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Factorizations of Some Variants of a Statistical Matrix

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Abstract. In this article, we define eight orthogonal matrices strongly linked to the well-known Helmert matrix. We derive LU factorizations by providing explicit closed-form formulas for the entries of L and U. Additionally, we factorize matrices by representing them in relation to diagonal matrices.

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1. Introduction

Matrix theory finds widespread application in various disciplines such as applied mathematics, computer science, engineering, statistics and more. Over time, researchers have defined and scrutinized various types of matrices, including determinants, inverses, and factorizations, [1–3, 7–9, 12–15, 18]. One of them is known as *Helmert matrix* which is introduced by Helmert with a prescribed first row and a triangle of zeroes above the diagonal, see [11]. An $n \times n$ standard Helmert matrix $\mathbf{H}$ is of the form

$$h_{ij} = 0 \text{ for } j > i > 1,$$

that is, the entries below the first row and above the main diagonal are all zero. $\mathbf{H}$ is called Helmertian in the strict sense if

$$h_{1j} = \sqrt{w_j} \text{ where } w_j > 0 \text{ and } \sum_{1}^{n} w_j = 1.$$

In [11], Helmert used the matrix with first row entries $h_{1j} = n^{-1/2}$. Thus, for example a 6×6 Helmert matrix is of the form

$$\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} & 0 & 0 \\ \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & -\frac{4}{\sqrt{20}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}.$$

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Dedicated to the memory of Prof. Dr. Ilker Akkus

In [13], Irwin used the matrix with a more general set of positive w_j. Usually, Helmert matrix is used in mathematical statistics [12, 16] especially for analysis of variance, see [5, 10, 17]. In recent years, there has been notable exploration into $q-$analogs of the Helmert matrix, see [3, 7].

The Pascal matrix stands out as one of the widely recognized matrices. Different matrix representations of the Pascal triangle are commonly encountered in contemporary literature. In [4], the authors embarks on a systematic exploration of the matrix representations of the Pascal triangle as independent mathematical entities. The focus of the paper is primarily on the $G-$matrices, comprising twelve triangular matrix forms $G_{1,n}, G_{2,n}, \ldots, G_{12,n}$ of order $n+1$ derived from expanding the Pascal triangle to level n, where n is a natural number greater than or equal to 2. By initiating with $G_{1,n}$ and $G_{12,n}$, for instance, the remaining eleven $G-$matrices can be produced by utilizing matrix transposition along with left and right actions of the permutation matrix R, as described in Definition 1.1, [4].

Definition 1.1 ([4]). A reflection matrix $R = [R_{i,j}]$ or order $n+1$ is defined by

$$R_{i,j} = \delta^{j}_{n-i}, \quad i, j = 0, 1, \ldots, n,$$

where δ^{j}_{n-i} is the Kronecker symbol.

In [6], the authors introduce thirty-six complete matrix configurations (referred to as $FP-$matrices) of the so-called $n-$greatest rhomboid sub-block extracted from the Pascal Triangle expanded to level $2n$ ($2 \leq n \in \mathbb{N}$).

In linear algebra, there are many different matrix decompositions, that means factorizing a matrix into a product of matrices. A factorization of an $n \times n$ Helmert matrix is given in [15] using the matrices R_k representing rotations in the plane of two coordinate axes. The $LU-$factorizations of $q-$analogs of the Helmert matrix are given in [3, 7] and for $q = 1$, one can obtain the factorization of the classical Helmert matrix.

In this paper, we will use the concept provided by [4], employing the reflection matrix R defined in Definition 1.1, and the Helmert matrix to define eight special $n \times n$ matrices. We provide the LU factorization of these matrices. Additional factorizations will be derived by expressing these matrices in relation to certain diagonal matrices.

2. Variants of the Helmert Matrix

In this section, we define eight square matrices. One of them is the well known Helmert matrix and the others are directly related to Helmert matrix. Throughout the section we will denote the transpose of a matrix A as A^T. Let us denote the classical Helmert matrix as $\mathbf{H}_1$.

Let R be the reflection matrix given in Definition 1.1 and $\mathcal{F} = \{f_i \mid i = 1, \ldots, 8\}$, where

$$\begin{aligned}
f_1(\mathbf{H}_1) &= \mathbf{H}_1 \\
f_2(\mathbf{H}_1) &= R\mathbf{H}_1 = \mathbf{H}_2 \\
f_3(\mathbf{H}_1) &= \mathbf{H}_1 R = \mathbf{H}_3 \\
f_4(\mathbf{H}_1) &= R\mathbf{H}_1 R = \mathbf{H}_4 \\
f_5(\mathbf{H}_1) &= \mathbf{H}_1^T = \mathbf{H}_5 \\
f_6(\mathbf{H}_1) &= R\mathbf{H}_1^T = \mathbf{H}_6 \\
f_7(\mathbf{H}_1) &= \mathbf{H}_1^T R = \mathbf{H}_7 \\
f_8(\mathbf{H}_1) &= R\mathbf{H}_1^T R = \mathbf{H}_8.
\end{aligned}$$

Then, $(\mathcal{F}, \circ)$ is a group under function composition $\circ$. This can be seen by constructing the group operation table of $(\mathcal{F}, \circ)$, [4]. We also see that the matrices $\mathbf{H}_i$ for $i = 1, 2, \ldots, 8$ are orthogonal.

The classical Helmert matrix of order n is given as

$$\mathbf{H}_1 = \begin{bmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & \cdots & 0 \\
\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{n-1}{\sqrt{n(n-1)}}
\end{bmatrix}.$$

The first row of $\mathbf{H}_1$ has the form

$$\left[\frac{1}{\sqrt{n}} \quad , \quad \frac{1}{\sqrt{n}} \quad , \quad \frac{1}{\sqrt{n}} \quad , \quad \cdots \quad , \quad \frac{1}{\sqrt{n}}\right]$$

and for $2 \leq i \leq n$ the form of the i−th row of $\mathbf{H}_1$ is

$$\left[\underbrace{\frac{1}{\sqrt{i(i-1)}} \ , \ \cdots \ , \ \frac{1}{\sqrt{i(i-1)}}}_{i-1 \text{ times}} \quad , \quad -\frac{i-1}{\sqrt{i(i-1)}} \quad , \quad \underbrace{0 \ , \ \cdots \ , \ 0}_{n-i \text{ times}}\right].$$

The matrix $\mathbf{H}_2$ of order n is of the form

$$\mathbf{H}_2 = \begin{bmatrix} \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{n-1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & 0 \\ \vdots & \vdots & \vdots & \iddots & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \iddots & \cdots & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & \cdots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}.$$

The n−th row of the matrix $\mathbf{H}_2$ has the form

$$\left[\frac{1}{\sqrt{n}} \quad , \quad \frac{1}{\sqrt{n}} \quad , \quad \frac{1}{\sqrt{n}} \quad , \quad \cdots \quad , \quad \frac{1}{\sqrt{n}}\right]$$

and for $1 \leq i \leq n-1$ the form of the i−th row of $\mathbf{H}_2$ is

$$\left[\underbrace{\frac{1}{\sqrt{(n-i+1)(n-i)]}} \ , \ \cdots \ , \ \frac{1}{\sqrt{(n-i+1)(n-i)}}}_{n-i \text{ times}} \quad , \quad -\frac{n-i}{\sqrt{(n-i+1)(n-i)}} \quad , \quad \underbrace{0 \ , \ \cdots \ , \ 0}_{i-1 \text{ times}}\right].$$

The matrix $\mathbf{H}_3$ of order n is of the form

$$\mathbf{H}_3 = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \cdots & 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \vdots & \iddots & \iddots & \vdots & \vdots & \vdots \\ 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} \\ -\frac{n-1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \end{bmatrix}.$$

The first row of the matrix $\mathbf{H}_3$ has the form

$$\left[\frac{1}{\sqrt{n}} \quad , \quad \frac{1}{\sqrt{n}} \quad , \quad \frac{1}{\sqrt{n}} \quad , \quad \cdots \quad , \quad \frac{1}{\sqrt{n}}\right]$$

and for $2 \leq i \leq n$ the form of the i−th row of $\mathbf{H}_3$ is

$$\left[\underbrace{0 \ , \ \cdots \ , \ 0}_{n-i \text{ times}} \ , \ -\frac{i-1}{\sqrt{i(i-1)}} \ , \underbrace{\frac{1}{\sqrt{i(i-1)}}, \ \cdots, \ \frac{1}{\sqrt{i(i-1)}}}_{i-1 \text{ times}}\right].$$

The matrix $\mathbf{H}_4$ of order n is of the form

$$\mathbf{H}_4 = \begin{bmatrix} -\frac{n-1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}.$$

The $n-$th row of the matrix $\mathbf{H}_4$ has the form

$$\left[\tfrac{1}{\sqrt{n}} \quad , \quad \tfrac{1}{\sqrt{n}} \quad , \quad \tfrac{1}{\sqrt{n}} \quad , \quad \cdots \quad , \quad \tfrac{1}{\sqrt{n}}\right]$$

and for $1 \le i \le n-1$ the form of the $i-$th row of $\mathbf{H}_4$ is

$$\left[\underbrace{0 \ , \ \cdots \ , \ 0}_{i-1 \text{ times}} \ , \ -\frac{n-i}{\sqrt{(n-i+1)(n-i)}} \ , \ \underbrace{\frac{1}{\sqrt{(n-i+1)(n-i)}} \ , \ \cdots \ , \ \frac{1}{\sqrt{(n-i+1)(n-i)}}}_{n-i \text{ times}}\right].$$

The matrix $\mathbf{H}_5$ of order n is of the form

$$\mathbf{H}_5 = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & -\frac{2}{\sqrt{6}} & \vdots & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & 0 & \cdots & 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & \cdots & \cdots & 0 & -\frac{n-1}{\sqrt{n(n-1)}} \end{bmatrix}.$$

The first column of the matrix $\mathbf{H}_5$ has the form

$$\left[\tfrac{1}{\sqrt{n}} \quad , \quad \tfrac{1}{\sqrt{n}} \quad , \quad \tfrac{1}{\sqrt{n}} \quad , \quad \cdots \quad , \quad \tfrac{1}{\sqrt{n}}\right]^T$$

and for $2 \le j \le n$ the form of the $j-$th column of $\mathbf{H}_5$ is

$$\left[\underbrace{\frac{1}{\sqrt{j(j-1)}} \ , \ \cdots \ , \ \frac{1}{\sqrt{j(j-1)}}}_{j-1 \text{ times}} \quad , \quad -\frac{j-1}{\sqrt{j(j-1)}} \quad , \quad \underbrace{0 \ , \ \cdots \ , \ 0}_{n-j \text{ times}}\right]^T.$$

The matrix $\mathbf{H}_6$ of order n is of the form

$$\mathbf{H}_6 = \begin{bmatrix} \frac{1}{\sqrt{n}} & 0 & 0 & \cdots & 0 & -\frac{n-1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & \cdots & \iddots & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & \vdots & 0 & -\frac{n-3}{\sqrt{(n-2)(n-3)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{n(n-1)}} \end{bmatrix}.$$

The first column of the matrix $\mathbf{H}_6$ has the form

$$\left[\tfrac{1}{\sqrt{n}} \quad , \quad \tfrac{1}{\sqrt{n}} \quad , \quad \tfrac{1}{\sqrt{n}} \quad , \quad \cdots \quad , \quad \tfrac{1}{\sqrt{n}}\right]^T$$

and for $2 \le j \le n$ the form of the $j-$th column of $\mathbf{H}_6$ is

$$\left[\underbrace{0 \ , \ \cdots \ , \ 0}_{n-j \text{ times}} \quad , \quad -\frac{j-1}{\sqrt{j(j-1)}} \quad , \quad \underbrace{\frac{1}{\sqrt{j(j-1)}} \ , \ \cdots \ , \ \frac{1}{\sqrt{j(j-1)}}}_{j-1 \text{ times}}\right]^T.$$

The matrix $\mathbf{H}_7$ of order n is of the form

$$\mathbf{H}_7 = \begin{bmatrix} \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \vdots & -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{n}} \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & 0 & \cdots & 0 & \frac{1}{\sqrt{n}} \\ -\frac{n-1}{\sqrt{n(n-1)}} & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n}} \end{bmatrix}.$$

The $n-$th column of the matrix $\mathbf{H}_7$ has the form

$$\left[\frac{1}{\sqrt{n}} \ , \ \frac{1}{\sqrt{n}} \ , \ \frac{1}{\sqrt{n}} \ , \ \cdots \ , \ \frac{1}{\sqrt{n}}\right]^T$$

and for $1 \leq j \leq n-1$ the form of the $j-$th column of $\mathbf{H}_7$ is

$$\left[\underbrace{\frac{1}{\sqrt{(n-j+1)(n-j)}} \ , \ \cdots \ , \ \frac{1}{\sqrt{(n-j+1)(n-j)}}}_{n-j \text{ times}} \ , \ -\frac{n-j}{\sqrt{(n-j+1)(n-j)}} \ , \ \underbrace{0 \ , \ \cdots \ , \ 0}_{j-1 \text{ times}}\right]^T.$$

The matrix $\mathbf{H}_8$ of order n is of the form

$$\mathbf{H}_8 = \begin{bmatrix} -\frac{n-1}{\sqrt{n(n-1)}} & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n(n-1)}} & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & 0 & \cdots & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{n-2}{\sqrt{(n-2)(n-3)}} & 0 & \cdots & \frac{1}{\sqrt{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \end{bmatrix}.$$

The $n-$th column of the matrix $\mathbf{H}_8$ has the form

$$\left[\frac{1}{\sqrt{n}} \ , \ \frac{1}{\sqrt{n}} \ , \ \frac{1}{\sqrt{n}} \ , \ \cdots \ , \ \frac{1}{\sqrt{n}}\right]^T$$

and for $1 \leq j \leq n-1$ the form of the $j-$th column of $\mathbf{H}_8$ is

$$\left[\underbrace{0 \ , \ \cdots \ , \ 0}_{j-1 \text{ times}} \ , \ -\frac{n-j}{\sqrt{(n-j+1)(n-j)}} \ , \ \underbrace{\frac{1}{\sqrt{(n-j+1)(n-j)}} \ , \ \cdots \ , \ \frac{1}{\sqrt{(n-j+1)(n-j)}}}_{n-j \text{ times}}\right]^T.$$

3. Factorizations of the Variants of the Helmert Matrix

Factorizing a matrix in terms of a lower triangular matrix L and an upper triangular matrix U is called an LU factorization. This factorization can be obtained by Gaussian elimination. The product sometimes includes a permutation matrix P and converts to PLU product. If L is a lower triangular with unit main diagonal and U is an upper triangular, the LU factorization of a matrix is unique. For all defined matrices $\mathbf{H}_i$, we are interested in PLU factorizations. We use the notation $\mathbf{H}_i = P_i L_i U_i$ and provide explicit expressions for P_i, L_i, U_i and inverses of these matrices.

We also consider these defined matrices in another perspective. The well known Helmert matrix, the matrix $\mathbf{H}_1$ with our representation, can be expressed in terms of a diagonal matrix as follows

$$\mathbf{H}_1 = D_1^{-1}\mathbf{S}_1 = D^{-1}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & -2 & 0 & \cdots & \vdots \\ 1 & 1 & 1 & -3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 1 & \cdots & \cdots & 1 & -(n-1) \end{bmatrix},$$

where D_1^2 is a diagonal matrix with entries $d_{1,ii}^2 = n,\ 1.2,\ 2.3, \cdots,\ n(n-1)$. We can obtain diagonal matrices D_i and the matrices $\mathbf{S}_i$ in a similar manner for $i = 2, 3, \ldots, n$.

Let $\mathbf{H}_i$ denote the square matrix of size n defined as follows for $i = 2, 3, \ldots, n$. In this paper, the identity matrix of size n is represented as I_n, and the matrices L_i for $i = 2, 3, \ldots, n$ are lower triangular matrices with a main diagonal consisting of units.

Firstly, for $2 \le i \le 4$, we will find the matrices D_i and $\mathbf{S}_i$ which satisfies $\mathbf{H}_i = D_i^{-1}\mathbf{S}_i$ and for $5 \le i \le 8$, we will find the matrices D_i and $\mathbf{S}_i$ which satisfies $\mathbf{H}_i = \mathbf{S}_i D_i^{-1}$.

i	D_i^2	S_i
2	$d_{2,ii}^2 = n(n-1), (n-1)(n-2), \cdots, 2.3, 1.2, n$	$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & -(n-1) \\ 1 & 1 & 1 & 1 & -(n-2) & 0 \\ 1 & 1 & \cdots & -(n-3) & 0 & 0 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$
3	$d_{3,ii}^2 = n, 1.2, 2.3, \cdots, (n-1)(n-2), n(n-1)$	$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 & 1 & 1 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots \\ 0 & -(n-2) & 1 & 1 & 1 & 1 \\ -(n-1) & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$
4	$d_{4,ii}^2 = n(n-1), (n-1)(n-2), \cdots, 2.3, 1.2, n$	$\begin{bmatrix} -(n-1) & 1 & 1 & 1 & \cdots & 1 \\ 0 & -(n-2) & 1 & 1 & 1 & 1 \\ 0 & 0 & -(n-3) & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$
5	$d_{5,ii}^2 = n, 1.2, 2.3, \cdots, (n-1)(n-2), n(n-1)$	$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \vdots & \vdots & 0 & -(n-2) & 1 \\ 1 & 0 & 0 & \cdots & 0 & -(n-1) \end{bmatrix}$
6	$d_{6,ii}^2 = n, 1.2, 2.3, \cdots, (n-1)(n-2), n(n-1)$	$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -(n-1) \\ 1 & 0 & \cdots & 0 & -(n-2) & 1 \\ 1 & 0 & \vdots & -(n-3) & 1 & 1 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots \\ 1 & -1 & 1 & \vdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$
7	$d_{7,ii}^2 = n(n-1), (n-1)(n-2), \cdots, 2.3, 1.2, n$	$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & -1 & 1 \\ 1 & 1 & \vdots & -2 & 0 & 1 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots \\ 1 & -(n-2) & 0 & \vdots & 0 & 1 \\ -(n-1) & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$

8	$d^2_{8,ii} = n(n-1), (n-1)(n-2), \cdots, 2.3, 1.2, n$	$\begin{bmatrix} -(n-1) & 0 & 0 & \cdots & 0 & 1 \\ 1 & -(n-2) & 0 & \cdots & 0 & 1 \\ 1 & 1 & -(n-3) & 0 & \vdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \vdots & -1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$

Table 1. Explicit forms of the matrices D^2_i and S_i.

Now, we will present the matrices $P, L_i, U_i, L_i^{-1}, U_i^{-1}$. For

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \vdots & 1 & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

we see that the matrices $\mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_6, \mathbf{H}_7$ inludes P in their LU factorization. The LU factorizations of the $q-$analogs of the Helmert matrix are provided in [3, 7]. Since obtaining the LU factorization of the matrix $\mathbf{H}_1$ for $q = 1$ is straightforward, we will not rewrite the LU factorization of $\mathbf{H}_1$ in this paper.

i	L_i	U_i
2	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{\sqrt{n(n-1)}}{\sqrt{2}} & 1 & 0 & \cdots & \cdots & 0 \\ \frac{\sqrt{n(n-1)}}{\sqrt{6}} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 & \ddots & \vdots \\ \frac{\sqrt{n(n-1)}}{\sqrt{(n-1)(n-2)}} & 0 & \vdots & 0 & \ddots & 0 \\ \sqrt{(n-1)} & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \cdots & \frac{1}{\sqrt{n(n-1)}} & -\frac{n-1}{\sqrt{n(n-1)}} \\ 0 & -\frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} & \frac{n-1}{\sqrt{2}} \\ 0 & 0 & -\frac{3}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \cdots & \frac{n-1}{\sqrt{6}} \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & \ddots & -\frac{n-1}{\sqrt{(n-1)(n-2)}} & \frac{n-1}{\sqrt{(n-1)(n-2)}} \\ 0 & 0 & \cdots & \cdots & 0 & \sqrt{n} \end{bmatrix}$
3	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ -\sqrt{n-1} & -\frac{\sqrt{n(n-1)}}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & -\frac{\sqrt{n(n-1)}}{\sqrt{2}} & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{n}} \\ 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} \\ 0 & 0 & -\frac{n-3}{\sqrt{(n-2)(n-3)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & \frac{1}{\sqrt{(n-2)(n-3)}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \cdots & \cdots & 0 & \sqrt{n(n-1)} \end{bmatrix}$
4	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ -\frac{\sqrt{n}}{\sqrt{n(n-1)}} & -\frac{\sqrt{n}}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & -\frac{\sqrt{n}}{\sqrt{2}} & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{n-1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} \\ 0 & 0 & -\frac{n-3}{\sqrt{(n-2)(n-3)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & \frac{1}{\sqrt{(n-2)(n-3)}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \cdots & \cdots & \cdots & 0 & \sqrt{n} \end{bmatrix}$

5	$\begin{bmatrix} 1&0&0&0&\cdots&0\\ 1&1&0&\cdots&\cdots&0\\ 1&\frac{1}{2}&1&\vdots&\vdots&0\\ 1&\frac{1}{2}&\frac{1}{3}&1&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 1&\frac{1}{2}&\frac{1}{3}&\cdots&\frac{1}{n-1}&1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{n}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&\cdots&\cdots&\frac{1}{\sqrt{n(n-1)}}\\ 0&-\frac{2}{\sqrt{2}}&0&\cdots&\cdots&0\\ 0&0&-\frac{3}{\sqrt{6}}&0&\cdots&0\\ \vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&0&-\frac{n-1}{\sqrt{(n-1)(n-2)}}&0\\ 0&\cdots&\cdots&\cdots&0&-\frac{n}{\sqrt{n(n-1)}} \end{bmatrix}$
6	$\begin{bmatrix} 1&0&0&0&\cdots&0\\ 1&1&0&\cdots&\cdots&0\\ 1&0&1&\vdots&\vdots&0\\ \vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\ 1&0&\cdots&0&1&0\\ 1&-1&-1&\cdots&-1&1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{n}}&0&0&\cdots&\cdots&-\frac{n-1}{\sqrt{n(n-1)}}\\ 0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&\cdots&\frac{1}{\sqrt{(n-1)(n-2)}}&\frac{n}{\sqrt{n(n-1)}}\\ 0&0&-\frac{2}{\sqrt{6}}&\cdots&\frac{1}{\sqrt{(n-1)(n-2)}}&\frac{n}{\sqrt{n(n-1)}}\\ \vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&0&-\frac{n-2}{\sqrt{(n-1)(n-2)}}&\frac{n}{\sqrt{n(n-1)}}\\ 0&\cdots&\cdots&\cdots&0&\sqrt{n(n-1)} \end{bmatrix}$
7	$\begin{bmatrix} 1&0&0&0&\cdots&0\\ 1&1&0&\cdots&\cdots&0\\ 1&0&1&\vdots&\vdots&0\\ \vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\ 1&0&\cdots&0&1&0\\ -(n-1)&-1&-1&\cdots&-1&1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{n(n-1)}}&\frac{1}{\sqrt{(n-1)(n-2)}}&\cdots&\cdots&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{n}}\\ 0&-\frac{n-1}{\sqrt{(n-1)(n-2)}}&-\frac{1}{\sqrt{(n-2)(n-3)}}&\cdots&-\frac{1}{\sqrt{2}}&0\\ \vdots&0&\ddots&\vdots&\vdots&\vdots\\ \vdots&\vdots&0&-\frac{3}{\sqrt{6}}&-\frac{1}{\sqrt{2}}&0\\ \vdots&\vdots&\vdots&\ddots&-\frac{2}{\sqrt{2}}&0\\ 0&\cdots&\cdots&\cdots&0&\sqrt{n} \end{bmatrix}$
8	$\begin{bmatrix} 1&0&0&0&\cdots&0\\ -\frac{1}{n-1}&1&0&\cdots&\cdots&0\\ -\frac{1}{n-1}&-\frac{1}{n-2}&1&\vdots&\vdots&0\\ -\frac{1}{n-1}&-\frac{1}{n-2}&-\frac{1}{n-3}&1&\vdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ -\frac{1}{n-1}&-\frac{1}{n-2}&\cdots&\cdots&-1&1 \end{bmatrix}$	$\begin{bmatrix} -\frac{n-1}{\sqrt{n(n-1)}}&0&0&\cdots&0&\frac{\sqrt{n}}{n}\\ 0&-\frac{n-2}{\sqrt{(n-1)(n-2)}}&0&\cdots&\cdots&\frac{\sqrt{n}}{n-1}\\ 0&0&-\frac{n-3}{\sqrt{(n-2)(n-3)}}&0&\cdots&\frac{\sqrt{n}}{n-2}\\ \vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&0&-\frac{1}{\sqrt{2}}&\frac{\sqrt{n}}{2}\\ 0&\cdots&\cdots&\cdots&0&\sqrt{n} \end{bmatrix}$

TABLE 2. Explicit forms of the matrices L_i and U_i.

The following table gives the explicit closed formulæ for the inverses of the matrices L_i and U_i for $i = 2, 3, \ldots, 8$.

i	L_i^{-1}	U_i^{-1}
2	$\begin{bmatrix} 1&0&0&0&\cdots&0\\ -\frac{\sqrt{n(n-1)}}{\sqrt{2}}&1&0&\cdots&\cdots&0\\ -\frac{\sqrt{n(n-1)}}{\sqrt{6}}&0&1&\vdots&\vdots&0\\ \vdots&\vdots&\ddots&1&\vdots&\vdots\\ -\frac{\sqrt{n(n-1)}}{\sqrt{(n-1)(n-2)}}&0&\vdots&\ddots&\ddots&0\\ -\sqrt{(n-1)}&0&\cdots&\cdots&0&1 \end{bmatrix}$	$\begin{bmatrix} \sqrt{n(n-1)}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&\cdots&\frac{1}{\sqrt{(n-1)(n-2)}}&\frac{1}{\sqrt{n}}\\ 0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&\vdots&\frac{1}{\sqrt{(n-1)(n-2)}}&\frac{1}{\sqrt{n}}\\ 0&0&-\frac{2}{\sqrt{6}}&\vdots&\vdots&\frac{1}{\sqrt{n}}\\ \vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&0&-\frac{n-2}{\sqrt{(n-1)(n-2)}}&\frac{1}{\sqrt{n}}\\ 0&\cdots&\cdots&\cdots&0&\frac{1}{\sqrt{n}} \end{bmatrix}$
3	$\begin{bmatrix} 1&0&0&0&\cdots&0\\ 0&1&0&\cdots&\cdots&0\\ 0&0&1&\vdots&\vdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&\cdots&0&1&0\\ \sqrt{n-1}&\frac{\sqrt{n(n-1)}}{\sqrt{(n-1)(n-2)}}&\cdots&\cdots&\frac{\sqrt{n(n-1)}}{\sqrt{2}}&1 \end{bmatrix}$	$\begin{bmatrix} \sqrt{n}&\frac{n-1}{\sqrt{(n-1)(n-2)}}&\frac{n-1}{\sqrt{(n-2)(n-3)}}&\cdots&\frac{n-1}{\sqrt{2}}&-\frac{n-1}{\sqrt{n}}\\ 0&-\frac{n-1}{\sqrt{(n-1)(n-2)}}&-\frac{1}{\sqrt{(n-1)(n-2)}}&\cdots&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{n(n-1)}}\\ 0&0&\ddots&\vdots&\vdots&\vdots\\ \vdots&\vdots&\ddots&-\frac{3}{\sqrt{6}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{n(n-1)}}\\ 0&\vdots&\vdots&0&-\frac{2}{\sqrt{2}}&\frac{1}{\sqrt{n(n-1)}}\\ 0&\cdots&\cdots&\cdots&0&\frac{1}{\sqrt{n(n-1)}} \end{bmatrix}$

4	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{\sqrt{n}}{\sqrt{n(n-1)}} & \frac{\sqrt{n}}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & \frac{\sqrt{n}}{\sqrt{2}} & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{n}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ 0 & -\frac{n-1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-2)(n-3)}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ 0 & 0 & -\frac{n-2}{\sqrt{(n-2)(n-3)}} & -\frac{1}{\sqrt{(n-2)(n-3)}} & \cdots & \frac{1}{\sqrt{n}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{\sqrt{n}} \end{bmatrix}$
5	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & \vdots & \vdots & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} & 1 \end{bmatrix}$	$\begin{bmatrix} \sqrt{n} & \frac{\sqrt{n}}{2} & \frac{\sqrt{n}}{3} & \cdots & \cdots & \frac{\sqrt{n}}{n} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -\frac{n-1}{\sqrt{n(n-1)}} \end{bmatrix}$
6	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & 1 & 0 \\ -(n-1) & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} \sqrt{n} & 0 & 0 & \cdots & \cdots & \frac{1}{\sqrt{n}} \\ 0 & -\frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{3}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{6}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & -\frac{n-1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{\sqrt{n(n-1)}} \end{bmatrix}$
7	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} \sqrt{n(n-1)} & \frac{\sqrt{n(n-1)}}{n-1} & \frac{\sqrt{n(n-1)}}{n-1} & \cdots & \cdots & -\frac{\sqrt{n(n-1)}}{n} \\ 0 & -\frac{n-2}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \vdots & \vdots & \vdots & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{\sqrt{n}} \end{bmatrix}$
8	$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{n-1} & 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{n-2} & \frac{1}{n-2} & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & 1 & 0 \\ 1 & \cdots & \cdots & \cdots & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{n}{\sqrt{n(n-1)}} & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n(n-1)}} \\ 0 & -\frac{n-1}{\sqrt{(n-1)(n-2)}} & 0 & \cdots & \vdots & \frac{1}{\sqrt{(n-1)(n-2)}} \\ 0 & 0 & -\frac{n-2}{\sqrt{(n-2)(n-3)}} & 0 & \vdots & \frac{1}{\sqrt{(n-2)(n-3)}} \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & 0 & -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{\sqrt{n}} \end{bmatrix}$

TABLE 3. Explicit forms of the matrices L_i^{-1} and U_i^{-1}.

Here, we just check that the matrices in the factorization of $P\mathbf{H}_2$ the matrices L_2, L_2^{-1} and U_2, U_2^{-1} are inverse. The others can be obtained similarly. From Table 2 and Table 3, we see that

- $L_2 = I_n + J$, where $J = [x_{ij}]$ is the matrix with $x_{11} = 0$ and $x_{ij} = 0$ for $j > 1$. For $i \neq n$, $x_{i1} = \frac{\sqrt{n(n-1)}}{\sqrt{i(i-1)}}$, $x_{n1} = \sqrt{n-1}$.
- $U_2 = [u_{ij}]$. Then we have $u_{ii} = -\frac{i}{\sqrt{i(i-1)}}$ for $1 < i < n$ and $u_{nn} = \sqrt{n}$, $u_{1j} = \frac{1}{\sqrt{n(n-1)}}$ for $1 \leq j < n$, $u_{1n} = -\frac{n-1}{\sqrt{n(n-1)}}$, $u_{ij} = -\frac{1}{\sqrt{i(i-1)}}$ for $n > j > i \geq 2$, $u_{in} = \frac{n-1}{\sqrt{i(i-1)}}$ for $1 < i < n$.
- $L_2^{-1} = I_n - J$ where J is the matrix defined above.
- $U_2^{-1} = [v_{ij}]$. Then we have $v_{ii} = -\frac{i-1}{\sqrt{i(i-1)}}$ for $1 < i < n$ and $v_{11} = \sqrt{n(n-1)}$, $v_{1j} = \frac{1}{\sqrt{j(j-1)}}$ for $1 < j < n$, $v_{in} = \frac{1}{\sqrt{n}}$ for $1 \leq i \leq n$, $v_{ij} = \frac{1}{\sqrt{j(j-1)}}$ for $n > j > i \geq 2$,

Proof.

- We should prove
$$\sum_{j\le k\le i} L_{2,ik}L^{-1}_{2,kj} = \delta_{ij},$$
where δ_{ij} is the Kronecker delta. But for $j > 1$, it can be easily seen by the definition of the matrices that this relation is true. So we only need to prove for $j = 1$. Since L_2 and L_2^{-1} are the lower triangular matrices with unit main diagonal, for $i = j$, we have a lower triangular matrix $L_2L_2^{-1}$ with unit on the main diagonal. For $i \neq j = 1$, we have
$$\sum_{1\le k<i} L_{2,ik}L^{-1}_{2,k1} = \frac{\sqrt{n(n-1)}}{\sqrt{i(i-1)}} - \frac{\sqrt{n(n-1)}}{\sqrt{i(i-1)}} = 0.$$
Hence, we get the result.
- For U_2 and U_2^{-1}, we will show again that
$$\sum_{j\le k\le i} U_{2,ik}U^{-1}_{2,kj} = \delta_{ij}.$$
We first consider the main diagonal. Since U_2 is an upper triangular matrix, the main diagonal entries come from
$$u_{11}v_{11} = \frac{1}{\sqrt{n(n-1)}}\sqrt{n(n-1)} = 1$$
and for $i > 1$
$$u_{ii}v_{ii} = \Big(-\frac{i}{\sqrt{i(i-1)}}\Big)\Big(-\frac{i-1}{\sqrt{i(i-1)}}\Big) = 1.$$
Otherwise, for $i = 1$, $j < n$, we have
$$\begin{aligned}\sum_{1\le k\le j} U_{2,1k}U^{-1}_{2,kj} &= -\frac{1}{\sqrt{n(n-1)}}\frac{j-1}{\sqrt{j(j-1)}} + \sum_{1\le k\le j-1}\frac{1}{\sqrt{n(n-1)}}\frac{1}{\sqrt{j(j-1)}}\\ &= -\frac{1}{\sqrt{n(n-1)}}\frac{j-1}{\sqrt{j(j-1)}} + (j-1)\frac{1}{\sqrt{n(n-1)}}\frac{1}{\sqrt{j(j-1)}}\\ &= 0\end{aligned}$$
and for $i = 1$, $j = n$, we have
$$\begin{aligned}\sum_{1\le k\le n} U_{2,1k}U^{-1}_{2,kn} &= -\frac{n-1}{\sqrt{n(n-1)}}\frac{1}{\sqrt{n}} + \sum_{1\le k\le j-1}\frac{1}{\sqrt{n(n-1)}}\frac{1}{\sqrt{n}}\\ &= -\frac{n-1}{\sqrt{n(n-1)}}\frac{1}{\sqrt{n}} + (n-1)\frac{1}{\sqrt{n(n-1)}}\frac{1}{\sqrt{n}}\\ &= 0.\end{aligned}$$
For $i > 1$ and $j < n$,
$$\begin{aligned}\sum_{i\le k\le j} U_{ik}U^{-1}_{kj} &= \left(-\frac{i}{\sqrt{i(i-1)}}\right)\left(\frac{1}{\sqrt{j(j-1)}}\right) + \left(-\frac{1}{\sqrt{i(i-1)}}\right)\left(-\frac{j-1}{\sqrt{j(j-1)}}\right) +\\ &\quad \sum_{i+1\le k\le j-1}\left(-\frac{1}{\sqrt{i(i-1)}}\right)\left(\frac{1}{\sqrt{j(j-1)}}\right)\\ &= \left(\frac{j-i-1}{\sqrt{ij(i-1)(j-1)}}\right) - (j-i-1)\left(\frac{1}{\sqrt{ij(i-1)(j-1)}}\right) = 0\end{aligned}$$
and for $i > 1$ and $j = n$ we have
$$\begin{aligned}\sum_{i\le k\le n} U_{ik}U^{-1}_{kn} &= \left(-\frac{i}{\sqrt{i(i-1)}}\right)\left(\frac{1}{\sqrt{n}}\right) + \left(\frac{n-1}{\sqrt{i(i-1)}}\right)\left(\frac{1}{\sqrt{n}}\right) + \sum_{i+1\le k\le n-1}\left(-\frac{1}{\sqrt{i(i-1)}}\right)\left(\frac{1}{\sqrt{n}}\right)\\ &= \left(\frac{n-i-1}{\sqrt{in(i-1)}}\right) - (n-i-1)\left(\frac{1}{\sqrt{in(i-1)}}\right) = 0\end{aligned}$$

□

Thus, the result follows.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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