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RESEARCH ARTICLE

Rotational Surfaces in $\mathbb{R}^4$ with New Approaches and Examples

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this paper, we give a new approach to the rotational minimal surfaces in 4-dimensional Euclidean space $\mathbb{R}^4$. One type of these surfaces is obtained by the composition of two families of rotations in orthogonal planes. For these surfaces, we give a new parameterization. Using this parametrization, we find new examples of rotational minimal surfaces and rotational surfaces with zero Gaussian curvature.

Keywords: *Rotational surface, minimal surface, Gauss curvature.*

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1. Rotational surfaces in $\mathbb{R}^4$

A rotation in the Euclidean space $\mathbb{R}^4$ is a positive isometry of $\mathbb{R}^4$ that leaves one point fixed [2]. This implies that a rotation of $\mathbb{R}^4$ leaves two absolutely orthogonal planes invariant as point-sets. If (x_1, x_2, x_3, x_4) denote the Cartesian coordinates in $\mathbb{R}^4$, these planes can be assumed to be the x_1x_2-plane and the x_3x_4-plane. Then a rotation of $\mathbb{R}^4$ can be written as

$$R_{m,n}(\theta) = \begin{pmatrix} \cos(m\theta) & -\sin(m\theta) & 0 & 0 \\ \sin(m\theta) & \cos(m\theta) & 0 & 0 \\ 0 & 0 & \cos(n\theta) & -\sin(n\theta) \\ 0 & 0 & \sin(n\theta) & \cos(n\theta) \end{pmatrix},$$

where $m, n \in \mathbb{R}$ are called the angles of rotation. There are two types of rotations in $\mathbb{R}^4$. A *special rotation* or *simple rotation* is a rotation which leaves pointwise fixed one of the two coordinate planes. This corresponds when some of the two angles of rotation is zero ($m = 0$ or $n = 0$). In case that m and n are not zero, the rotation is called a *general rotation*. Notice that a general rotation is the product of two simple rotations, $R_{m,n}(\theta) = R_{m,0}(\theta) \cdot R_{0,n}(\theta) = R_{0,n}(\theta) \cdot R_{m,0}(\theta)$.

A rotational (regular) surface Σ of $\mathbb{R}^4$ is a surface invariant by a one parameter group of rotations [7]. If $G_{m,n} = \{R_{m,n}(\theta) : \theta \in \mathbb{R}\}$ is a such a group where m, n are not simultaneously zero, we will say that Σ is a *special* (resp. *general*) rotational surface if m or n is zero (resp. $m, n \neq 0$). Notice that the orbit of a point of Σ by the group $G_{m,n}$ is not a circle in general. However, the orbit of a point $p = (x_1, x_2, x_3, x_4) \in \Sigma$ under the action of the group $G_{m,0}$ is a circle if $x_1^2 + x_2^2 \neq 0$, or the point is fixed if $x_1 = x_2 = 0$. It is immediate that any rotational surface of $\mathbb{R}^3$ when it is included in an affine 3-dimensional subspace, it is a special rotational surface.

We find a parametrization of a general rotational surface Σ. It is cleat that if $\gamma = \gamma(t)$, $t \in I \subset \mathbb{R}$, is any curve contained in Σ, then the action of the group $G_{m,n}$ on γ is an open set of Σ. This gives a parametrization of Σ depending on the parameter of γ and the parameter of $G_{m,n}$. However a better and suitable parametrization of Σ is considering the two curves γ_1 and γ_2 obtained by intersecting Σ with the x_1x_2-plane and the x_3x_4-planes, respectively. After a reparametrization, we can assume that $t \in I \subset \mathbb{R}$ is a common parameter for γ_1 and γ_2. If

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we write $\gamma_1(t) = (x_1(t), x_2(t))$ and $\gamma_2(t) = (x_3(t), x_4(t))$, then a local parametrization of Σ is

$$\begin{aligned}\Psi(t,\theta) = R_{m,n}(\theta)(\gamma_1(t),\gamma_2(t)) &= \begin{pmatrix} x_1(t)\cos(m\theta) - x_2(t)\sin(m\theta) \\ x_1(t)\sin(m\theta) + x_2(t)\cos(m\theta) \\ x_3(t)\cos(n\theta) - x_4(t)\sin(n\theta) \\ x_3(t)\sin(n\theta) + x_4(t)\cos(n\theta) \end{pmatrix} \\ &= R_{m,0}(\theta)(\gamma_1(t),0,0) + R_{0,n}(\theta)(0,0,\gamma_2(t)),\end{aligned} \tag{1.1}$$

where $\theta \in \mathbb{R}$. In the following result we show a parametrization of Σ without the use of the coordinates $x_i(t)$, $1 \le i \le 4$ of γ_1 and γ_2.

Proposition 1.1. *If Σ is a general rotational surface of $\mathbb{R}^4$, then Σ can be parametrized by*

$$\Psi(t,\theta) = (f(t)\cos(m\theta + t), f(t)\sin(m\theta + t), g(t)\cos(\theta), g(t)\sin(\theta)), \tag{1.2}$$

or

$$\Psi(t,\theta) = (f(t)\cos(m\theta), f(t)\sin(m\theta), g(t)\cos(\theta + \theta_0), g(t)\sin(\theta + \theta_0)), \tag{1.3}$$

where $m, \theta_0 \in \mathbb{R}$ and f and g are smooth positive functions defined in some interval $I \subset \mathbb{R}$. Regularity is equivalent to $\left(m^2f^2 + g^2\right)\left(f'^2 + g'^2\right) + f^2g^2 \neq 0$ for (1.2) and $\left(m^2f^2 + g^2\right)\left(f'^2 + g'^2\right) \neq 0$ for (1.3).

Proof. Consider the parametrization of Σ given by (1.1). After the change of parameter $n\theta \to \theta$ and $m \to m/n$, we can assume that the parametrization of Σ is

$$\Psi(t,\theta) = \begin{pmatrix} x_1(t)\cos(m\theta) - x_2(t)\sin(m\theta) \\ x_1(t)\sin(m\theta) + x_2(t)\cos(m\theta) \\ x_3(t)\cos(\theta) - x_4(t)\sin(\theta) \\ x_3(t)\sin(\theta) + x_4(t)\cos(\theta) \end{pmatrix}. \tag{1.4}$$

Let

$$\begin{aligned}\gamma_1(t) &= (x_1(t), x_2(t)) = f(t)(\cos\eta(t), \sin\eta(t)), \\ \gamma_2(t) &= (x_3(t), x_4(t)) = g(t)(\cos\phi(t), \sin\phi(t)),\end{aligned}$$

for some smooth functions f, g, η, ϕ with $f, g > 0$. Using these identities in (1.4), we obtain

$$\begin{aligned}\Psi(t,\theta) &= R_{m,n}(\theta)(\gamma_1(t),\gamma_2(t)) \\ &= (f(t)\cos(m\theta + \eta(t)), f(t)\sin(m\theta + \eta(t)), g(t)\cos(\theta + \phi(t)), g(t)\sin(\theta + \phi(t)))\end{aligned}$$

Define the local diffeomorphism

$$(\bar{t}, \bar{\theta}) \mapsto (t, \theta + \phi(t)).$$

Then, renaming t and θ, the parametrization Ψ becomes

$$\Psi(t,\theta) = (f(t)\cos(m\theta + \eta(t) - m\phi(t)), f(t)\sin(m\theta + \eta(t) - m\phi(t)), g(t)\cos(\theta), g(t)\sin(\theta)).$$

Let $\varphi(t) = \eta(t) - m\phi(t)$. Then we have

$$\Psi(t,\theta) = (f(t)\cos(m\theta + \varphi(t)), f(t)\sin(m\theta + \varphi(t)), g(t)\cos(\theta), g(t)\sin(\theta)).$$

We distinguish two cases.

1. If φ is not a constant function, then we work around $t_0 \in I$ where $\varphi'(t_0) \neq 0$. Consider the inverse function φ^{-1} of φ. Using the change $s = \varphi(t)$, the functions $f(t)$ and $g(t)$ change to $f(\varphi^{-1}(s))$ and $g(\varphi^{-1}(s))$. Finally the change $s \to t$ yields (1.2) renaming $f(\varphi^{-1}(s))$ and $g(\varphi^{-1}(s))$ by f and g, respectively. This gives (1.2).
2. If φ is a constant function, $\varphi(t) = \varphi_0$, then the change $\theta \mapsto \theta - \varphi_0/m$ yields (1.3) for constant $\theta_0 = \varphi_0/m$.

□

For surfaces parametrized by (1.3) and in the calculations of curvatures of surface, the fact that θ_0 is 0 or not is irrelevant. Therefore, the constant θ_0 can be assumed to be zero. This gives the following definition.

Definition 1.1. A *pure* general rotational surface of $\mathbb{R}^4$ is a surface parametrized by (1.3) when $\theta_0 = 0$.

From this definition, we separate the general rotational surfaces in two classes: the pure general rotational surfaces and the not pure general rotational surface, which are parametrized by (1.2). For special rotational surfaces, we also obtain suitable parametrizations as shown the following result.

Proposition 1.2. *Let Σ be a special rotational surface of $\mathbb{R}^4$ and assume that Σ is not included in an affine 3-subspace of $\mathbb{R}^4$ of equation $x_4 = ct$. Then there is a parametrization of Σ given by*

$$\Psi(t,\theta) = (f(t)\cos(\theta), f(t)\sin(\theta), g(t), t), \tag{1.5}$$

where f and g are smooth functions defined in some interval $I \subset \mathbb{R}$ with $f > 0$.

Proof. We know that $n = 0$ in (1.1). Let us use γ_1 and γ_2 as in the proof of Prop. 1.1. After the change $m\theta \to \theta$, the parametrization of a special rotational surface

$$\Psi(t,\theta) = R_{1,0}(\theta)(\gamma_1(t), \gamma_2(t)) = (x_1(t)\cos(\theta), x_2(t)\sin(\theta), x_3(t), x_4(t)).$$

Let $\gamma(t) = f(t)(\cos\varphi(t), \sin\varphi(t))$ with $f > 0$. Then the parametrization of a special rotational surface Σ is

$$\Psi(t,\theta) = (f(t)\cos(\theta + \varphi(t)), f(t)\sin(\theta + \varphi(t)), x_3(t), x_4(t)).$$

The local diffeomorphism $(\bar{t}, \bar{\theta}) \mapsto (t, \theta + \varphi(t))$ changes this parametrization into $\Psi(t,\theta) = (f(t)\cos(\theta), f(t)\sin(\theta), x_3(t), x_4(t))$. Since Σ is not included in a subspace of equation $x_4 = ct$, then $x_4'(t) \neq 0$ in some subinterval of I. Then we can take the inverse of x_4 parametrizing the surface by (1.5) after renaming f and x_3. □

In order to have examples of rotational surfaces in $\mathbb{R}^4$ assuming conditions on the curvature of the surface, we can observe that the number of degrees of freedom of the (special or general) rotational surfaces is 2 because they depend on the functions f and g. For general rotational surfaces, however, the constant m in (1.2) must also be taken into account.

2. Rotational minimal surfaces in $\mathbb{R}^4$

In this section we investigate rotational surfaces in $\mathbb{R}^4$ with zero mean curvature everywhere, that is, rotational minimal surfaces. For pure general rotational surfaces, the classification was done by Moore when $m = 1$ [7], proving that the generating curve is contained in a particular hyperbola (see Cor. 2.1 below). If $m \neq 1$, the mean curvature equation was obtained in [3]: see also [1, 4, 5, 6, 8]. First, we recall the notion of a minimal surface in $\mathbb{R}^4$.

Let Σ be a regular surface immersed in $\mathbb{R}^4$ by means of the parametrization $\Psi\colon \Sigma \to \mathbb{R}^4$. Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connection of $\mathbb{R}^4$ and the induced on Σ, respectively. If $X, Y \in \mathfrak{X}(\Sigma)$ are two tangent vector fields on Σ with local extensions $\overline{X}$ and $\overline{Y}$ on $\mathbb{R}^4$, then $\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^\top$. Let $\nabla^\perp$ be the connection on the normal bundle $N\Sigma$ of Σ defined as follows. If X is a tangent vector field on Σ and N is a normal vector field, with local extensions $\overline{X}$ and $\overline{N}$, then $\nabla^\perp_X N = (\overline{\nabla}_{\overline{X}}\overline{N})^\perp$. The second fundamental form σ of Σ is defined by

$$\sigma(X,Y) = (\overline{\nabla}_{\overline{X}}\overline{Y})^\perp.$$

The mean curvature vector $\vec{H}$ of Σ is defined by

$$\vec{H} = \frac{1}{2}(\sigma(X_1,X_1) + \sigma(X_2,X_2)),$$

where $\{X_1, X_2\}$ is an orthonormal basis of $\mathfrak{X}(\Sigma)$. This vector $\vec{H}$ does not depend on the orthonormal basis chosen in $\mathfrak{X}(\Sigma)$. The surface Σ is said to be minimal if $\vec{H} = 0$.

We now express $\vec{H}$ in local coordinates, $\Psi = \Psi(t,s)$. As usual, the coefficients of the first fundamental form are

$$E = \langle \Psi_t, \Psi_t\rangle, \quad F = \langle \Psi_t, \Psi_s\rangle, \quad G = \langle \Psi_s, \Psi_s\rangle.$$

Let $W = \sqrt{EG - F^2}$. If $\{N_1, N_2\}$ is an orthonormal frame in $N\Sigma$, then we have

$$\begin{aligned}
(\overline{\nabla}_{\Psi_t}\Psi_t)^\perp &= \sigma_{11}^1 N_1 + \sigma_{11}^2 N_2,\\
(\overline{\nabla}_{\Psi_t}\Psi_s)^\perp &= \sigma_{12}^1 N_1 + \sigma_{12}^2 N_2,\\
(\overline{\nabla}_{\Psi_s}\Psi_s)^\perp &= \sigma_{22}^1 N_1 + \sigma_{22}^2 N_2.
\end{aligned}$$

After some computations, the mean curvature vector $\vec{H}$ is

$$\vec{H} = \frac{1}{2W}\left((G\sigma_{11}^1 - 2F\sigma_{12}^1 + E\sigma_{22}^1)N_1 + (G\sigma_{11}^2 - 2F\sigma_{12}^2 + E\sigma_{22}^2)N_2\right).$$

Thus Σ is minimal if and only if the following two equations hold in Σ:

$$G\sigma_{11}^1 - 2F\sigma_{12}^1 + E\sigma_{22}^1 = 0,$$
$$G\sigma_{11}^2 - 2F\sigma_{12}^2 + E\sigma_{22}^2 = 0,$$

which in turn is equivalent to

$$G\langle\Psi_{tt}, N_1\rangle - 2F\langle\Psi_{ts}, N_1\rangle + E\langle\Psi_{ss}, N_1\rangle = 0, \tag{2.1}$$
$$G\langle\Psi_{tt}, N_2\rangle - 2F\langle\Psi_{ts}, N_2\rangle + E\langle\Psi_{ss}, N_2\rangle = 0. \tag{2.2}$$

Notice that if $\{N_1, N_2\}$ is an orthogonal basis of $N\Sigma$, the minimality condition $\vec{H} = 0$ also writes as (2.1)-(2.2).

By completeness, we give the classification of pure general rotational surfaces and special rotational surfaces of $\mathbb{R}^3$ with zero mean curvature.

Theorem 2.1. *Let Σ be a pure general rotational surface generated by $\gamma(t) = (f(t), 0, g(t), 0)$. Then Σ is a minimal surface if and only if the curvature κ of the $2-$planar curve γ satisfies*

$$\kappa = \frac{1}{\sqrt{f'^2 + g'^2}}\frac{gf' - m^2 fg'}{m^2 f^2 + g^2}. \tag{2.3}$$

Proof. We know that the parametrization of Σ is (1.3). An orthogonal basis $\{N_1, N_2\}$ of the normal bundle $N\Sigma$ is

$$N_1(t,\theta) = (g'(t)\cos(m\theta), g'(t)\sin(m\theta), -f'(t)\cos(\theta), -f'(t)\sin(\theta)),$$
$$N_2(t,\theta) = (-g(t)\sin(m\theta), g(t)\cos(m\theta), mf(t)\sin(\theta), -mf(t)\cos(\theta)).$$

Equation (2.2) is a trivial identity $0 = 0$. Equation (2.1) becomes

$$\left(f'^2 + g'^2\right)\left(gf' - m^2 fg'\right) + \left(m^2 f^2 + g^2\right)\left(f''g' - f'g''\right) = 0.$$

Since the curvature κ of γ is

$$\kappa = \frac{f'g'' - g'f''}{(f'^2 + g'^2)^{3/2}},$$

we obtain (2.3). □

Equation (2.3) can be explicitly solved when $m = 1$ (see also [7]).

Corollary 2.1. *Let Σ be a pure general rotational surface of $\mathbb{R}^4$ parametrized by (1.3). Suppose $m = 1$. If Σ is a minimal surface, then the generating curve is parametrized by $\gamma(t) = (t, g(t))$, where*

$$g(t) = -ct \pm \sqrt{(1 + c^2)t^2 + d}, \quad c, d \in \mathbb{R}.$$

In particular, the generating curve $t \mapsto (t, g(t))$ lies in a hyperbola of $\mathbb{R}^2$.

Proof. From (2.3), we have

$$\kappa = \frac{gf' - fg'}{\sqrt{f'^2 + g'^2}(f^2 + g^2)}.$$

Without lose of generality we can assume that $\gamma(t) = (t, g(t))$. Then this above equation is

$$\frac{g''}{1 + g'^2} = \frac{g - tg'}{t^2 + g^2}.$$

This gives

$$-\tan^{-1}(g') = \tan^{-1}(\frac{g}{t}) + \lambda, \quad \lambda \in \mathbb{R}.$$

Hence

$$g' = \frac{-g+ct}{t+cg}, \quad c = -\tan(\lambda).$$

Changing c by another constant we get

$$g(t)^2 - t^2 + 2ctg(t) - d = 0, \quad d \in \mathbb{R}.$$

This proves the result. □

The classification of the minimal special rotational surfaces is the following.

Theorem 2.2. *Let Σ be a special rotational surface. If Σ is minimal then Σ is included in an affine 3-subspace of $\mathbb{R}^4$ and it is a minimal surface as surface of $\mathbb{R}^3$.*

Proof. Suppose that Σ is not included in any affine 3-subspace of $\mathbb{R}^4$ of equation $x_4 = ct$. Then Σ can be parametrized using (1.5). We know that $1 + f'(t)^2 + g'(t)^2 \neq 0$ and $f > 0$. An orthogonal basis of $N\Sigma$ is

$$\begin{aligned} N_1 &= (\cos(\theta), \sin(\theta), 0, -f'), \\ N_2 &= \left(-f'g'\cos(\theta), -f'g'\sin(\theta), 1 + f'^2, -g'\right). \end{aligned}$$

Then equations (2.1)-(2.2) are, respectively,

$$\begin{aligned} f^2 f'' - f\left(f'^2 + g'^2 + 1\right) &= 0, \\ f\left(f'^2+1\right)g'' + f'g'\left(-ff'' + f'^2 + g'^2 + 1\right) &= 0. \end{aligned}$$

Multiplying the first equation by $f'g'$ and adding the second one multiplied by f^2, we obtain

$$f^3(1+f'^2)g'' = 0.$$

Thus $g''(t) = 0$ for all $t \in I$, proving that $g(t) = at + b$, $a, b \in \mathbb{R}$. Then the parametrization (1.5) is

$$\Psi(t,\theta) = (f(t)\cos(\theta), f(t)\sin(\theta), at + b, t).$$

This proves that Σ is included in the affine 3-subspace of equation $x_3 - ax_4 - b = 0$, obtaining the result. □

Once we have proved Thm. 2.2, we can rediscover the minimal surfaces of $\mathbb{R}^3$. Suppose $g(t) = at$ after a translation along the x_3-axis. By repeating again the computations, the minimal surface equations (2.1)-(2.2) are

$$\begin{aligned} a^2 - ff'' + f'^2 + 1 &= 0, \\ af'\left(a^2 - ff'' + f'^2 + 1\right) &= 0. \end{aligned}$$

From the second equation, we distinguish three cases. If f is constant, the first equation yields a contradiction. If $a = 0$, then Σ is included in the 3-space $x_3 = 0$, that is, $\mathbb{R}^3$ after renaming coordinates. If $a \neq 0$, then $a^2 - ff'' + f'^2 + 1 = 0$ whose solution is

$$f(t) = \frac{\sqrt{1+a^2}}{c}\cosh(ct + d), \quad c, d \in \mathbb{R}, c > 0.$$

This proves that the generating curve is a catenary.

From now, we study general rotational surfaces with zero mean curvature. In the case of a not pure general rotational surface, the minimal surface equations (2.1)-(2.2) are not easy to solve because of the presence of the parameter t in $\cos(m\theta + t)$ and $\sin(m\theta + t)$ in (1.2). In order to give examples, we are interested in particular cases.

A first situation is when f and g are constant, namely, $f(t) = f_0$ and $g(t) = g_0$.

Theorem 2.3. *Let Σ be a general rotational surface parametrized by (1.2). If f and g are constant, then Σ is not minimal.*

Proof. Suppose that $f(t) = f_0$ and $g(t) = g_0$ are constant, with $f_0 g_0 \neq 0$. Regularity of Σ is equivalent to $\varphi' \neq 0$. An orthogonal basis of $N\Sigma$ is

$$N_1 = (1, \tan(m\theta + t), 0, 0),$$
$$N_2 = (0, 0, 1, \tan(\theta)).$$

Then (2.1) and (2.2) become

$$f_0 g_0^2 \cos^{-1}(m\theta + t) = 0,$$
$$f_0^2 g_0 \cos^{-1}(\theta) = 0.$$

Since θ and t are arbitrary, we have $f_0 = g_0 = 0$, obtaining a contradiction. □

Other case to study is when one of the functions f or g are constant. We prove that this is not possible.

Theorem 2.4. *Let Σ be a general rotational surface parametrized by* (1.2)*. If f or g is constant, then Σ is not minimal.*

Proof. We distinguish two cases.

1. Case g is constant, say $g(t) = g_0$. Regularity of the surface is equivalent to

   $$(g_0^2 + m^2 f^2) f'^2 + g_0^2 f^2 \neq 0, \quad \forall t \in I.$$

   On the other hand, after computing an orthogonal basis $\{N_1, N_2\}$, equation (2.1) is of type $A_1(t)\sin\theta + A_2(t)\cos\theta = 0$. Thus $A_1 = A_2 = 0$. The computation of A_2 yields $A_2 = -mg_0 f f'(f'^2 + f^2)$. In particular, f is a constant function. By Thm. 2.3, we conclude that Σ is not minimal.

2. Case f is constant, say $f(t) = f_0$. Then, regularity yields

   $$g'(t)^2(g(t)^2 + m^2 f_0^2) + f_0^2 g(t)^2 \neq 0, \quad \forall t \in I.$$

   Following the same steps as in previous case, we obtain that equation (2.1) is of type

   $$B_1(t)\sin(m\theta + t) + B_2(t)\cos(m\theta + t) = 0,$$

   where $B_2 = -f_0(m^2 g'^2 + g^2)$. In particular, we arrive to $m^2 g'(t)^2 = g(t) = 0$ for all $t \in I$, obtaining a contradiction again.

□

An interesting case is when $f(t) = g(t)$. Under this assumption, in the following result we classify general rotational minimal surfaces when $m = 1$.

Theorem 2.5. *Let Σ be a general rotational surface parametrized by* (1.2) *with $m = 1$. If $f(t) = g(t)$ then Σ is minimal if and only if*

$$f(t) = \frac{c_1}{\sqrt{\cos(t + c_2)}}, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 \neq 0. \tag{2.4}$$

Proof. The metric components are

$$E = 2f'(t)^2 + f(t)^2, \quad F = f(t)^2, \quad G = 2f(t)^2,$$

where the regularity yields

$$4f(t)^2(f'(t)^2 + f(t)^2) \neq 0, \quad \forall t \in I.$$

After computing $\{N_1, N_2\}$, Equations (2.1) and (2.2) are

$$\begin{aligned} f(t)^2 \sin(\theta + t) B(t) &= 0, \\ f(t)^2 \left(f(t)\sin(\theta + t) - 2f'(t)\cos(\theta + t)\right) B(t) &= 0, \end{aligned} \tag{2.5}$$

where

$$B(t) = f(t)\left(f(t) - 2f''(t)\right) + 6f'(t)^2.$$

Hence, the equation to solve is

$$f(t)\left(f(t) - 2f''(t)\right) + 6f'(t)^2 = 0.$$

It is easy to prove that the solution is (2.4). □

3. Rotational surfaces in $\mathbb{R}^4$ with zero Gaussian curvature

In this section we give examples of rotational surfaces in $\mathbb{R}^4$ with constant Gaussian curvature K. However, the computations when K is non-zero for the parametrizations of Props. 1.1 and 1.2 are cumbersome. For this reason, we will only focus when $K = 0$. Following with the notation of Sect. 2, the Gaussian curvature of Σ is

$$K = \frac{1}{W^2}\sum_{k=1}^{2}(\sigma_{11}^k\sigma_{22}^k - (\sigma_{12}^k)^2). \tag{3.1}$$

First we begin with the analog of Thm. 2.2. The fact that the parametrization (1.5) of Σ is given in terms of three functions make difficult to find examples unless we restrict the functions that appear in (1.5). A first approach is assuming that the generating curve of Σ is a graph on the x_4-axis. Then this curve becomes $t \mapsto (f(t), g(t), t)$.

Theorem 3.1. *Let Σ be a special rotational surface parametrized by* (1.5). *If $K = 0$ everywhere, then f and g satisfy*

$$f'(t) = c\sqrt{1 + g'(t)^2}, \tag{3.2}$$

for some constant $c \in \mathbb{R}$.

Proof. Here we need to compute an orthonormal basis B for $\mathfrak{X}(\Sigma)$ by using the Gram-Schmidt method. Then from the expression of K in (3.1), the equation $K = 0$ is equivalent to

$$-f''(t)g'(t)^2 - f''(t) + f'(t)g'(t)g''(t) = 0.$$

An integration yields (3.2). □

Theorem 3.1 allows to give many examples for surfaces with $K = 0$ in $\mathbb{R}^4$ of special rotational type by taking g as particular functions.

1. Case $g(t) = t$. Then $f(t) = \sqrt{2}ct + c_1$, $c_1 \in \mathbb{R}$. This surface is included in the 3-subspace of equation $x_3 - x_4 = 0$. It is not difficult to see that Σ is a cone of revolution in that subspace.
2. Case $g(t) = \sqrt{1 - t^2}$. Then $f(t) = -2c\tan^{-1}\left(\frac{\sqrt{1-t^2}}{t+1}\right) + c_1$, $c_1 \in \mathbb{R}$. This surface is included in the hypersurface $x_3^2 + x_4^2 = 1$.
3. Case $g(t) = t^2$. Then $f(t) = c\left(\frac{1}{2}t\sqrt{4t^2+1} - \frac{1}{4}\log\left(\sqrt{4t^2+1} - 2t\right)\right) + c_1$, $c_1 \in \mathbb{R}$.
4. Case $g(t) = e^t$. Then $f(t) = c\left(\sqrt{e^{2t}+1} - \tanh^{-1}\left(\sqrt{e^{2t}+1}\right)\right) + c_1$, $c_1 \in \mathbb{R}$.

We now consider the study of general rotational surfaces with zero Gaussian curvature.

Theorem 3.2. *Let Σ be a general rotational surface parametrized by* (1.2). *If f and g are constant, then $K = 0$.*

Proof. This is a direct consequence of the computations given in the proof of Thm. 2.3, where the values of σ_{ij}^k were obtained. □

Theorem 3.3. *Let Σ be a general rotational surface parametrized by* (1.2) *with zero Gaussian curvature.*

1. *If $g(t) = g_0$ is constant, then*

$$f(t) = \pm\tan^{-1}(\sqrt{c_1t^2 - 1}) \mp \sqrt{c_1t^2 - 1} + c_2, \quad c_1, c_2 \in \mathbb{R}, c_1 \neq 0. \tag{3.3}$$

2. *If $f(t) = f_0$ is constant, then t is a function of g given by*

$$t(g) = \pm m\tan^{-1}\left(\sqrt{c_1g^2 - 1}\right) \mp \sqrt{c_1g^2 - m^2} + c_2, \quad c_1, c_2 \in \mathbb{R}, c_1 \neq 0. \tag{3.4}$$

Proof. As in Thm. 2.4, we distinguish if g or f is constant.

1. Suppose $g(t) = g_0 \neq 0$ is a constant function. After calculating an orthonormal basis of $N\Sigma$, the expression of K is
$$K = -\frac{g_0^2 m^2 \left(f^3 f'' + f'^4\right)}{\left(f'^2 \left(m^2 f^2 + g_0^2\right) + g_0^2 f^2\right)^3}.$$
Then $K = 0$ is equivalent to
$$f^3 f'' + f'^4 = 0.$$
Solving this ODE, we obtain (3.3)

2. Suppose $f(t) = f_0 \neq 0$ is a constant function. The Gaussian curvature is
$$K = -\frac{f_0^2 \left(g^3 g'' + m^2 g'^4\right)}{g'^2 \left(g^2 + m^2 f_0^2\right) + f_0^2 g^2}.$$
By the assumption, the equation to solve is
$$g^3 g'' + m^2 g'^4 = 0.$$
The solution is (3.4).

□

As in the minimal case, consider the case $f(t) = g(t)$. In the following result, we classify rotational surfaces with $K = 0$ when $m = 1$.

Theorem 3.4. *Let Σ be a general rotational surface parametrized by (1.2) with $f(t) = g(t)$ and $m = 1$. Then the Gaussian curvature Σ is $K = 0$ if and only if Σ can be parametrized by*
$$\Psi(t,\theta) = c_1 e^{c_2 t} \left(\cos(\theta + t), \sin(\theta + t), \cos\theta, \sin\theta\right). \quad c_1, c_2 \in \mathbb{R}, c_1 c_2 \neq 0. \tag{3.5}$$

Proof. By a direct calculation, the Gaussian curvature writes as
$$K = -\frac{2f^2 \left(f f'' - f'^2\right)}{4f'^2 + f^2}.$$
Thus, our equation to solve is
$$f f'' - f'^2 = 0,$$
or equivalently, $(f'/f)' = 0$. After solving this ODE, we find $f(t) = c_1 e^{c_2 t}$, for some nonzero constants c_1, c_2, completing the proof. □

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Competing interests

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Author's contributions

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