Journal of Universal Mathematics
Vol.7 No.2 pp.75-84 (2024)
ISSN-2618-5660
DOI: 10.33773/jum.1428190

NECESSARY CONDITION FOR IA-AUTOMORPHISMS IN LEIBNIZ ALGEBRAS

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Abstract. Let F denote the free Leibniz algebra, which is generated by the set $X = \{x_1, ..., x_n\}$ over the field K with characteristic 0. Let R be an ideal of F. This investigation begins by obtaining a specific matrix representation for the IA-automorphisms of the Leibniz algebra F/R'. Following this, we establish a necessary condition for an IA-endomorphism of F/R' to qualify as an IA-automorphism. This method is explicitly based on Dieudonné determinant.

1. Introduction

Consider the Leibniz algebra F, the free algebra of finite rank n over a field K. Let R be an ideal of F, and denote by R' the commutator subalgebra of R. The Leibniz algebra F/R' of rank n is defined in the usual way.

In their work [2], Bahturin and Nabiev established an explicit matrix representation for automorphisms of a Lie algebra L/R' that are congruent modulo R/R', where L is a free Lie algebra of rank n and R is an ideal of L. Shpilrain, in [9], provided a necessary condition for the invertibility of a matrix over the integral group ring of a free group, utilizing a non-commutative determinant. Initially given for free Lie algebras in [3], this condition was based on a non-commutative determinant.

Furthermore, in [14], the author and Ekici gave a criterion grounded in the Dieudonné determinant with some applications. Recently, [11] addressed the computation of valuations of Dieudonné determinants of matrices over discrete valuation skew fields, exploring two applications stemming from this problem.

Leibniz algebras, serving as potential non-(anti)commutative extensions of Lie algebras, were thoroughly examined in terms of homological algebra by Loday and Pirashvili in [7]. Numerous findings in Leibniz algebras highlight their close relationship with Lie algebras, prompting attempts to extend specific combinatorial results from varieties of Lie algebras to their Leibniz counterparts.

Date: **Received:** 2024-01-30; **Accepted:** 2024-07-16.

Key words and phrases. Leibniz algebra, Automorphisms, Dieudonné determinant.

The author was supported by the Cukurova University BAP Coordination Council [FBA-2020-13173].

Mathematics Subject Classification: 17A32, 17A36, 17A50.

In [8], Mikhalev and Umirbaev derived significant results regarding subalgebras of free Leibniz algebras. The author investigated the automorphisms of free Leibniz algebras with rank two in the work documented in [12]. Additionally, Papistas and Drensky, in their work [5] in 2005, examined automorphisms within the domain of a free left nilpotent Leibniz algebra with finite rank. Meanwhile, free metabelian Leibniz algebras were characterized in the reference [6].

On another note, explicit matrix forms for IA-automorphisms of free metabelian Leibniz algebras were established for rank 3 in [15] and for rank n in [16]. A recent study by the author in [13] contributed a necessary and sufficient condition for a set of n elements in F/R' to function as a generating set.

This study initially derives a matrix representation for the IA-automorphisms of the Leibniz algebra F/R', employing similar techniques as presented in [**?**]. Subsequently, we provide a necessary condition for the invertibility of a matrix belonging to $UL(F/R')$. This condition establishes a means for identifying non-automorphisms within the Leibniz algebra F/R'. Notably, our approach is explicitly grounded in a non-commutative determinant: the Dieudonné determinant. Furthermore, we present several applications of this methodology.

2. Preliminaries

Loday and Pirashvili described free Leibniz algebras in [7]. Consider the Leibniz algebra F generated freely by a set $\{x_1, ..., x_n\}$ over a field K of characteristic 0. Let $Ann(F)$ represent the ideal of F generated by elements $\{[x, x] : x \in F\}$. The algebra $F_{Lie} = F/Ann(F)$ is identified as a Lie algebra. The notation $Aut(F)$ refers to the automorphism group of F, while $IAut(F)$ designates the IA-automorphisms of F. These automorphisms induce the identity mapping on the quotient algebra F/F', where F' is the commutator ideal of F. Let R be a subalgebra of F, and designate R' as the derived subalgebra of R. The paper [7] introduces the universal enveloping algebra for the Leibniz algebra F. Denote by $UL(F)$, the universal enveloping algebra of F, i.e., the free associative algebra with the generating set $\{r_1, ..., r_n, l_1, ..., l_n\}$, where $l_i = l_{x_i}$ and $r_i = r_{x_i}$ the universal operators of left and right multiplication on x_i. These elements satisfy the following relations

$$(r_{x_i} + l_{x_i})l_{x_j} = 0$$

Denoted by Δ, the kernel of the homomorphism $\varepsilon : UL(F) \to K$ defined by $\varepsilon(r_{x_i}) = 0$, $\varepsilon(l_{x_i}) = 0$ for $i = 1, 2, ..., n$, that is augmentation ideal of $UL(F)$. That is also an $UL(F)$-module generated by r_{x_i}, l_{x_i}, where $i = 1, 2, ..., n$. We represent the mth associative power of Δ as Δ^m. Denoted by Δ_R, the ideal of $UL(F)$ is defined as the kernel of the natural homomorphism $\sigma_R : UL(F) \to UL(F/R)$.

Let $\widehat{a}$ represent the image of $a \in F/R$ under the natural homomorphism $F/R \to (F/R)_{Lie}$. Utilizing this homomorphism, we establish the mapping

$$\widehat{\ }: UL(F/R) \to U((F/R)_{Lie})$$

where $U((F/R)_{Lie})$ denotes the universal enveloping algebra of $(F/R)_{Lie}$. Throughout the subsequent discussion, we define the Lie algebra $(F/R)_{Lie}$ alongside its corresponding subalgebra in $U((F/R)_{Lie})$. This results in $\widehat{r_x} = \widehat{x}$ and $\widehat{l_x} = -\widehat{x}$. It is evident that the kernel of the homomorphism $\widehat{\ }$ is generated by $r_x + l_x$, $x \in F/R$. This kernel is denoted as $\Delta_{Ann(F/R)}$. According to the reference [7], the mapping

$$\delta : U((F/R)_{Lie}) \to UL(F/R)$$

is defined as $\delta(\widehat{x}) = r_x$. Notably, due to the equality $\widehat{\delta(\widehat{x})} = \widehat{r_x} = \widehat{x}$, we establish the identification of the algebra $U((F/R)_{Lie})$ with its corresponding subalgebra in $UL(F/R)$.

3. Automorphisms of F/R'

Consider the abelian Leibniz algebra R/R' that is freely generated by a set $\{a_1, a_2, \dots, a_n\}$ as a free K-module. Let F/R be a Leibniz algebra over K, functioning as a free K-module. The wreath product of Leibniz algebras R/R' and F/R is defined in a standard manner, akin to the case of Lie algebras [10]. Denoted as $W = (R/R')wr(F/R)$, it takes the form $W = F/R \oplus I_{R/R'}$, where it is the semidirect sum of F/R and the free F/R-module $I_{R/R'}$ with the free generating set $\{a_1, a_2, \dots, a_n\}$. Furthermore, R/R' is not only a module on F/R but also a $UL(F/R)$-module, where the module action is given by

$$\begin{aligned} u * r_v &= [u, v] \\ u * l_v &= [v, u] \end{aligned}$$

for $u \in R/R'$, $v \in F/R$ and $r_v, l_v \in UL(F/R)$. Let $\overline{x}$ represent $x + R' \in F/R'$, and $\overline{\overline{x}}$ denote $x + R \in F/R$.

The proof of the following theorem is identical to the one presented in the case of Lie algebras, as detailed in [10].

Theorem 3.1. *The mapping $\overline{x_j} \longrightarrow \overline{\overline{x_j}} + a_j$, $j \in \{1, 2, ..., n\}$ extends to a monomorphism $\mu : F/R' \longrightarrow (R/R')wr(F/R)$.*

Let $AutW$ represent the automorphism group of W. Consider a subgroup of $AutW$ denoted as $\overline{AutW}$. The elements of $\overline{AutW}$ are characterized by their invariance of $I_{R/R'}$ and F/R. In other words, if $\alpha \in \overline{AutW}$, then the automorphism $\alpha : W \to W$ satisfies $\alpha(I_{R/R'}) \subset I_{R/R'}$ and $\alpha(F/R) \subset F/R$.

The subsequent theorem analogies the embedding concept in Lie algebras, initially established by Bahturin and Nabiyev in [**?**]. The same arguments are employed to prove this theorem in the case of Leibniz algebras.

Theorem 3.2. *An embedding denoted by $\vartheta : \overline{Aut(F/R')} \to \overline{Aut((R/R')wr(F/R))}$ exists, such that if $\alpha \in \overline{Aut(F/R')}$ preserves R/R', and $\widetilde{\alpha} = \vartheta(\alpha)$, then $\widetilde{\alpha}\mu = \mu\alpha$, where μ represents the embedding defined in Theorem 1.*

The proof of the theorem at hand mirrors the demonstration employed by Bahturin and Nabiyev in establishing their result for Lie Algebras [**?**]. The author and Tas Adiyaman have already given similar proofs in [15, 16] to obtain the explicit matrix form of IA-automorphisms of the free metabelian Leibniz algebras, and the theorem is a generalization of the corresponding result in [16].

Theorem 3.3. *Let F/R' be a Leibniz algebra of finite rank. Let G be the group of invertible matrices of the form $E + AQ$, where E is the identity matrix, $A = [a_{kj}]_{n\times m}$ is a fix matrix, $Q = [q_{ji}]_{m\times n}$ is an arbitrary matrix both with coefficients in $UL(F/R), 1 \le i, k \le n$, $1 \le j \le m$. Then $IAut(F/R') \cong G$.*

4. The Dieudonné Determinant

Every invertible square matrix belonging to $U((F/R)_{\text{Lie}})$ can be expressed as a multiplication of elementary and diagonal matrices, as detailed in [3]. In this context, elementary matrices differ from the identity matrix by, at most, a single element outside the diagonal. Consider an algebra

$$(UL(F)/\Delta_R)/(\Delta^m/\Delta_R), m \geq 2.$$

Denote by H_m the image of this algebra under the homomorphism $\widehat{\ }$ and take the multiplicative group H_m^* of all invertible elements of H_m. Since

$$(a+u+\widehat{\Delta_R})(a^{-1}-a^{-2}u+\ldots+(-1)^{m-1}a^{-m}u^{m-1}+\widehat{\Delta_R}) = 1+\widehat{\Delta_R} \text{ modulo } \widehat{\Delta^m/\Delta_R},$$

we have

$$(a+u+\widehat{\Delta_R})^{-1} = a^{-1} - a^{-2}u + ... + (-1)^{m-1}a^{-m}u^{m-1} + \widehat{\Delta_R} \text{ modulo } \widehat{\Delta^m/\Delta_R}.$$

Therefore, the invertible elements in H_m can be expressed as

$$a + u + \widehat{\Delta_R} + \widehat{\Delta^m/\Delta_R}$$

with $u \in \widehat{\Delta}$ and $0 \neq a \in K$. Next, consider the commutator subgroup $[H_m^*, H_m^*]$ within the group H_m^*. This subgroup is generated, modulo $\widehat{\Delta^m/\Delta_R}$, by elements characterized by the following expression

$$(1-u+\widehat{\Delta_R})(1-w+\widehat{\Delta_R})(1-u+\widehat{\Delta_R})^{-1}(1-w+\widehat{\Delta_R})^{-1}$$

Here, u and w belong to the set Δ. Let S_m be the subsemigroup of $\widehat{UL(F)/\Delta_R}$ generated by all such elements. For a matrix A belonging to the general linear group $GL_n(H_m)$ over H_m, its Dieudonné determinant is defined by exploiting the property that every invertible matrix over H_m can be diagonalized. For any arbitrary permutation $\sigma \in S_n$, we link it with the permutation matrix $P(\sigma) = (\delta_{i,\sigma(j)})$, where δ represents the Kronecker symbol.

For every invertible matrix A over a skew field, a decomposition A can be expressed as $A = TDP(\sigma)V$ known as the Bruhat Normal Form, where

$$T = \begin{bmatrix} 1 & * & * \\ & ... & * \\ 0 & & 1 \end{bmatrix},\ D = diag(a_1, ..., a_n),\ V = \begin{bmatrix} 1 & . & 0 \\ * & .. & . \\ * & * & 1 \end{bmatrix},$$

σ is a permutation, $P(\sigma)$ is the permutation matrix corresponding to σ. The matrices D and σ are unique with these properties (refer to [4]). The Dieudonné determinant of A is given by

$$D_m(A) = \pi(sgn(\sigma)a_1...a_n),$$

where π is the canonical mapping $H_m^* \to H_m^*/[H_m^*, H_m^*]$.

Theorem 4.1. *Consider R as an ideal and F/R' as a finitely generated Leibniz algebra. Let $M \in GL_n(UL(F)/\Delta_R)$ and $det_m(M)$ represent any preimage of $D_m(\widehat{M})$ in $UL(F)/\Delta_R$, where $\Delta_R \subset \Delta^m$ for $m \geq 2$. Then, for any arbitrary m,*

$$det_m(M) = (a + r_u)r_{g_m} \quad \textit{modulo } (\Delta^m/\Delta_R + \Delta_{Ann(F/R)})$$

where $a \in K \setminus \{0\}, u \in \widehat{\Delta_R}$, $g_m \in S_m$.

Proof. Let $M \in GL_n(UL(F)/\Delta_R)$. Since M is invertible over $U(F)/\Delta_R$, then $\widehat{M}$, the image of M under the homomorphism $\widehat{\ }: UL(F/R) \to U((F/R)_{Lie})$, is an invertible matrix over $\widehat{UL(F)/}\Delta_R$ and it can be written as $\widehat{M} = E.D$, where E is the product of elementary matrices and

$$D = diag(a_1 + \widehat{\Delta_R}, a_2 + \widehat{\Delta_R}, ..., a_n + \widehat{\Delta_R})$$

where $0 \neq a_i \in K$ by [14]. Given that the sole invertible elements within $U(F_{Lie})$ are the elements belonging to the field K, the invertible elements within $\widehat{UL(F)/}\Delta_R$ can be expressed as

$$a_1 + \widehat{\Delta_R}, a_2 + \widehat{\Delta_R}, \ldots, a_n + \widehat{\Delta_R}$$

where the elements $a_1, \ldots, a_n$ are constrained to lie within the field K. Consider the algebra $H_m = (\widehat{UL(F)/}\Delta_R)/(\widehat{\Delta^m/\Delta_R})$. The image of $\widehat{M}$ over H_m remains invertible. Consequently, the Dieudonné determinant of $\widehat{M}$ can be expressed as follows

$$D_m(\widehat{M}) = a_1.a_2...a_n + \widehat{\Delta_R} + (\widehat{\Delta^m/\Delta_R}).$$

This representation implies that the Dieudonné determinant of $\widehat{M}$ can be further written as $a + u + w$, where $a = a_1 \cdot a_2 \ldots a_n \in K$, $u \in \widehat{\Delta_R}$, and $w \in \widehat{\Delta^m/\Delta_R}$. Consider the algebra $H_m = (\widehat{UL(F)/}\Delta_R)/(\widehat{\Delta^m/\Delta_R})$. The image of $\widehat{M}$ over H_m is also invertible. Therefore, the Dieudonné determinant of $\widehat{M}$ takes the form

$$D_m(\widehat{M}) = a_1.a_2...a_n + \widehat{\Delta_R} + (\widehat{\Delta^m/\Delta_R}).$$

This implies that the Dieudonné determinant of $\widehat{M}$ can be expressed as

$$a + u + w$$

where $a = a_1.a_2 \ldots a_n \in K$, $u \in \widehat{\Delta_R}$, $w \in \widehat{\Delta^m/\Delta_R}$. An arbitrary preimage $det_m(\widehat{M})$ of $D_m(\widehat{M})$ in $\widehat{UL(F)/}\Delta_R$ is equal to

$$(a + u)g_m \ \text{ modulo}(\widehat{\Delta^m/\Delta_R}),$$

where, $a = a_1.a_2...a_n$, $u \in \widehat{\Delta_R}$, $g_m \in S_m$. Through the homomorphism $\delta : U((F/R)_{Lie}) \to UL(F/R)$ defined as $\delta(x) = r_x$, for $x \in (F/R)_{Lie}$, it is clear that any preimage $det_m(M)$ of $det_m(\widehat{M})$ in $UL(F)/\Delta_R$ can be expressed as

$$(a + r_u)r_{g_m} \text{ modulo } (\Delta^m/\Delta_R \ + \Delta_{Ann(F/R)}),$$

where $\Delta_{Ann(F/R)}$ is an ideal of $UL(F/R)$ generated by the element $r_v + l_v$ for $v \in F/R$. □

Now we have

Theorem 4.2. *Let ψ be an element of $IAut(F/R')$. Consider $\widetilde{\psi}$ as the restricted automorphism of ψ to $I_{R/R'}$, as defined in Theorem 3.2. Denote by M the matrix corresponding to $\widetilde{\psi}$, and let $det_m(M)$ represent an arbitrary preimage of $D_m(\widehat{M})$ in $UL(F)/\Delta_R$. It holds*

$$det_m(M) = (1 + r_u)r_{g_m} \mod (\Delta^m/\Delta_R + \Delta_{Ann(F/R)})$$

where $u \in \widehat{\Delta_R}$ and $g_m \in S_m$ for any $m \geq 2$.

Proof. Given an IA-automorphism ψ of F/R'. By the equality $\mu\psi = \widetilde{\psi}\mu$ from Theorem 3.2 and the definition of the $\overline{AutW}$, there exists an automorphism $\widetilde{\psi}$ restricted to $I_{R/R'}$ with an invertible corresponding matrix M over $UL(F)/\Delta_R$. Through the homomorphism

$$\widehat{\ }: UL(F/R) \to U((F/R)_{Lie}),$$

$\widehat{M}$ is also invertible over $\widehat{UL(F)/\Delta_R}$, expressed as

$$\widehat{M} = E.D,$$

where E is the product of elementary matrices and $D = diag(1+\widehat{\Delta_R}, 1+\widehat{\Delta_R}, ..., 1+\widehat{\Delta_R})$. Consequently, this implies

$$D_m(\widehat{M}) = (1+u)g_m \ modulo \ \widehat{\Delta^m/\Delta_R}$$

where, $u \in \widehat{\Delta_R}$ and $g_m \in S_m$. Thus, according to Theorem 4.1, the arbitrary preimage of $det_m(M)$ in $UL(F)/\Delta_R$ is given by

$$(1+r_u)r_{g_m} \ \text{ modulo } (\Delta^m/\Delta_R \ + \Delta_{Ann(F/R)}).$$

□

Remark 4.3. Theorem 4.2 establishes a necessary condition for an IA-endomorphism of F/R' to qualify as an $IA-$automorphism. This condition provides a means to identify the non-invertibility of a square matrix M over $UL(F)/\Delta_R$. The process involves computing $detm(M)$, initiating from $m = 1$, and proceeding until the condition outlined in the Theorem 4.2 is contradicted.

Example 4.4. Let $R = \gamma_m(F)$, m-th term of the lower central series of F, for $m \geq 4$ and ψ be the endomorphism of $F/\gamma_m(F)'$ defined as

$$\begin{aligned} \psi \quad : \quad & \overline{x_1} \to \overline{x_1} + [\overline{x_1}, \overline{x_2}] + [[\overline{x_1}, [\overline{x_2}, \overline{x_3}]], \overline{x_4}] \\ & \overline{x_i} \to \overline{x_i} + w_i, i \neq 1 \end{aligned}$$

where $w_i \in \gamma_m(F)$. Through the verification of Theorem 3.3, it is determined that the restriction of $\widetilde{\psi}$ to $I_{R/R'}$ is associated with the matrix M of the form

$$\begin{bmatrix} 1+r_{\overline{\overline{x_2}}} & r_{\overline{\overline{x_1}}} + r_{\overline{\overline{x_3}}} l_{\overline{\overline{x_1}}} r_{\overline{\overline{x_4}}} & l_{\overline{\overline{x_2}}}.l_{\overline{\overline{x_1}}}.r_{\overline{\overline{x_4}}} & ... & 0 \\ u_{21} & 1+u_{22} & u_{23} & ... & u_{2n} \\ u_{31} & u_{32} & 1+u_{33} & ... & u_{3n} \\ . & . & . & ... & . \\ u_{n1} & u_{n2} & u_{n3} & ... & 1+u_{nn} \end{bmatrix}$$

where $u_{ij} \in \Delta^3$. Let M be invertible in $UL(F)/\Delta_{\gamma_m(F)}$. Then, $\widehat{M}$ is also invertible and which is of the form

$$\begin{bmatrix} 1+\overline{\overline{x_2}} & \overline{\overline{x_1}} - \overline{\overline{x_3 x_1 x_4}} & \overline{\overline{x_2 x_1 x_4}} & ... & 0 \\ \widehat{u_{21}} & 1+\widehat{u_{22}} & \widehat{u_{23}} & ... & \widehat{u_{2n}} \\ \widehat{u_{31}} & \widehat{u_{32}} & 1+\widehat{u_{33}} & ... & \widehat{u_{3n}} \\ . & . & . & ... & . \\ \widehat{u_{n1}} & \widehat{u_{n2}} & \widehat{u_{n3}} & ... & 1+\widehat{u_{nn}} \end{bmatrix}.$$

Since $\Delta_{\gamma_m(F)} \subset \Delta^3$ for $m \geq 4$, consider $H_3 = \widehat{U(F)/\Delta_{\gamma_m(F)}}/\widehat{\Delta^3/\Delta_{\gamma_m(F)}}$. The image of elements of $\widehat{M}$ in H_3 determines

$$\begin{bmatrix} 1+\overline{\overline{x_2}} & \overline{\overline{x_1}} & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}.$$

Then, we obtain

$$D_3(\widehat{M}) = 1 + \overline{\overline{x_2}} \text{ modulo } (\widehat{\Delta_{\gamma_m(F)}} + \widehat{\Delta^3/\Delta_{\gamma_m(F)}}).$$

Therefore,

$$det_3(M) = 1 + r_{\overline{\overline{x_2}}} + \Delta_{\gamma_m(F)} + \Delta^3/\Delta_{\gamma_m(F)} + \Delta_{Ann(F/R)}.$$

By Theorem 4.2,

$$1+r_{\overline{\overline{x_2}}}+\Delta_{\gamma_m(F)}+\Delta^3/\Delta_{\gamma_m(F)}+\Delta_{Ann(F/R)} = 1+\Delta_{\gamma_m(F)}+\Delta^3/\Delta_{\gamma_m(F)}+\Delta_{Ann(F/R)}.$$

Hence, it follows that $r_{\overline{\overline{x_2}}} \in \Delta_{\gamma_m(F)} + \Delta^3/\Delta_{\gamma_m(F)} + \Delta_{Ann(F/R)}$. This is impossible, thus, $\widetilde{\psi}$ cannot be an automorphism.

Example 4.5. Let $R = F'$, and consider the endomorphism ψ on F/R' defined by the following mappings

$$\begin{aligned} \psi \quad : \quad & \overline{x_1} \to \overline{x_1} + [[\overline{x_1}, \overline{x_2}], \overline{x_3}] + w_1, \\ & \overline{x_i} \to \overline{x_i} + w_i, i \neq 1. \end{aligned}$$

where $w_i \in F'', i = 1, \cdots, n$. The associated matrix M is given in the form

$$\begin{bmatrix} 1+r_{\overline{\overline{x_2}}}r_{\overline{\overline{x_3}}}+u_{11} & l_{\overline{\overline{x_1}}}r_{\overline{\overline{x_3}}}+u_{12} & u_{13} & .. & u_{1n} \\ u_{21} & 1+u_{22} & u_{23} & ... & u_{2n} \\ u_{31} & u_{32} & 1+u_{33} & ... & u_{3n} \\ . & . & . & ... & . \\ u_{n1} & u_{n2} & u_{n3} & ... & 1+u_{nn} \end{bmatrix},$$

where $w_{ij} \in \Delta^3$. Let M be invertible in $UL(F)/\Delta_{F''}$. Hence, $\widehat{M}$ is

$$\begin{bmatrix} 1+\overline{\overline{x_2 x_3}}+\widehat{u_{11}} & -\overline{\overline{x_1 x_3}}+\widehat{u_{12}} & \widehat{u_{13}} & ... & \widehat{u_{1n}} \\ \widehat{u_{21}} & 1+\widehat{u_{22}} & \widehat{u_{23}} & ... & \widehat{u_{2n}} \\ \widehat{u_{31}} & \widehat{u_{32}} & 1+\widehat{u_{33}} & ... & \widehat{u_{3n}} \\ . & . & . & ... & . \\ \widehat{u_{n1}} & \widehat{u_{n2}} & \widehat{u_{n3}} & ... & 1+\widehat{u_{nn}} \end{bmatrix}$$

Since, $\Delta_{F''} \subset \Delta^3$, take $H_3 = \widehat{UL(F)/\Delta_{F''}}/\widehat{\Delta^3/\Delta_{F''}}$. $\widehat{M}$ in H_3 is

$$\begin{bmatrix} 1+\overline{\overline{x_2 x_3}} & -\overline{\overline{x_1 x_3}} & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}.$$

Then, we obtain

$$det_3(M) = 1 + r_{\overline{\overline{x_2}}}r_{\overline{\overline{x_3}}} + \Delta_{F''} + \Delta^3/\Delta_{F''} + \Delta_{Ann(F/R)}.$$

By Theorem 4.2

$$1 + r_{\overline{\overline{x_2}}} r_{\overline{\overline{x_3}}} + \Delta_{F''} + \Delta^3/\Delta_{F''} + \Delta_{Ann(F/R)} = 1 + \Delta_{F''} + \Delta^3/\Delta_{F''} + \Delta_{Ann(F/R)}.$$

This yields $r_{\overline{\overline{x_2}}} r_{\overline{\overline{x_3}}} \in \Delta_{F''} + \Delta^3/\Delta_{F''} + \Delta_{Ann(F/R)}$. However, this is impossible. Thus, $\widetilde{\psi}$ and ψ are not automorphisms.

Example 4.6. Given an endomorphism ψ of F/F'' defined as

$$\begin{aligned} \psi \quad : \quad & \overline{x_1} \to \overline{x_1} + [[\overline{x_2}, \overline{x_2}] + [\overline{x_2}, \overline{x_1}], \overline{x_1}] \\ & \overline{x_i} \to \overline{x_i} + [\overline{x_i}, \overline{x_i}] + [x_i, x_1], i \neq 1 \end{aligned}$$

Its associated matrix M is

$$\begin{bmatrix} 1 + l_{\overline{\overline{x_2}}} r_{\overline{\overline{x_1}}} & (r_{\overline{\overline{x_2}}} + l_{\overline{\overline{x_2}}}) r_{\overline{\overline{x_1}}} & 0 & ... & 0 \\ 0 & 1 + r_{\overline{\overline{x_2}}} + l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_1}}} & 0 & ... & 0 \\ 0 & 0 & 1 + r_{\overline{\overline{x_3}}} + l_{\overline{\overline{x_3}}} + r_{\overline{\overline{x_1}}} & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 + r_{\overline{\overline{x_n}}} + l_{\overline{\overline{x_n}}} + r_{\overline{\overline{x_1}}} \end{bmatrix}$$

Then, $\widehat{M}$ is

$$\begin{bmatrix} 1 - \overline{\overline{x_2 x_1}} & 0 & 0 & ... & 0 \\ 0 & 1 + \overline{\overline{x_1}} & 0 & ... & 0 \\ 0 & 0 & 1 + \overline{\overline{x_1}} & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 + \overline{\overline{x_1}} \end{bmatrix}$$

Since, $\Delta_{F''} \subset \Delta^2$, consider $H_2 = U\widehat{(F)/\Delta}_{F''}/\widehat{\Delta^2/\Delta}_{F''}$. $\widehat{M}$ in H_2 is

$$\begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 + \overline{\overline{x_1}} & 0 & ... & 0 \\ 0 & 0 & 1 + \overline{\overline{x_1}} & .. & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 + \overline{\overline{x_1}} \end{bmatrix}$$

Then,

$$D_3(\widehat{M}) = 1 + n.\overline{\overline{x_1}} + ... + (\overline{\overline{x_1}})^n \text{modulo } (\widehat{\Delta_{F''}} + \widehat{\Delta^2/\Delta_{F''}}).$$

Hence, we obtain

$$det_2(M) = 1 + nr_{\overline{\overline{x_1}}} + ... + (r_{\overline{\overline{x_1}}})^n + \Delta_{F''} + \Delta^2/\Delta_{F''} + \Delta_{Ann(F/R)}.$$

By Theorem 4.2, we can express the equation as follows

$$1 + nr_{\overline{\overline{x_1}}} + ... + (r_{\overline{\overline{x_1}}})^n + \Delta_{F''} + \Delta^2/\Delta_{F''} + \Delta_{Ann(F/R)} = 1 + \Delta_{F''} + \Delta^2/\Delta_{F''} + \Delta_{Ann(F/R)}.$$

This yields $nr_{\overline{\overline{x_1}}} + ... + (r_{\overline{\overline{x_1}}})^n \in \Delta_{F''} + \Delta^2/\Delta_{F''} + \Delta_{Ann(F/R)}$ and consequently, $nr_{\overline{\overline{x_1}}} \in \Delta_{F''} + \Delta^2/\Delta_{F''} + \Delta_{Ann(F/R)}$. However, this is impossible. Therefore, ψ is not an automorphism.

Conclusion

This study initially derives a matrix representation of the IA-automorphisms on the Leibniz algebra F/R'. Following this, we set forth a prerequisite for an IA-endomorphism of F/R' to qualify as an IA-automorphism. In this criterion, we identify the non-invertibility of a square matrix M over $UL(F)/\Delta_R$. This approach explicitly relies on the Dieudonn'e determinant.

5. Acknowledgments

The author would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author was supported by the Cukurova University BAP Coordination Council [FBA-2020-13173].

The Declaration of Conflict of Interest/ Common Interest
The author declared that no conflict of interest or common interest.

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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