Research Article GU J Sci, Part A, 11(1): 210-224 (2024) 10.54287/gujsa.1427944

Gazi University
Journal of Science
PART A: ENGINEERING AND INNOVATION
http://dergipark.org.tr/gujsa

Falling Body Motion in Time Scale Calculus

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Keywords

Time Scale Calculus
Falling Body Motion
Delta Derivative
H-Calculus
Q-Calculus

Abstract

The falling body problem for different time scales, such as $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$, $q\mathbb{N}0$, $\mathbb{P}c,d$ is the subject of this study. To deal with this problem, we use time-scale calculus. Time scale dynamic equations are used to define the falling body problem. The exponential time scale function is used for the solutions of these problems. The solutions of the falling body problem in each of these time scales are found. Moreover, we also test our mathematical results with numerical simulations.

Cite

Pelen, N. N., & Kayar, Z. (2024). Falling Body Motion in Time Scale Calculus. *GU J Sci, Part A, 11*(1), 210-224. doi:10.54287/gujsa.1427944

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Article Process

Submission Date	29.01.2024
Revision Date	20.02.2024
Accepted Date	08.03.2024
Published Date	21.03.2024

1. INTRODUCTION

Limits in calculating derivatives of real functions are the essential tools of regular calculus. However, many real-life problems are discussed in both continuous and discrete domains. To unify discrete and continuous domains Time Scale Calculus was established in the thesis of Stephan Hilger (Hilger, 1988). After that many studies were done about this issue. Bohner and Peterson (2001) is one of the most important books that explains the significant part of this calculus. The following articles show the results on the dynamic equations in time scale calculus; Akın and Bohner (2003), Akın et al. (2020), Anderson (2005), Kayar et al. (2022), Kayar and Kaymakçalan (2022a) and Kayar and Kaymakçalan (2022b). In the literature, many different kinds of calculus were defined. For instance, one can find the classical calculus, discrete calculus, h-calculus and q-calculus in the literature. Time scale calculus contains all of these in itself. Therefore, the results that we have found in time-scale calculus are general ones and one can reduce these results to these specific calculus types. In Alanazi et al. (2020), the falling body problem was studied by using q-calculus whose domain is $q^{\mathbb{N}_0}$. The basic formula for q-calculus is formerly obtained by Euler, nevertheless, its calculus was introduced by Jackson (1910). In this study, we generalize these results by using time scale calculus and we also show that our results coincide with the results in Alanazi et al. (2020). Moreover, we support our results with the numerical simulations.

2. METHODS AND PRELIMINERIES

The information in that section is taken from Bohner and Peterson (2001). The nonempty closed subset of real numbers is defined as a time scale and denoted by $\boldsymbol{T}$. The *forward jump* operator $\sigma: \boldsymbol{T} \to \boldsymbol{T}$ is defined as

$$\sigma(t) := inf\{s \in \boldsymbol{T}: s > t\},$$

for $t \in \boldsymbol{T}$ while the *backward jump* operator $\rho: \boldsymbol{T} \to \boldsymbol{T}$ is defined by

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$$\rho(t) := sup\{s \in \boldsymbol{T} : s < t\}.$$

If $t < sup\ \{\boldsymbol{T}\}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > inf\{\boldsymbol{T}\}$ and $\rho(t) = t$, then t is called *left-dense*. Besides, if $\rho(t) < t$, we say that t is *left-scattered*. The graininess function $\mu : \boldsymbol{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 1: We say that a function $p : \boldsymbol{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \boldsymbol{T}^k = \boldsymbol{T}/(\rho(sup\{T\}), sup\{T\}]$ holds. The set of all regressive and rd-continuous functions $p : \boldsymbol{T} \to \mathbb{R}$ is denoted by

$$\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}(\boldsymbol{T}) = \boldsymbol{\mathcal{R}}(\boldsymbol{T}, \mathbb{R}).$$

Definition 2: If $p \in \boldsymbol{\mathcal{R}}$, then we define the exponential function by

$$e_p(t,s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for} \quad s,t \in \boldsymbol{T}$$

where the cylinder transformation is defined by

$$\xi_h(z) := \begin{cases} \dfrac{Log(1+zh)}{h}, h \neq 0 \\ z, h = 0 \end{cases}$$

Theorem 1: Time scale exponential function has following properties.

i. $e_p(t,s) = \frac{1}{e_p(s,t)}$
ii. $\left(\frac{1}{e_p(.,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$
iii. $e_{\ominus p}(t,\tau) = \frac{1}{e_p(t,\tau)}$

Definition 3: If $p \in \boldsymbol{\mathcal{R}}$ then the first order linear dynamic equation

$$y^{\Delta} = p(t)y(t) \tag{1}$$

is called regressive.

Theorem 2:*(Variations of Constants Formula)*

Suppose (1) is regressive. Let $t_0 \in \boldsymbol{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem

$$y^{\Delta} = p(t)y + f(t), y(t_0) = y_0 \tag{2}$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau \tag{3}$$

Theorem 3: *(Variations of Constants Formula)*

Suppose (1) is regressive. Let $t_0 \in \boldsymbol{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem

$$y^{\Delta} = -p(t)y^{\sigma} + f(t), y(t_0) = y_0 \tag{4}$$

is given by

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau, \tag{5}$$

where $p(t) = \frac{p(t)}{1+\mu(t)p(t)}$.

2.1. Preliminaries for Falling Body Problem

Let us assume that in a constant gravitational field, a particle of mass m falls through the air from a height h with an initial speed v_0. It then encounters a resistance force that opposes the relative motion by which the particle moves relative to the air. It is known that this resistance force is related to relative speed. For slow speeds, the magnitude of the resistance force is proportional to the speed. But in other cases it may be proportional to the square of the velocity (or some other force). By applying Newton's second law we get

$$m\frac{dv}{dt} = -mg - mkv, \tag{6}$$

where k represents a positive constant indicating the strength of the retarding force, the inverse of seconds is its dimensionality and, $-mkv$, is a positive upward force because we take z and $v = z'$ to be positive upward, and the motion is downward, that is, $v < 0$, so that, $-kmv > 0$. The solution of the equation (6) with the initial condition $v(0) = v_0$ is given by

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}. \tag{7}$$

By using the initial condition $z(0) = h$ and integrating (7), we get

$$z(t) = h - \frac{gt}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right), \tag{8}$$

see (Thornton, 2004).

3. MAIN RESULTS

3.1 Main Results for Falling Body Problem in Time Scale Calculus

The equation of motion (6) in view of the time scale calculus becomes

$$mv^{\Delta} = -mg - mkv \tag{9}$$

Or

$$v^{\Delta} = -g - kv. \tag{10}$$

Let

$$v^{\Delta} = -kv \tag{11}$$

be regressive. Then the solution of initial value problem

$$v^{\Delta} = -g - kv, \qquad v(0) = v_0 \tag{12}$$

can be obtained by Variation of Constants Formula (3) as

$$v(t) = e_{-k}(t,0)v_0 - g\int_0^t e_{-k}\left(t,\sigma(\tau)\right)\Delta\tau.$$

By using the properties given in Theorem 1 we have

$$e_{-k}(t,\sigma(\tau)) = \frac{1}{e_{-k}(\sigma(\tau),t)} = \frac{1}{k}\frac{k}{e_{-k}(\sigma(\tau),t)} = \frac{1}{k}\left(\frac{1}{e_{-k}(\tau,t)}\right)^{\Delta}.$$

Therefore,

$$\int_0^t e_{-k}\left(t,\sigma(\tau)\right)\Delta\tau = \int_0^t \frac{1}{k}\left(\frac{1}{e_{-k}(\tau,t)}\right)^{\Delta}\Delta\tau = \frac{1}{k}\left[1 - \frac{1}{e_{-k}(0,t)}\right] = \frac{1}{k}\left[1 - e_{-k}(t,0)\right]$$

implies that

$$v(t) = e_{-k}(t,0)\left[v_0 + \frac{g}{k}\right] - \frac{g}{k}. \tag{13}$$

The vertical distance $z(t)$ in time-scale calculus is governed by

$$z^{\Delta} = e_{-k}(t,0)\left[v_0 + \frac{g}{k}\right] - \frac{g}{k} \tag{14}$$

where $v(t) = z^{\Delta}(t)$. Integrating (14) implies that

$$z(t) = h - \frac{gt}{k} + \left[v_0 + \frac{g}{k}\right]\int_0^t e_{-k}(s,0)\,\Delta s = h - \frac{gt}{k} - \frac{1}{k}\left[v_0 + \frac{g}{k}\right][1 - e_{-k}(t,0)] = h - \frac{gt}{k} + \frac{1}{k}\left[v_0 + \frac{g}{k}\right][1 - e_{-k}(t,0)].$$

Remark 1: Let us choose the equation of motion (6) in view of the time scale calculus as

$$mv^{\Delta} = -mg - mkv^{\sigma}$$

or

$$v^{\Delta} = -g - kv^{\sigma}.$$

Let us assume that these equations are regressive. Then the solution of the initial value problem

$$v^{\Delta} = -g - kv^{\sigma}, \qquad v(0) = v_0$$

can be obtained by Variation of Constants Formula (5) as

$$v(t) = e_{\ominus(-k)}(t,0)v_0 - g\int_0^t e_{\ominus(-k)}(t,\tau)\,\Delta\tau.$$

By using the properties given in Theorem 1 we have

$$e_{\ominus(-k)}(t,\tau) = \frac{1}{e_{-k}(t,\tau)} = e_{-k}(\tau,t).$$

Therefore,

$$\int_0^t e_{\ominus(-k)}(t,\tau)\Delta\tau = \int_0^t e_{-k}(\tau,t)\Delta\tau = -\frac{1}{k}\int_0^t -ke_{-k}(\tau,t)\Delta\tau = -\frac{1}{k}[1-e_{-k}(0,t)] = \frac{1}{k}\left[e_{\ominus(-k)}(t,0)-1\right]$$

implies that

$$v(t) = e_{\ominus(-k)}(t,0)v_0 - \frac{g}{k}\left[e_{\ominus(-k)}(t,0)-1\right] = e_{\ominus(-k)}(t,0)\left[v_0-\frac{g}{k}\right]+\frac{g}{k}.$$

The vertical distance $z(t)$ in time-scale calculus is governed by

$$z^{\Delta} = e_{\ominus(-k)}(t,0)\left[v_0-\frac{g}{k}\right]+\frac{g}{k},$$

where $v(t) = z^{\Delta}(t)$. After integration, we get

$$z(t) = h + \frac{gt}{k} + \left[v_0-\frac{g}{k}\right]\int_0^t e_{\ominus(-k)}(s,0)\,\Delta s = h + \frac{gt}{k} + \frac{1}{k}\left[v_0-\frac{g}{k}\right]\left[1-e_{\ominus(-k)}(t,0)\right].$$

3.2 Special Cases

Below the falling body problems for the important special time scales is considered.

3.2.1. T=$\mathbb{R}$

We consider the special case where $\mu(t) = 0$ for all $t \in \boldsymbol{T}$. In this case equation (9) becomes as equation (6) and solution (13) reduces to solution (7), see (Thornton, 2004).

3.2.2. T=$\mathbb{Z}$

We consider the special case where $\mu(t) = 1$ for all $t = n \in \boldsymbol{T}$. In this case equation (10) becomes as

$$v(n+1) - v(n) = \Delta v = -mg - mkv$$

and solution (13) reduces to

$$v(n) = (1-k)^n v_0 - g\sum_{i=0}^{n-1}(1-k)^{n-i-1} = (1-k)^n v_0 - g\sum_{i=0}^{n-1}(1-k)^i$$

$$= (1-k)^n v_0 - g\,\frac{1-(1-k)^n}{k} = (1-k)^n\left[v_0+\frac{g}{k}\right]-\frac{g}{k},$$

see (Elaydi (2005), page 4).

The vertical distance $z(t)$ in time scale calculus is governed by

$$\Delta z = (1-k)^n\left[v_0+\frac{g}{k}\right]-\frac{g}{k}, \tag{15}$$

where $v(n) = \Delta z(n)$. Summation (15) implies that

$$z(n) = h - \frac{gn}{k} + \frac{1}{k}\left[v_0+\frac{g}{k}\right]\left[1-(1-k)^n\right]$$

3.2.3. T=$\boldsymbol{h}\mathbb{Z}$, where $\mathbf{0} < \boldsymbol{h} \in \mathbb{R}$

We consider the special case where $\mu(t) = h$ for all $t = n \in \boldsymbol{T}$. In this case equation (10) becomes as

$$\frac{v(n+h)-v(n)}{h} = v^{\Delta} = -g-kv$$

and solution (13) reduces to

$$v(t) = (1-kh)^{\frac{t}{h}}\left[v_0+\frac{g}{k}\right]-\frac{g}{k}.$$

The vertical distance $z(t)$ becomes

$$z(t) = h-\frac{gt}{k}+\frac{1}{k}\left[v_0+\frac{g}{k}\right][1-(1-kh)^{t/h}]$$

Moreover, one can see falling body motion and its velocity behavior when $\boldsymbol{T}$ is taken as $\mathbb{Z}, \boldsymbol{h}\mathbb{Z}$ and $\mathbb{R}$ in Figure 1, Figure 2, Figure 3 and Figure 4 for different initial velocities and initial heights.

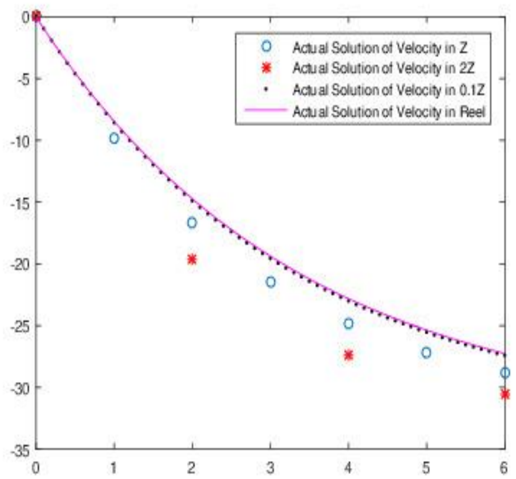

Figure 1. Graphs when $v_0 = 0$

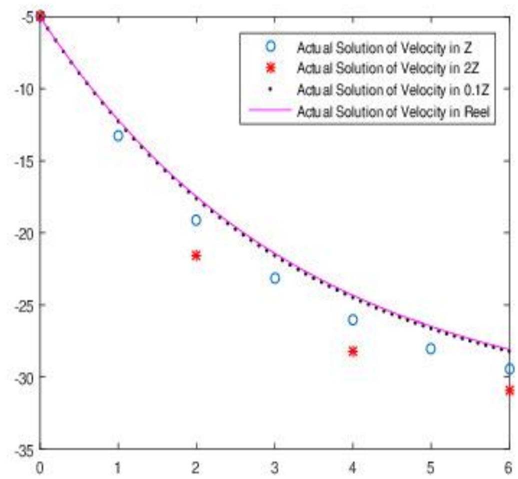

Figure 2. Graphs when $v_0 = -5$

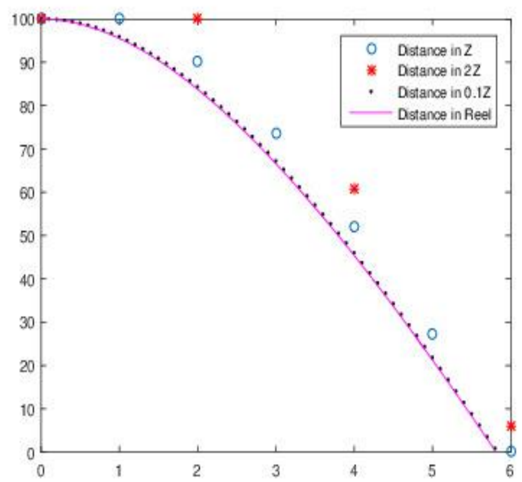

Figure 3. Graphs when $v_0 = 0$ and $h_0 = 116,7$

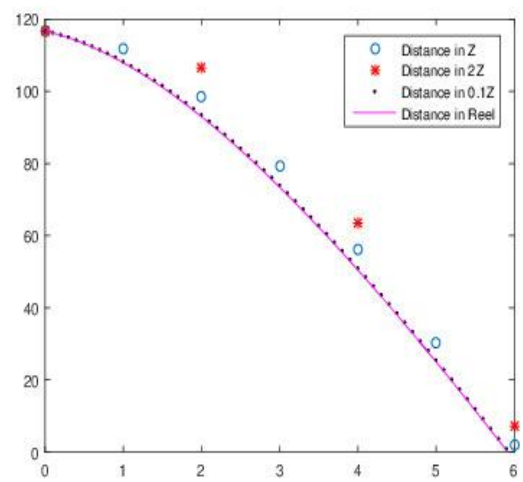

Figure 4. Graphs when $v_0 = -5$ and $h_0 = 120$

3.2.4. T=$q^{\mathbb{N}_0} = \{1, q, q^2, q^3, \dots\}$, where $1 < q \in \mathbb{R}$

We consider the special case where $\sigma(t) = qt$ and $\mu(t)$ depends on t for all $t \in \boldsymbol{T}$. In this case equation (9) becomes as equation (14) in Alanazi et al. (2020). and solution (13) reduces to equation (26) in Alanazi et al. (2020). In Figure 5 and Figure 6 one can see the velocity and distance behavior when T=$1.5^{\mathbb{N}_0}$.

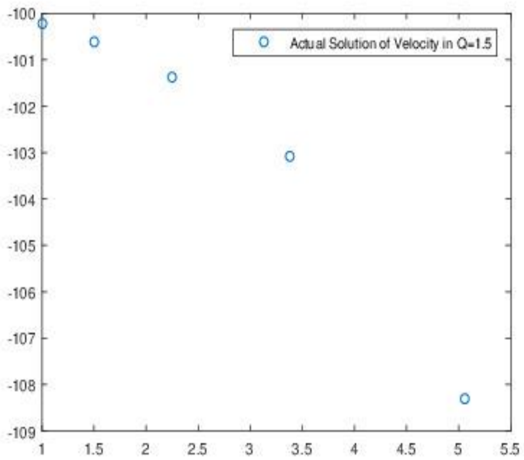

Figure 5. Velocity graphs for $1.5^{\mathbb{N}_0}$

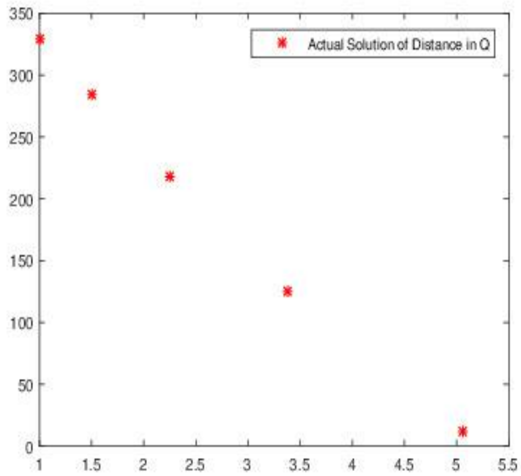

Figure 6. Distance graphs for $1.5^{\mathbb{N}_0}$

3.2.5. T=$\mathbb{P}_{c,d} = \cup_{i=0}^{\infty}[i(c+d), i(c+d)+c]$, where $0 < c, d \in \mathbb{R}, i \in \mathbb{N}_0$

We consider the special case where

$$\sigma(t) = \begin{cases} t, if t \in \cup_{i=0}^{\infty}[i(c+d), i(c+d)+c], \\ d, if t \in \cup_{i=0}^{\infty}\{i(c+d)+c\}, \end{cases}$$

$$\mu(t) = \begin{cases} 0, if t \in \cup_{i=0}^{\infty}[i(c+d), i(c+d)+c], \\ b, if t \in \cup_{i=0}^{\infty}\{i(c+d)+c\}, \end{cases}$$

In this case equation (10) becomes as

$$\frac{dv}{dt} = -g - kv, t \in [i(c+d), i(c+d)+c].$$

We will find the solution interval by interval.

First Case: Let us assume that the particle stops at the end of each interval $[i(c+d), i(c+d)+c]$ for all $i = 0,1,2,..$ automatically, i.e without a force.

In this case the initial velocity at the beginning of each interval should be

$$v_i = \frac{g}{k}[e^{kc} - 1]$$

for all $i = 0,1,2,\cdots$. Indeed, for the interval $t \in [0, c]$, we have

$$\frac{dv}{dt} = -g - kv,$$

$$v(0) = v_0$$

and the solution becomes

$$v(t)=e_{-k}^{P}(t,0)\left[v_0+\frac{g}{k}\right]-\frac{g}{k}=e^{-kt}\left[v_0+\frac{g}{k}\right]-\frac{g}{k},$$

where $e_{\alpha}^{P}(t,t_0)$ is exponential function for $\mathbb{P}_{c,d}$ defined by $e_{\alpha}^{P}(t,t_0)=e^{\alpha(t-t_0)}$.

If the particle stops at the point when $t=c$, then its velocity should be zero at the point when $t=c$ and so, $v(c)=0$. This implies that $v_0=\frac{g}{k}[e^{kc}-1]$.

For each interval $t\in[i(c+d),i(c+d)+c]$, we can find the solution as

$$v(t)=e_{-k}^{P}(t,i(c+d))\left[v_i+\frac{g}{k}\right]-\frac{g}{k}=e^{-k(t-i(c+d))}\left[v_i+\frac{g}{k}\right]-\frac{g}{k}$$

and the initial velocity of each interval as $v_i=\frac{g}{k}[e^{kc}-1]$ at the point $t=i(c+d)$ for all $i=0,1,2,\cdots$.

In this case at the points when $t=i(c+d)+c$, we have

$$\Delta v(i(c+d)+c)=v\big((i+1)(c+d)\big)-v(i(c+d)+c)=v\big((i+1)(c+d)\big)=v_{i+1}=\frac{g}{k}[e^{kc}-1]$$

Let us consider the displacement of the particle. Different than the other time scales, in this case z does not denote the vertical distance but it denotes the displacement of the particle.

For the interval $t\in[0,c]$, we have

$$\frac{dz}{dt}=v=e^{-kt}\left[v_0+\frac{g}{k}\right]-\frac{g}{k}$$

$$z(0)=h$$

where $v_0=v_i=\frac{g}{k}[e^{kc}-1]$ for all $i=0,1,2,\ldots$ and the solution becomes

$$z(t)=h-\frac{gt}{k}+\frac{1}{k}\left(v_0+\frac{g}{k}\right)(1-e^{-kt})$$

which shows the displacement of the particle over the first c seconds.

For the displacement of the particle at the point when $t=c$, we need to compute

$$z(c)=h-\frac{gc}{k}+\frac{1}{k}\left(v_0+\frac{g}{k}\right)(1-e^{-kc})=h_0.$$

Therefore, the particle is falling from the point $z=h$ and stops at the point $z=h-h_0$ at $t=c$.

For the interval $t\in[c+d,2c+d]$, we have

$$\frac{dz}{dt}=v=e^{-kt}\left(v_0+\frac{g}{k}\right)-\frac{g}{k},$$

$$z(c+d)=h-h_0$$

and the solution becomes

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$$z(t) = h - h_0 - \frac{g}{k}(t - c - d) + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(e^{-k(c+d)} - e^{-kt})$$

which shows the displacement of the particle over the interval $[c + d, 2c + d]$.

For the displacement of the particle at the point when $t = 2c + d$, we need to compute

$$z(2c + d) = h - h_0 - \frac{gc}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})e^{-k(c+d)} = \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})(e^{-k(c+d)} - 1) = h_1.$$

Therefore, the particle is falling from the point $z = h - h_1$ and stops at the point $z = h - h_0 - h_1$ at $t = 2c + d$.

For the interval $t \in [2c + 2d, 3c + 2d]$, we have

$$\frac{dz}{dt} = v = e^{-kt}\left(v_0 + \frac{g}{k}\right) - \frac{g}{k}$$

$$z(2c + 2d) = h - h_0 - h_1$$

and the solution becomes

$$z(t) = h - h_0 - h_1 - \frac{g}{k}(t - 2c - 2d) + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(e^{-k(2c+2d)} - e^{-kt})$$

which shows the displacement of the particle over the interval $[2c + 2d, 3c + 2d]$.

For the displacement of the particle at the point when $t = 3c + 2d$, we need to compute

$$z(3c + 2d) = h - h_0 - h_1 - \frac{gc}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})e^{-k(2c+2d)}$$

$$= \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kc}\right)\left(e^{-k(c+d)\}} - 1\right)e^{-k(c+d)} = h_2.$$

Therefore, the particle is falling from the point $z = h - h_1$ and stops at the point $z = h - h_0 - h_1 - h_2$ at $t = 3c + 2d$.

For the interval $t \in [3c + 3d, 4c + 3d]$, we have

$$\frac{dz}{dt} = v = e^{-kt}\left[v_0 + \frac{g}{k}\right] - \frac{g}{k}$$

$$z(3c + 3d) = h - h_0 - h_1 - h_2$$

and the solution becomes

$$z(t) = h - h_0 - h_1 - h_2 - \frac{g}{k}(t - 3c - 3d) + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(e^{-k(3c+3d)} - e^{-kt})$$

which shows the displacement of the particle over the interval $[3c + 3d, 4c + 3d]$.

For the displacement of the particle at the point when $t = 4c + 3d$, we need to compute

$$z(4c + 3d) = h - h_0 - h_1 - h_2 - \frac{gc}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})e^{-k(3c+3d)}$$

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$$= \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kc}\right)\left(e^{-k(c+d)} - 1\right)e^{-2k(c+d)} = h_3$$

Therefore, the particle is falling from the point $z = h - h_1 - h_2$ and stops at the point $z = h - h_0 - h_1 - h_2 - h_3$ at $t = 4c + 3d$.

Thus, the displacement $z(t)$ for the interval $t \in [i(c+d), i(c+d) + c]$ is

$$z(t) = h - \sum_{j=0}^{i-1} h_j - \frac{g}{k}(t - i(c+d)) + \frac{1}{k}\left[v_0 + \frac{g}{k}\right]\left[e^{-k(i(c+d))} - e^{-kt}\right]$$

$$= \frac{g}{k}[i(c+d) - t] + \frac{1}{k}\left[v_0 + \frac{g}{k}\right]\left[e^{-k(i-1)(c+d)} + e^{-k[(i-1)(c+d)+c]} + e^{-ki(c+d)} - e^{-kt}\right],$$

where

$$h_i = h - \sum_{j=0}^{i-1} h_j - \frac{gc}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})e^{-ki(c+d)} = \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})e^{-k(i-1)(c+d)}[e^{-k(c+d)} - 1]$$

for all $i = 0,1,2,\cdots$ with $h_0 = h - \frac{gc}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kc})$ and the displacement $z(t)$ for the interval $t \in [0, c]$ is

$$z(t) = h - \frac{gt}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kt}).$$

In this case, $\Delta z(i(c+d) + c) = 0$ for $i = 0,1,2,\cdots$.

Second Case: Let us assume that the particle stops at the end of each interval $[i(c+d), i(c+d) + c]$ for all $i = 0,1,2,\cdots$ by a force.

In this case the initial velocity at the beginning of each interval changes but the final velocities will be zero. Let the initial velocities at the beginning of each interval $t \in [i(c+d), i(c+d) + c]$ be $v(i(c+d)) = v_i$. Let us find the solution interval by interval.

For the interval $t \in [0, c]$, we have

$$\frac{dv}{dt} = -g - kv,$$

$$v(0) = v_0$$

and the solution becomes

$$v(t) = e_{-k}^{P}(t, 0)\left[v_0 + \frac{g}{k}\right] - \frac{g}{k} = e^{-kt}\left[v_0 + \frac{g}{k}\right] - \frac{g}{k}$$

where $e_\alpha^P(t, t_0)$ is exponential function for $\mathbb{P}_{c,d}$ defined by $e_\alpha^P(t, t_0) = e^{\alpha(t - t\_0)}$.

For the interval $t \in [c + d, 2c + d]$, we have

$$\frac{dv}{dt} = -g - kv,$$

$$v(c+d)=v_1$$

and the solution becomes

$$v(t)=e^{-k(t-c-d)}\left[v_1+\frac{g}{k}\right]-\frac{g}{k}.$$

For each interval $t\in[i(c+d),i(c+d)+c]$, we have

$$\frac{dv}{dt}=-g-kv,$$

$$v(i(c+d))=v_i$$

and we can find the solution as

$$v(t)=e_{-k}^{P}(t,i(c+d))\left[v_i+\frac{g}{k}\right]-\frac{g}{k}=e^{-k[t-i(c+d)]}\left[v_i+\frac{g}{k}\right]-\frac{g}{k}$$

for the initial velocity of each interval v_i at the point $t=i(c+d)$ for all $i=0,1,2,\ldots$.

In this case at the points when $t=i(c+d)+c$, we have

$$\Delta v(i(c+d)+c)=v((i+1)(c+d))-v(i(c+d)+c)=v((i+1)(c+d))=v_{i+1}$$

Let us consider the displacement of the particle. Different than the other time scales, in this case, z does not denote the vertical distance but it denotes the displacement of the particle. For the interval $t\in[0,c]$, we have

$$\frac{dz}{dt}=v=e^{-kt}\left[v_0+\frac{g}{k}\right]-\frac{g}{k},$$

$$z(0)=h$$

and the solution becomes

$$z(t)=h-\frac{gt}{k}+\frac{1}{k}\left(v_0+\frac{g}{k}\right)(1-e^{-kt})$$

which shows the displacement of the particle over the first c seconds. For the displacement of the particle at the point when $t=c$, we need to compute

$$z(c)=h-\frac{gc}{k}+\frac{1}{k}\left(v_0+\frac{g}{k}\right)(1-e^{-kc})=h_0.$$

Hence, the particle is falling from the point $z=h$ and stops at the point $z=h-h_0$ at $t=c$.

For the interval $t\in[c+d,2c+d]$, we have

$$\frac{dz}{dt}=v=e^{-kt}\left[v_1+\frac{g}{k}\right]-\frac{g}{k}$$

$$z(c+d)=h-h_0,$$

and the solution becomes

$$z(t)=h-h_0-\frac{g}{k}(t-c-d)+\frac{1}{k}\left(v_1+\frac{g}{k}\right)(e^{-k(c+d)}-e^{-kt})$$

which shows the displacement of the particle over the interval $[c+d, 2c+d]$. For the displacement of the particle at the point when $t=2c+d$, we need to compute

$$z(2c+d)=h-h_0-\frac{gc}{k}+\frac{1}{k}\left(v_1+\frac{g}{k}\right)(1-e^{-kc})e^{-k(c+d)}=\frac{1}{k}(1-e^{-kc})\left[-\left(v_0+\frac{g}{k}\right)+\left(v_1+\frac{g}{k}\right)e^{-k(c+d)}\right]=h_1.$$

Hence, the particle is falling from the point $z=h-h_1$ and stops at the point $z=h-h_0-h_1$ at $t=2c+d$. For the interval $t\in[2c+2d, 3c+2d]$, we have

$$\frac{dz}{dt}=v=e^{-kt}\left[v_2+\frac{g}{k}\right]-\frac{g}{k}$$

which shows the displacement of the particle over the interval $[2c+2d, 3c+2d]$. For the displacement of the particle at the point when $t=3c+2d$, we need to compute

$$z(3c+2d)=h-h_0-h_1-\frac{gc}{k}+\frac{1}{k}\left[v_2+\frac{g}{k}\right](1-e^{-kc})e^{-k(2c+2d)}$$

$$=\frac{1}{k}(1-e^{-kc})\left[-\left(v_1+\frac{g}{k}\right)e^{-k(c+d)}+\left(v_2+\frac{g}{k}\right)e^{-2k(c+d)}\right]=h_2$$

Thus, the particle falls from the point $z=h-h_1$ and stops at the point $z=h-h_0-h_1-h_2$ at $t=3c+2d$. For the interval $t\in[3c+3d, 4c+3d]$, we have

$$\frac{dz}{dt}=v=e^{-kt}\left[v_3+\frac{g}{k}\right]-\frac{g}{k}$$

$$z(3c+3d)=h-h_0-h_1-h_2$$

and the solution becomes

$$z(t)=h-h_0-h_1-h_2-\frac{g}{k}(t-3c-3d)+\frac{1}{k}\left(v_3+\frac{g}{k}\right)(e^{-k(3c+3d)}-e^{-kt})$$

which shows the displacement of the particle over the interval $[3c+3d, 4c+3d]$. For the displacement of the particle at the point when $t=4c+3d$, we need to compute

$$z(4c+3d)=h-h_0-h_1-h_2-\frac{gc}{k}+\frac{1}{k}\left(v_3+\frac{g}{k}\right)(1-e^{-kc})e^{-k(3c+3d)}$$

$$=\frac{1}{k}(1-e^{-kc})\left[-\left(v_2+\frac{g}{k}\right)e^{-2k(c+d)}+\left(v_3+\frac{g}{k}\right)e^{-3k(c+d)}\right]=h_3.$$

Therefore, the particle is falling from the point $z=h-h_1-h_2$ and stops at the point $z=h-h_0-h_1-h_2-h_3$ at $t=4c+3d$. Thus, the displacement $z(t)$ for the interval $t\in[i(c+d), i(c+d)+c]$ is

$$z(t)=h-\sum_{j=0}^{i-1}h_j-\frac{g}{k}(t-i(c+d))+\frac{1}{k}\left[v_i+\frac{g}{k}\right]\left[e^{-k(i(c+d))}-e^{-kt}\right]$$

$$=\frac{g}{k}[i(c+d)+c-t]-\frac{1}{k}(1-e^{-kc})\left[v_{i-1}+\frac{g}{k}\right]e^{-(i-1)k(c+d)}+\frac{1}{k}\left[v_i+\frac{g}{k}\right]\left[e^{-ki(c+d)}-e^{-kt}\right],$$

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where

$$h_i = h - \sum_{j=0}^{i-1} h_j - \frac{gc}{k} + \frac{1}{k}\left(v_i + \frac{g}{k}\right)(1 - e^{-kc})e^{-ki(c+d)} = \frac{1}{k}(1 - e^{-kc})\left[-\left(v_{i-1} + \frac{g}{k}\right)e^{-(i-1)k(c+d)} + \left(v_i + \frac{g}{k}\right)e^{-ik(c+d)}\right]$$

for all $i = 0,1,2,\cdots$ with $h_0 = h - \frac{gc}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(v\_0 + (1 - e^{-kc})$ and the displacement $z(t)$ for the interval $t \in [0, c]$ is

$$z(t) = h - \frac{gt}{k} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - e^{-kt}).$$

In this case, $\Delta z(i(c + d) + c) = 0$ for $i = 0,1,2,\cdots$. In Figure 7 and Figure 9, the velocity and distance graph for the first case of $\mathbb{P}_{5,2}$ is drawn and in Figure 8 and Figure 10, their behaviors are given in the graph for the second case of $\mathbb{P}_{5,2}$.

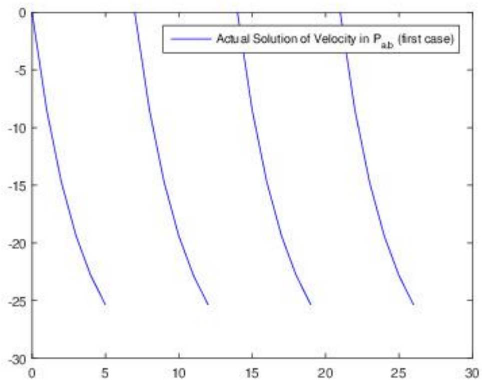

Figure 7. Velocity graphs for first case of $\mathbb{P}_{5,2}$

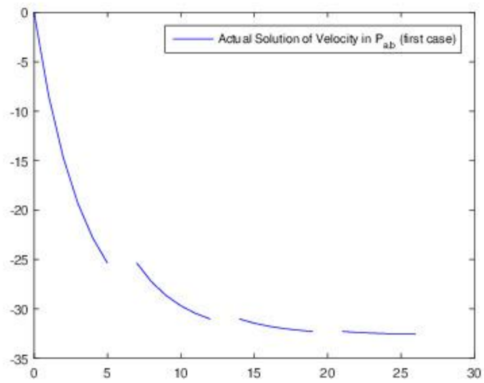

Figure 8. Velocity graphs for second case of $\mathbb{P}_{5,2}$

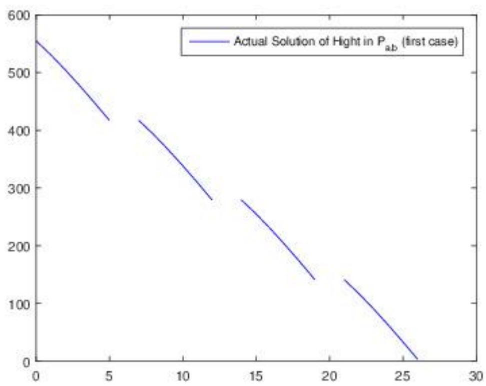

Figure 9. Distance graphs for first case of $\mathbb{P}_{5,2}$

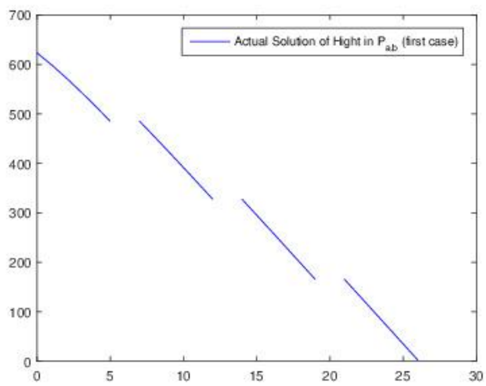

Figure 10. Distance graphs for first case of $\mathbb{P}_{5,2}$

4. DISCUSSIONS AND CONCLUSIONS

In Newtonian physics, free fall is any motion of an object in which the only force acting on it is gravity. Gravity reduces to the curvature of space-time in general relativity context, there is no force acting on a freely falling object and gravity acts almost equally in a roughly uniform gravitational field on every part of an object. When the object starts to move, it encounters a resistance force that opposes the relative motion by which the particle moves relative to the air.

In this study, we have mainly concentrated on the falling body motion in different kind of time scales. We obtained the above results for each of them. Additionally, numerical simulation of these results was obtained by using determined coefficients.

We have compared velocity and distance in $\mathbb{R}$ with the results obtained in $2\mathbb{Z}$, $0.1\mathbb{Z}$ and $\mathbb{Z}$. Based on the findings, the results of the discrete model on $0.1\mathbb{Z}$ yield the best fit to that of $\mathbb{R}$. We can conclude that the time scale h$\mathbb{Z}$ with $0 < h < 1$ provides the best approximation to $\mathbb{R}$, as expected. Moreover, in the time scale $\mathbb{P}_{c,d}$ if the particle stops at the end of each interval $[i(c+d), i(c+d)+c]$ for all $i = 0,1,2,\cdots$ automatically, i.e without a force, the initial velocity at the beginning of each interval is the same and the velocity function is periodic. In the second case where the particle stops at the end of each interval $[i(c+d), i(c+d)+c]$ for all $i = 0,1,2,\cdots$ by a force, the initial velocity at the beginning of each interval changes and the velocity function is decreasing and not periodic. The distance functions for both cases show similar behaviors.

Considering falling body motion under different kind of time scales, helps us to evaluate this physical event under different kind of media. In other words, different kind of time scales can explain the different media in real-world applications. As a result, these kinds of results can be useful for the media with different kind of physical properties, which shows the importance of this study.

AUTHOR CONTRIBUTIONS

Conceptualization, Z.K. and N.N.P.; methodology, Z.K. and N.N.P; fieldwork, Z.K.; software, N.N.P.; title, Z.K.; validation, Z.K. and N.N.P; formal analysis, Z.K. and N.N.P; research, Z.K. and N.N.P; sources, Z.K. and N.N.P; manuscript-original draft, Z.K. and N.N.P; manuscript-review and editing, Z.K. and N.N.P; visualization, Z.K. and N.N.P; supervision, Z.K. and N.N.P. All authors have read and legally accepted the final version of the article published in the journal.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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