x^9 + 5189895782400tx^9 -$
$18317588889600it^2x^9 + 348109864960ix^9 - 533433600000t^5x^9 + 7823692800000it^4x^9 + 382763335680x^8 + 2469035520000it^5x^8 +$
$26900729856000it^3x^8 - 31411357286400t^2x^8 + 32591268864000t^4x^8 - 6944204390400itx^8 - 7696731340800tx^7 - 352835338240ix^7 +$
$365783040000it^6x^7 + 43797970944000it^2x^7 - 98712649728000it^4x^7 + 731566080000t^5x^7 + 48709140480000t^3x^7 - 269269073920x^6 +$
$10241925120000t^6x^6 + 6988022415360itx^6 - 69432665702400it^3x^6 + 49230603878400t^2x^6 - 221030498304000t^4x^6 +$
$40967700480000it^5x^6 - 77097310617600t^3x^5 - 43966424678400it^2x^5 + 366953545728000it^4x^5 + 5111579934720tx^5 -$
$92177326080000it^6x^5 + 167132528640ix^5 + 202790117376000t^5x^5 - 512096256000000it^5x^4 - 409677004800000t^6x^4 +$
$65374912512000it^3x^4 - 2939132313600itx^4 + 82239815680x^4 - 30490125926400t^2x^4 - 76814438400000it^7x^4 +$
$447068160000000t^4x^4 - 768144384000000t^7x^3 - 30870077440ix^3 - 389128126464000it^4x^3 - 1279367577600tx^3 +$
$15834651033600it^2x^3 + 40924348416000t^3x^3 + 1024192512000000it^6x^3 - 784238837760000t^5x^3 + 5796790272000t^2x^2 -$
$17836277760000it^3x^2 - 229419122688000t^4x^2 - 8304721920x^2 + 737418608640000it^5x^2 + 1474837217280000t^6x^2 +$
$2765319782400000it^7x^2 + 396361728000itx^2 + 1426063360ix - 4835613081600t^3x + 82195513344000it^4x + 77888225280tx +$
$4301608550400000t^7x - 1147095613440000it^6x - 1334417817600it^2x + 394070261760000t^5x - 460886630400000it^8x -$
$7298088960it - 2458062028800000it^7 - 374561832960000t^6 - 460886630400000t^8 + 117440512 + 13525843968000t^4 -$
$145332633600t^2 - 92079783936000it^5 + 614360678400it^3$

and,

$d(x,t) = (-x^{10} + 20ix^9 + 180x^8 - 180tx^7 - 960ix^7 + 2520itx^6 - 3360x^6 + 15120tx^5 + 8064ix^5 + 13440x^4 - 50400itx^4 - 15360ix^3 -$
$100800tx^3 - 11520x^2 + 120960itx^2 + 5120ix - 302400t^3x + 80640tx + 1024 - 23040it + 604800it^3)(-ix^{15} - 30x^{14} + 420ix^{13} -$
$420itx^{12} + 3640x^{12} - 21840ix^{11} - 10080tx^{11} - 96096x^{10} + 110880itx^{10} + 320320ix^9 - 25200it^2x^9 + 739200tx^9 - 453600t^2x^8 -$
$3326400itx^8 + 823680x^8 + 3628800it^2x^7 - 10644480tx^7 - 1647360ix^7 - 2116800it^3x^6 - 2562560x^6 + 24837120itx^6 + 16934400t^2x^6 -$
$50803200it^2x^5 + 3075072ix^5 + 42577920tx^5 - 25401600t^3x^5 + 2795520x^4 - 53222400itx^4 - 101606400t^2x^4 + 127008000it^3x^4 +$
$254016000it^4x^3 + 135475200it^2x^3 + 338688000t^3x^3 - 1863680ix^3 - 47308800tx^3 - 860160x^2 + 28385280itx^2 + 116121600t^2x^2 -$
$508032000it^3x^2 + 1524096000t^4x^2 - 3048192000it^4x - 58060800it^2x + 10321920tx + 245760ix - 406425600t^3x - 12902400t^2 +$
$135475200it^3 + 32768 - 2032128000t^4 - 1720320it + 1524096000it^5)$

is a rational solution to the Gardner equation (1.1).

4. Conclusion

Two types of solutions to the Gardner equation have given in this work.
We first construct solutions in terms of Wronskians of order N depending on $2N$ real parameters with trigonometric or hyperbolic functions.
Using a passage to the limit when one parameter goes to 0, we get rational solutions to the Gardner equation depending on $2N$ real parameters.
So we construct very easily rational solutions.
This work presents different new representations of the solutions to the Gardner equation.
These results give also an efficient method to construct an infinite hierarchy of multi-parametric families of rational solutions to the Gardner equation as a quotient of polynomials in x and t depending on $2N$ real parameters.

Article Information

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of interest disclosure: The author declare that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Supporting/Supporting organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical approval and participant consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism statement: This article was scanned by the plagiarism program.

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