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RESEARCH ARTICLE

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Buckling behavior of simply supported tapered square plates

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Abstract

Buckling is one of the important design parameters for the design of plate structures. The critical buckling loads and buckling coefficients for a series of different tapered plates are investigated. Buckling analyses of square plates are performed by changing different types of thickness variation by using ANSYS finite element software. Plates are assumed to be thin plate. The loading types and taper ratio of the simply supported square plates considerably affect the critical buckling loads. The analysis results are compared with existing literature findings and presented graphically and in tabular form. It is concluded that the analyses results are in good harmony with the literature results. The graphics proposed in this study could be useful for designers. The novelty of this study is the presentation of new straightforward formulas that provide critical buckling load coefficients separately for designers, specifically for simply supported tapered plates under uniaxial and biaxial axial loading conditions.

Keywords: Tapered plate; finite element method; critical buckling load; buckling coefficient

1. Introduction

Plates are plane structural elements whose one dimension is smaller than the other two dimensions. They utilized various areas such as structural, mechanical, marine, and aerospace engineering. These types of structural elements generally bear loads perpendicular to their plane and are also subjected to many different types of loads. In the design of plate structures, in-plane buckling loads must be taken into account, as well as other loads acting on the plate.

Tapered plates are utilized as structural elements in certain applications to achieve objectives such as weight reduction and to enhance architectural designs. Researchers have made significant studies about the buckling of variable thickness plates in the past [1-4]. Based on findings from these studies, tapered plates provide various benefits in engineering and design, including improved performance in buckling, reduced weight, optimized stiffness, enhanced aesthetics, efficient material use, and versatile functional capabilities.

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Researchers used different solution methods to calculate the critical buckling load of the plates with variable thickness. Klein developed a simple formulation for the buckling of bi-axially loaded simply supported plates that are tapered in thickness and in plan simultaneously in his report. He also presented some design curves as the result of the study [5]. Pope gives an analysis for the buckling of tapered rectangular plates in thickness. The considered plates are loaded under uniform loads in the direction of taper with different boundary conditions [6]. Harik et al., presented a semi-numerical and a semi-analytical method for the analysis of rectangular plates with varying rigidities. Investigated plates are subjected to different in-plane loads and results were presented for different boundary conditions [7]. Nerantzaki and Katsikadelis applied the analog equation method in the buckling analysis of plates of variable thickness and demonstrated that the obtained numerical results from the method were effective in his study [8]. Saeidifar et al. studied the numerical calculation of buckling loads for an elastic rectangular plate with variable thickness in one direction. In the study, the numerical examples have been presented and, also compared to some numerical results from the literature [9]. Viswanathan et al. investigated the buckling of rectangular plates of variable thickness under uniaxial in-plane load using the quintic spline approximation technique. The thickness of the plates they examined varies linearly, exponentially, and sinusoidally in one direction [10].

The finite element method, which is frequently used in studies, has also been preferred by researchers in the problem of plate buckling. Hassan and Kurgan carried out a study on the modeling and buckling analysis of rectangular plates in ANSYS [11]. Kacimi et al. have investigated the critical shear buckling stresses for simply supported plates. The plates are loaded with different ratios of shear loads. Results are also compared with analytical ones in the study [12]. Uslu et al. have studied the buckling of perforated square plates by using ANSYS software [13]. Baran and Balkan have investigated the thermal buckling of plates. In the study, they discussed the response of thermal buckling conditions under different orientation angles [14]. Deniz has studied the buckling behavior of composite plates with curved surface experimentally. In the study he evaluates the results at the view of the critical buckling load versus the hole diameter and fiber orientations [15].

In this study, the buckling behavior of simply supported tapered square plates is investigated by using finite element software. For this purpose, by tapering different thickness models simply supported on all four edges square plates were formed with a series, and the critical buckling loads of these plates under uniaxial and bi-axial loadings were calculated. For designers, simple formulas that give the critical buckling load coefficients of simply supported tapered square plates are developed for both loading cases separately.

2. Formulation of the problem

According to the classical plate theory or thin plate theory assumptions, the ratio of the short side of the plate to its thickness should be greater than 10 and less than 100. When this ratio is less than 10, it is necessary to use first order shear deformation theory or higher order theories, where different effects are also taken into account. If this ratio is greater than 100, then the structural element will need to be defined as a membrane.

According to [16], the methods used for uniform thickness plates can also be applied in the calculations of the variable thickness plates if the rigidities of the plates change slightly. Otherwise, more detailed methods should be used.

The geometry and loading condition of a thin square tapered plate is given in Fig. 1. The square tapered plate is simply supported at four edges. The square tapered plate is subject to uniaxial buckling load. This load is an in-plane compressive N load acting on and normal to the edges $x = 0$ and $x = a$. The governing equation of the plate is given in Equation 1[17].

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]+N\frac{\partial^2 w}{\partial x^2}=0 \quad (1)$$

In this equation $D=D(x,y)=\frac{Eh^3\,(x,y)}{12(1-\upsilon)^2}$ is the variable flexural stiffness of the tapered plate, $h=h(x,y)$ is the variable thickness, υ is Poisson's ratio, E is Young's modulus, $w(x,y)$ is the transverse deflection, and N is the distributed normal buckling load.

If the solution of the above equation which is a linear partial differential equation with constant coefficients is turned into a two-harmonic equation, it becomes as follows:

$$w_0=\sum_m^\infty\sum_n^\infty a_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \quad (2)$$

In this equation, m represents the half wave in longitudinal direction and n represents the half wave in transverse direction. The function $w(x,y)$ satisfies the simply supported boundary condition for the displacement. Bending moment M_n can be described as follows:

$$M_n=-D\left[\left(\frac{m\pi}{a}\right)^2+\nu\left(\frac{n\pi}{b}\right)^2\right] sin\frac{m\pi x}{b} sin\frac{n\pi y}{b} \quad (3)$$

In a simply supported on four edges square plate, the bending moment at the edges of the plate is $M_n=0$. If this situation is substituted into the $w(x,y)$ equation, the following expression is obtained:

$$\left\{D\left[\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right]-N\left(\frac{m\pi}{a}\right)^2\right\} sin\frac{m\pi x}{b} sin\frac{n\pi y}{b}=0. \quad (4)$$

When the equation is simplified, the following expression is obtained:

$$N=\frac{\pi^2 a^2 D}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \quad (5)$$

When the denominator of one of the terms in Equation 5 becomes equal to zero, the critical buckling load, which is the smallest value of N, occurs. In this equation m must be chosen so as to make the equation minimum [12]. This critical value is obtained, by taking $n=1$ in the equation. Hence the Equation 5 becomes as in Equation 6.

$$N_{cr}=k\frac{\pi^2 D}{a^2} \quad (6)$$

The bending rigidity of the tapered rectangular plate D can be defined by using the function as in Equation 7.

$$D = D_m \left[\left(\frac{h_0 + h_1}{2} \right) \left(h_0 + \frac{h_1 - h_0}{a} x \right)^{-3} \right] \tag{7}$$

In the equation Dm is the bending rigidity value at the middle section of the plate at $x = a/2$. The critical buckling coefficient can be stated in terms of Dm in Equation 8.

$$k = \frac{a^2 \, N_{cr}}{\pi^2 \, D_m} \tag{8}$$

The critical buckling coefficient k, defined in terms of D_m, is independent of the thickness variation direction. This coefficient for the plates with a small taper ratio h_1 / h_0 will be almost the same as the uniform thickness plates [4].

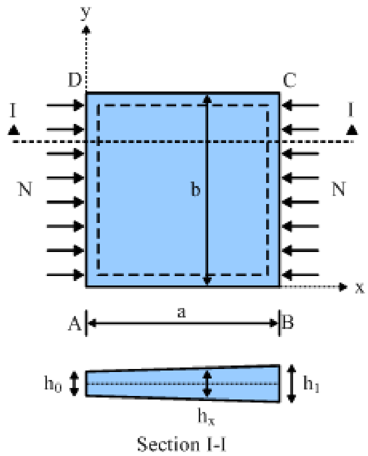

Fig. 1. Geometry and loading condition of tapered square plate.

3. Finite element software

Buckling analyses were conducted using the well-established finite element software ANSYS. Initially, the material properties and geometric parameters were defined within the ANSYS package. Subsequently, the finite element mesh was generated by selecting an appropriate element from the library. Following mesh creation, the plate thickness variation was implemented using APDL (Ansys Parametric Design Language) within the program. Boundary conditions and loads were then specified, and both static and buckling analyses were performed accordingly. The choice of element from the library plays a crucial role in determining result accuracy and computational efficiency. In this study, the SHELL181 element was utilized.

The SHELL181 element is specifically designed for analyzing thin plate structures. The reliability and accuracy of the SHELL181 element for buckling analysis have been validated through numerous studies in the literature [11]. It features four nodes positioned at the corners, with each node having six degrees of freedom: translations in the x, y, and z directions, and rotations about the x, y, and z axes. SHELL181 is well-suited for applications involving linear, large rotation, and/or large strain nonlinear behavior [18]. Various element properties can be adjusted using key option (KEYOPT) settings in the software. This element is suitable for conducting both bending and buckling analyses of plates. The geometry of the SHELL181 element is illustrated in Fig. 2.

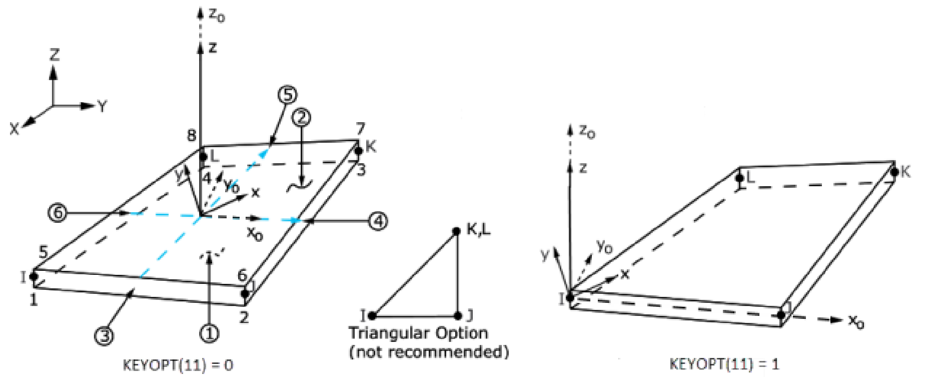

Fig. 2. SHELL181 finite element [11].

4. Material, model and loading conditions

The numerical analyses employ structural steel material. The Young's modulus and Poisson's ratios of the structural steel are 2.10^5 MPa, and 0.30, respectively.

The dimensions of the square tapered plates used in the numerical analyses are $a = 1000\ mm$ and $b = 1000$ mm, respectively. The plate thickness varies linearly along the x-axis as shown in Fig. 1. In this study, $h_0(\textit{thickness at the } x = 0)$ is constant and equals to 10 mm. And $h_1(\textit{thickness at the } x = a)$ varies according to the taper ratio. According to the thin plate assumptions, taper ratio h_1 / h_0 is up to a maximum of 10. In this study, the taper ratio varies from 1 to 10 in increments of 0.125. The boundary conditions of the square tapered plate are given in the Table 1. In the Table 1, $u, v,$ *and* w denote the deflections at the x, y, and z directions.

Analyses are performed for tapered square plates with different in-plane loadings as uniaxial and biaxial as shown in Fig. 3. In uniaxial loading, in-plane loads of uniform compressive forces N per unit length is subjected parallel to the tapered direction x. In biaxial loading, in-plane loads of the same uniform compressive forces N per unit length is subjected perpendicular to the tapered direction x in addition to the uniaxial buckling loadings.

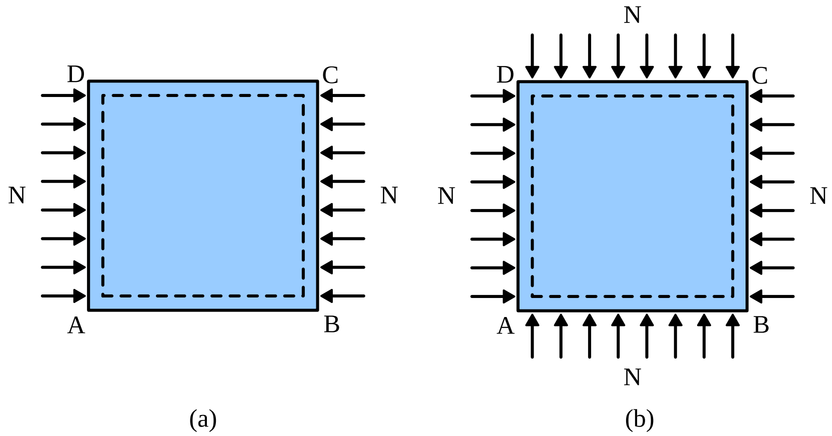

Fig. 3. The loading conditions of the square tapered plate: (a) uniaxial load and (b) biaxial load.

Table 1. Boundary conditions of the square tapered plate.

Edge or Point	Boundary Condition 1	Boundary Condition 2
Edge AB	$w(x,0)=0$	$\frac{\partial w(x,0)}{\partial x}=0$
Edge BC	$w(a,y)=0$	$\frac{\partial w(a,y)}{\partial y}=0$
Edge CD	$w(x,a)=0$	$\frac{\partial w(x,a)}{\partial x}=0$
Edge DA	$w(0,y)=0$	$\frac{\partial w(0,y)}{\partial y}=0$
Point A	$v(0,0)=0$	-
Point B	$v(a,0)=0$	-
Midpoint of edge AB	$u\left(\frac{a}{2},0\right)=0$	-

5. Results and discussion

The buckling coefficients obtained by finite element models for simply supported square tapered plates with different taper ratios, along with the corresponding values from Wittrick and Ellen [4], are listed in Table 2. As shown in Table 2, the results from the finite element model closely match those from Wittrick and Ellen.

Table 2. The verification of the buckling coefficients of uniaxial loaded tapered square plates.

h_1/h_0	$k^{Wittrick}$	k^{ANSYS}	$k^{Diference}$(%)

1.125	3.966	3.982	0.41
1.250	3.882	3.900	0.47
1.500	3.638	3.662	0.65
1.750	3.364	3.393	0.86
2.000	3.100	3.132	1.03

The calculated buckling coefficients for tapered plates under uniaxial and biaxial loadings are provided in Table 3. As seen from Table 3, the buckling coefficient decreases as the taper ratio increases. For a given taper ratio, the uniaxial buckling coefficient is roughly twice as large as the biaxial buckling coefficient. Figure 4 illustrates the buckling coefficients for both loading cases obtained from the numerical models. As shown in Fig. 4, the buckling coefficient decreases as the taper ratio increases for both uniaxial and biaxial loadings. Additionally, the uniaxial and biaxial buckling coefficients become closer to each other as the taper ratio increases.

Table 3. The buckling coefficients of simply supported tapered square plates under uniaxial and biaxial loadings

h_1/h_0	Buckling Coefficient		h_1/h_0	Buckling Coefficient		h_1/h_0	Buckling Coefficient	
	Uniaxial	Biaxial		Uniaxial	Biaxial		Uniaxial	Biaxial
1.000	3.998	1.999	4.125	1.713	1.158	7.250	0.957	0.736
1.125	3.965	1.991	4.250	1.664	1.134	7.375	0.939	0.724
1.250	3.882	1.970	4.375	1.618	1.110	7.500	0.922	0.714
1.375	3.770	1.940	4.500	1.574	1.087	7.625	0.905	0.703
1.500	3.642	1.905	4.625	1.532	1.065	7.750	0.889	0.693
1.625	3.508	1.866	4.750	1.491	1.044	7.875	0.874	0.683
1.750	3.373	1.825	4.875	1.453	1.023	8.000	0.858	0.673
1.875	3.240	1.783	5.000	1.416	1.003	8.125	0.844	0.664
2.000	3.111	1.740	5.125	1.381	0.984	8.250	0.830	0.655
2.125	2.988	1.697	5.250	1.347	0.965	8.375	0.816	0.646
2.250	2.871	1.655	5.375	1.315	0.947	8.500	0.802	0.637
2.375	2.760	1.614	5.500	1.284	0.930	8.625	0.790	0.628
2.500	2.655	1.574	5.625	1.254	0.913	8.750	0.777	0.620
2.625	2.555	1.535	5.750	1.226	0.897	8.875	0.765	0.612
2.750	2.461	1.497	5.875	1.199	0.881	9.000	0.753	0.604
2.875	2.373	1.461	6.000	1.172	0.866	9.125	0.741	0.596
3.000	2.289	1.425	6.125	1.147	0.851	9.250	0.730	0.588
3.125	2.210	1.391	6.250	1.123	0.837	9.375	0.719	0.581
3.250	2.136	1.358	6.375	1.099	0.823	9.500	0.708	0.574
3.375	2.065	1.326	6.500	1.077	0.809	9.625	0.698	0.566
3.500	1.998	1.296	6.625	1.055	0.796	9.750	0.688	0.560
3.625	1.935	1.266	6.750	1.034	0.783	9.875	0.678	0.553
3.750	1.875	1.238	6.875	1.013	0.771	10.000	0.668	0.546
3.875	1.818	1.210	7.000	0.994	0.759			
4.000	1.764	1.184	7.125	0.975	0.747			

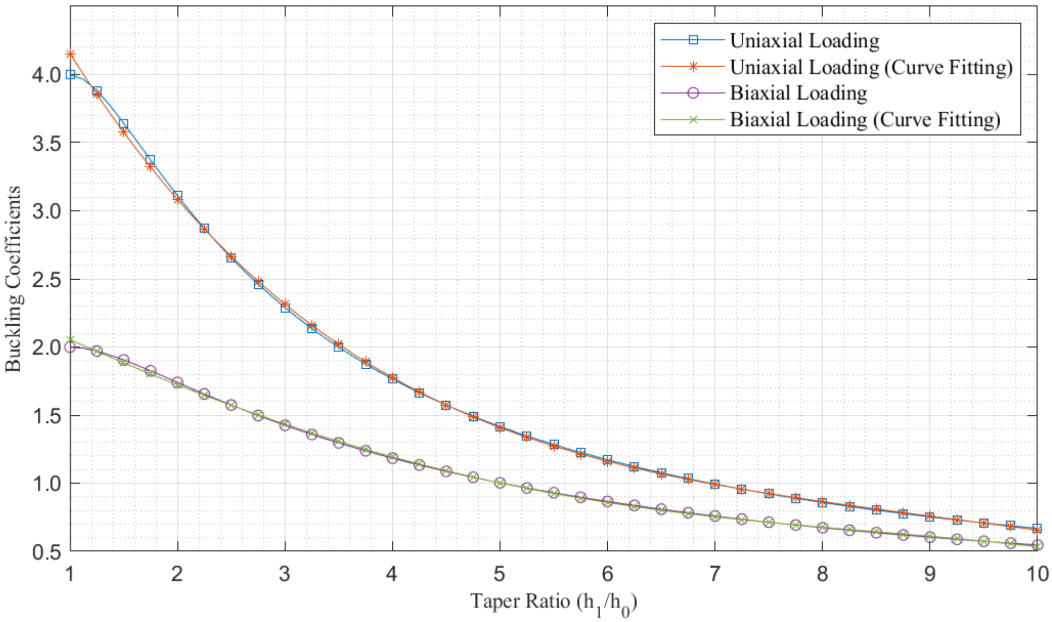

Fig. 4. The buckling coefficients for uniaxial and biaxial loadings.

MATLAB was used to fit curves to the numerical results, as depicted in Fig. 4. The formulas of these fitted curves are given by Equation 9 and Equation 10 for uniaxial and biaxial loadings, respectively. These formulas can be utilized in the design of simply supported tapered square plates.

$$k_u = 0.0003741\left(\frac{h_1}{h_0}\right)^4 - 0.0153\left(\frac{h_1}{h_0}\right)^3 + 0.2328\left(\frac{h_1}{h_0}\right)^2 - 1.668\left(\frac{h_1}{h_0}\right) + 5.603 \quad (9)$$

$$k_b = -0.0001848\left(\frac{h_1}{h_0}\right)^4 + 0.002808\left(\frac{h_1}{h_0}\right)^3 + 0.00918\left(\frac{h_1}{h_0}\right)^2 - 0.3763\left(\frac{h_1}{h_0}\right) + 2.419 \quad (10)$$

6. Conclusion

The critical buckling coefficient of tapered square plate under uniaxial and biaxial loadings are calculated by using general purpose finite element software. Plate is simply supported in all four edges. The plates are tapered in only x direction and these non-uniform plates with linear thickness variations are loaded in-plane in the direction of thickness variations. The key findings from the research can be summarized as below:

- While creating the simply supported steel plate models, the thickness at the (x=0) AD edge of the plate was taken as constant h_0.
- Plates have a small taper ratio of h_1 / h_0 along the x-axis.
- Plate thickness varies slightly, linearly and the taper ratio varies from 1 to 10.

- Formulas have been developed separately for the critical buckling load coefficients for uniaxial and biaxial loading cases.
- Curves for design are presented for uniaxial and biaxial loadings.
- In tapered plates, it has been determined that when the thickness ratio or the slope increases by a very small amount the critical buckling load change more rapidly.
- While the difference in buckling coefficient between uniaxial and biaxial loading is approximately 2 for uniform thickness, this difference decreases to 0.122 as the taper ratio increases.

In future research, different parameters could be used for the critical buckling investigation of non-uniform plates under in plane loads. These parameters could be aspect ratios, materials, boundary conditions, thickness functions, etc.

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References

[1] S. S. Hwang, "Stability of plates with piecewise varying thickness," *J. Appl. Mech.*, vol. 40, no. 4, pp. 1127–1129, 1973
[2] P. V. Navaneethakrishnan, "Buckling of nonuniform plates: spline method," *J. Eng. Mech.*, vol. 114, no. 5, 1988.
[3] C. M. Wang, G. M. Hong, and T. J. Tan, "Elastic buckling of tapered circular plates," *Comput. Struct,* vol. 55, no. 6, pp. 1055–1061, 1995.
[4] W. H. Wittrick and C. H. Ellen, "Buckling of tapered rectangular plates in compression," *Aeronautical Quarterly*, vol. 13, no. 4, pp. 308–326, Nov. 1962, doi: 10.1017/s0001925900002547.
[5] B. Klein, "The buckling of tapered plates in compression compressive buckling under varying loading of simply supported plates simultaneously tapered in plan form and in thickness," *Aircraft Engineering and Aerospace Technology*, vol. 28, no. 12, pp. 427-430, 1956, doi: 10.1108/eb032774.
[6] G. G. Pope, "The buckling of plates tapered in thickness," 1963.
[7] I. E. Harik, X. Liu, and R. Ekambaram, "Elastic stability of plates with varying rigidities," *Computers and Structures*, vol. 38, no. 2, pp. 161–168, 1991.
[8] M. S. Nerantzaki and J. T. Katsikadelis, "Buckling of plates with variable thickness—an analog equation solution," *Eng Anal Bound Elem*, vol. 18, no. 2, pp. 149–154, Sep. 1996.
[9] M. Saeidifar, S. N. Sadeghi, and M. R. Saviz, "Analytical solution for the buckling of rectangular plates under uni-axial compression with variable thickness and elasticity modulus in the y-direction," *Proc Inst Mech Eng C J Mech Eng Sci*, vol. 224, no. 1, pp. 33–41, Aug. 2009, doi: 10.1243/09544062JMES1562.
[10] K. K. Viswanathan, P. V. Navaneethakrishnan, and Z. A. Aziz, "Buckling analysis of rectangular plates with variable thickness resting on elastic foundation," in *IOP Conference Series: Earth and Environmental Science*, Institute of Physics Publishing, 2015. doi: 10.1088/1755-1315/23/1/012006.
[11] A. Hassan and N. Kurgan, "Modeling and buckling analysis of rectangular plates in ANSYS," *International Journal of Engineering and Applied Sciences*, vol. 11, no. 1, pp. 310–329, May 2019, doi: 10.24107/ijeas.531011.
[12] A. Kacimi, G. Ikhenazen, F. Mohri, M. Saidani and M. Kilardj, "Linear and nonlinear stability analysis of thin rectangular plates subjected to local in-plane shearing," *Structural Engineering International*, vol. 33, no. 4, pp. 689–698, 2023, doi: 10.1080/10168664.2023.2208143.
[13] F. Uslu, M.H. Saraçoğlu and U. Albayrak, "Buckling of Square and Circular Perforated Square Plates under Uniaxial Loading," *Journal of Innovations in Civil Engineering and Technology*, vol. 4, no. 2, pp. 61–75, 2022.
[14] F. Baran and D. Balkan, "Thermal buckling analysis of laminated plates with variable angle fiber orientation," *Niğde Ömer Halisdemir University Journal of Engineering Sciences*, vol. 12, no. 3, pp. 912–918, 2023, doi: 10.28948/ngumuh.1241416.
[15] M. E. Deniz, "Buckling behavior of curved composite plates with a central circular hole," Dicle University Journal of Engineering, vol. 8, no. 1, pp. 203–212, 2017.
[16] Timoshenko, S., Woinowsky-Krieger, S., Theory of Plates and Shells. McGraw-Hill, Inc., 1959.
[17] D. Çıraklı, M.H. Saraçoğlu, U. Albayrak, "The effect of layer thicknesses on buckling behavior in foam core sandwich plates with various materials," *International Journal of Technological Sciences*, vol. 15, no. 3, pp. 111–117, Dec 2023, doi: 10.55974/utbd.1373786.
[18] A. Swanson Analysis System Inc., "ANSYS User's manual." 2005.