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Araştırma Makalesi - Research Article

Fibonacci Numbers Sequence Derived From Suborbital Graphs for the Modular Group Γ

Γ Modüler Grubunun Alt Yörüngesel Graflarından Üretilmiş Fibonacci Sayı Dizisi

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ABSTRACT

The Fibonacci sequence, a special number sequence studied a lot recently and plays an important role with its applications in many fields of science, can be obtained in different areas of mathematics and with different methods. In this study, Fibonacci numbers are obtained with using suborbital graphs of the Modular group **Γ** and some special matrices.

Keywords- Fibonacci Numbers, Suborbital Graphs, Modular Group, Imprimitive Action, Congruence Subgroup

ÖZ

Son zamanlarda üzerinde çokça çalışılan ve bilimin birçok alanındaki uygulamalarıyla önemli bir rol oynayan özel bir sayı dizisi olan Fibonacci dizisi, matematiğin farklı alanlarında ve farklı yöntemlerle elde edilebilmektedir. Bu çalışmada, Fibonacci sayıları Modüler grubun alt yörüngesel grafları ve bazı özel matrisler kullanılarak elde edilmiştir.

Anahtar Kelimeler- Fibonacci Sayıları, Alt Yörüngesel Graflar, Modüler Grup, İmpirimitif Hareket, Kongrüans Altgrup

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I. INTRODUCTION

In the 13th century, the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, ...\}$ is constructed inductively so that each member is the sum of the preceding two members, was introduced by Leonardo Fibonacci. These numbers play a significant role not only in all areas of Mathematics but also in statistics, finance, architecture ext. Additionally, the golden ratio is a mathematical principle that originates from the Fibonacci sequence, calculated by dividing each Fibonacci number by one that directly precedes it. Further, [1] has more thorough details regarding this sequence.

The modular group is a fundamental group in number theory. The group acts on the upper half plane

$\mathcal{U} := \{z \in \mathbb{C}: Im(z) > 0\}$ and its elements correspond to fractional linear transformations of this plane as follows. It plays a crucial role in the study of modular forms, automorphic forms, and their connections to elliptic curves, quadratic forms, and many other areas of mathematics. For more information, see [2].

Suborbital graphs are an interesting topic within graph theory that has received significant attention in the literature. Suborbital graphs are classes of graphs that arise in the study of group actions on sets, particularly when considering the orbits of vertices or edges under the action of a group. These graphs have applications in various fields, including algebraic graph theory and combinatorics.

The concept of suborbital graphs for a permutation group acting on a set was first introduced in [3]. Based on this idea in [4], suborbital graphs for the Modular group were given as follows:

The modular group Γ consists of the pairs of matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, where\ \ a, b, c, d\ \epsilon \mathbb{Z}\ \ ve\ ad - bc = 1.$$

In the paper, we omit the symbol $\pm$, and identify each matrix with its negative. Γ acts on the extended rational

$\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ by

$$z \to \frac{az + b}{cz + d},\ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon\Gamma.$$

More precisely, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \frac{x}{y} = \frac{ax + by}{cx + dy}.$$

We summarize imprimitive action of a group G on a set X and suborbital graphs concerning the pair (G, X). For each member g of G, if $g: X \to X$ one to one onto, then g is called a permutation of X. If all members of G are permutations of X, then pair (G, X), or only G for short, is said to be a permutation group.

Now suppose (G, X) is a permutation group and the relation " $\approx$ " on X is an equivalence relation. Whenever $x \approx y$ implies that $g(x) \approx g(y)$, for all $g \in G$, the relation $\approx$ is called G −invariant. In this case, each equivalence class is said to be a block. Some of G −invariant equivalence relations are below:

(i) Identity relation: $\quad x \approx y \ \Leftrightarrow x = y,$

(ii) Universal relation: $x \approx y\ \ for\ all\ x, y \epsilon X.$

These two relations are called trivial. If there is a relation other than the above two, on X then (G, X) is called an imprimitive permutation group. If G is transitive on X, that is, if $x, y \in X$ then $y = g(x)$ for some $g \in G$, then (G, X) is an imprimitive transitive permutation group.

In the paper, we take $G = \Gamma$ and $X = \widehat{\mathbb{Q}}$. Of course, in this case, $(\Gamma, \widehat{\mathbb{Q}})$ is a transitive permutation group, the imprimitive relation is defined in [3] as follows:

The stabilizer Γ_∞ of the Modular group Γ is the group

$$\Gamma_0(n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \middle| \ c \equiv 0\ mod n\right\}.$$

If $n > 1$, then $\Gamma_\infty < \Gamma_0(n) < \Gamma$. So the relation, defined by

$$\frac{r}{s} \approx \frac{x}{y} \ \ \Leftrightarrow\ \ ry - sx \equiv 0\ (mod n),$$

is imprimitive. So $(\Gamma, \widehat{\mathbb{Q}})$ is an imprimitive transitive permutation group.

Now we give suborbitals, and suborbital graphs from [4]. Γ acts, as well, on the set $\widehat{\mathbb{Q}} \times \widehat{\mathbb{Q}}$, by $g: (\alpha, \beta) \rightarrow (g(\alpha), g(\beta))$. The orbits of the action are called suborbitals.

From the suborbital $O(\alpha, \beta)$ containing (α, β), we construct the suborbital graph $\mathcal{G}(\alpha, \beta)$: the vertices α are in $\widehat{\mathbb{Q}}$, while there is a directed edge from u to v, shown

$$u \rightarrow v \ \text{ if } (u, v) \in O(\alpha, \beta).$$

If we chose, as a vertex set, the block

$$[\infty] = [1/0] = \left\{\frac{x}{y} \in \mathbb{Q} \middle| \ y \equiv 0 \ (modn)\right\},$$

we get a subgraph $\mathcal{G}\left(\alpha, \frac{u}{n}\right)$, $F_{u,n}$ for short.

Let $v_0, v_1, v_2, \dots, v_k$ be in $[\infty]$. The configurations $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$ and $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots$ are called a path and infinite path, respectively.

Theorem 1.1 [4] There is an edge $\frac{r}{s} \rightarrow \frac{x}{y} \epsilon F_{u,n}$ if and only if either

$a)$ $x \equiv ur \ modn, \quad ry - sx = n$ or

$b)$ $x \equiv -ur \ modn, \quad ry - sx = -n$

One of the studies about suborbital graphs can be reviewed in [5]. After that several authors' work on these subjects facilitated some branches of Mathematics such as number theory, group theory ext. Obtaining the Fibonacci sequence is one of the most attractive outcomes within them. This sequence has been obtained in many ways. For example, in [6-9], some modular subgroups of Γ and related suborbital graphs are used to construct the sequence using infinite paths, and furthermore, in [10-12], some continued fractions are taken to produce the sequence, and in [13] some special matrices are used.

In this work, we believe in outlining the suborbital graph for a congruence subgroup $\Gamma_0(n)$ to find a relation of these graphs and the Fibonacci sequence.

II. MAIN CALCULATIONS

Let u be a natural number. Then,

$$T = \begin{pmatrix} -u & 1 \\ -(u^2 + 3u + 1) & u + 3 \end{pmatrix}$$

is in Γ.

If $k_0 = 3$, is the minimal positive integer for which $u^2 + k_0 \, u + 1 \equiv 0 (modn)$ in [5], and $n = u^2 + 3u + 1$, then $T \in \Gamma_0(n)$.

Further, from

$$T(z) = \frac{-uz + 1}{-(u^2 + 3u + 1)z + (u + 3)} = z \qquad (1)$$

we get that

$$\frac{-(2u + 3) + \sqrt{5}}{-2(u^2 + 3u + 1)} = z_1 \ , \quad \frac{-(2u + 3) - \sqrt{5}}{-2(u^2 + 3u + 1)} = z_2$$

are fixed points of T.

Theorem 2.1 Let k be a natural number. Then,

i) $T^k\left(\frac{1}{0}\right) \longrightarrow T^k\left(\frac{u}{n}\right)$ in $F_{u,n}$

ii) $T^k\left(\frac{1}{0}\right) \longrightarrow T^{k+1}\left(\frac{1}{0}\right)$ in $F_{u,n}$

iii) $\left\{T^k\left(\frac{1}{0}\right)\right\}_{k \epsilon \mathbb{N}}$ is an increasing sequence and the path

$$T\left(\frac{1}{0}\right) \longrightarrow T^2\left(\frac{1}{0}\right) \longrightarrow T^3\left(\frac{1}{0}\right) \longrightarrow \cdots$$

is an infinite.

Proof. i) We will prove this by using the principle of mathematical induction and Theorem 1.1.

We observe that

$$T\left(\frac{1}{0}\right) = \frac{u}{n} \longrightarrow \frac{3u+1}{3n} = T\left(\frac{u}{n}\right)$$

holds for $k = 1$.

Assume that

$$T^k\left(\frac{1}{0}\right) \longrightarrow T^k\left(\frac{u}{n}\right) \tag{2}$$

Since the expression (2) is true for k, we have

$$T\left(T^k\left(\frac{1}{0}\right)\right) \longrightarrow T\left(T^k\left(\frac{u}{n}\right)\right)$$

So the proof is completed.

ii) Using the above condition (i), we get

$$T^k\left(\frac{1}{0}\right) \longrightarrow T^k\left(\frac{u}{n}\right) = T^k\left(T\left(\frac{1}{0}\right)\right).$$

iii) Since T with $T(z) = \dfrac{-uz+1}{(u^2+3u+1)z+(u+3)}$ is strictly increasing, we have the results.

Theorem 2.2 Let $a, b \epsilon \mathbb{N}$ and $\dfrac{1}{n} \leq \dfrac{a}{nb} < \dfrac{(2u+3)-\sqrt{5}}{2n}$. Then,

i) $\dfrac{a}{nb} < T\left(\dfrac{a}{nb}\right) < \dfrac{(2u+3)-\sqrt{5}}{2n}$,

ii) $\dfrac{a}{nb} \longrightarrow T\left(\dfrac{a}{nb}\right)$ in $F_{u,n}$ if and only if $a = \dfrac{(2u+3)b-\sqrt{5b^2+4}}{2}$ and there exists some $t \epsilon \mathbb{N}$ such that $5b^2+4 = t^2$.

Proof. i) Given $\dfrac{a}{nb} < \dfrac{(2u+3)-\sqrt{5}}{2n}$ we have $(2u+3)b - 2a > \sqrt{5}b \Rightarrow \left((2u+3)b-2a\right)^2 > 5b^2$.

Then, $a^2 - (2u+3)ab + (u^2+3u+1)b^2 > 0$ and let us take $n = u^2+3u+1$, so

$$na^2 - (2u+3)abn + n^2b^2 > 0$$

Then, we obtain that

$$\frac{a}{nb} < T\left(\frac{a}{nb}\right) = \frac{-au+nb}{(-a+(u+3)b)n} \tag{3}$$

Furthermore, since

$$\frac{a}{nb} < \frac{(2u+3)-\sqrt{5}}{2n}$$

and T is increasing on

$$\left[\frac{1}{n}, \frac{(2u+3)-\sqrt{5}}{2n}\right) \cap \mathbb{Q}$$

and by (1), then we get that

$$T\left(\frac{a}{nb}\right) < \frac{(2u+3)-\sqrt{5}}{2n} \tag{4}$$

From (3) and (4), we see that

$$\frac{a}{nb} < T\left(\frac{a}{nb}\right) < \frac{(2u+3)-\sqrt{5}}{2n}.$$

ii) Let $\frac{a}{nb} \longrightarrow T\left(\frac{a}{nb}\right)$ be an edge in $F_{u,n}$. Then,

it follows that

$$a^2-(2u+3)ab+nb^2>0 \quad \text{and} \quad a^2-(2u+3)ab+nb^2=1.$$

Since

$$4a^2-4(2u+3)ab+4nb^2=4, \quad 4a^2-4(2u+3)ab+4nb^2+5b^2=4+5b^2,$$

so

$$|(2u+3)b-2a| = \sqrt{5b^2+4} \text{ holds.}$$

Since $\frac{a}{nb} < \frac{(2u+3)-\sqrt{5}}{2n}$ this shows that $(2u+3)b-2>0$ and

$$a = \frac{(2u+3)b-\sqrt{5b^2+4}}{2}$$

which means for $a, b \epsilon \mathbb{N}$ there exists some $t \epsilon \mathbb{N}$ such that $5b^2+4=t^2$.

Conversely, if $a = \frac{(2u+3)b-\sqrt{5b^2+4}}{2}$, $t \epsilon \mathbb{N}$ and $5b^2+4=t^2$, then

$$\frac{a}{nb} = \frac{\frac{(2u+3)b-\sqrt{5b^2+4}}{2}}{nb}$$

and

$$\frac{-\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)u+nb}{\left(-\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)+(u+3)b\right)n} = T\left(\frac{a}{nb}\right).$$

This implies that

$$\left(-\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)+(u+3)b\right)\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)$$
$$-b\left(-\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)u+nb\right)=1$$

and

$$\left(-\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)u+nb\right) \equiv -\left(\left(-\left(\frac{(2u+3)b-\sqrt{5b^2+4}}{2}\right)u+nb\right)\right)u \ (mod\ n).$$

According to Theorem 1.1, we have that $\frac{a}{nb} \longrightarrow T\left(\frac{a}{nb}\right)$ in $F_{u,n}$.

Here are two major corollaries without proof.

Corollary 2.3

i) $\frac{1}{0} \longrightarrow \frac{u}{n}+\frac{0}{1.n} \longrightarrow \frac{u}{n}+\frac{1}{3.n} \longrightarrow \frac{u}{n}+\frac{3}{8.n} \longrightarrow \cdots \longrightarrow \frac{u}{n}+\frac{a_k}{b_k.n} \longrightarrow \frac{u}{n}+\frac{b_k}{(3b_k-a_k).n} \longrightarrow \cdots$

is an infinite path in $F_{u,n}$,

ii) Every vertex in (i) is less than $\dfrac{(2u+3)-\sqrt{5}}{2n}$,

iii) The numbers $5a_k^2+4,\ 5b_k^2+4$ are perfect squares for the natural numbers $a_k, b_k \in \mathbb{N}$ in (i) .

Proof. Theorem 2.1 and Theorem 2.2 conclude the proof.

Corollary 2.4 The numbers $k \in \mathbb{Z}^+$ making $5k^2+4$ perfect squares are

$$0, 1, 3, 8, \dots, x, y, 3y-x \tag{5}$$

Proof. Corollary 2.3 concludes the proof.

Now, we will investigate the numbers obtained from the inverse matrix of T .

The inverse matrix of T is $\begin{pmatrix} u+3 & -1 \\ (u^2+3u+1) & -u \end{pmatrix}$, and for $u=1$ we have $S := \begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix}$.

So we get the following.

Theorem 2.5

i) For all $k \in \mathbb{N}, S^k\left(\frac{3}{5}\right) \longrightarrow S^{k+1}\left(\frac{3}{5}\right)$,

ii) $\left\{S^k\left(\frac{3}{5}\right)\right\}_{k\epsilon\mathbb{N}}$ is an increasing sequence and the path $\frac{3}{5} \longrightarrow S\left(\frac{3}{5}\right) \longrightarrow S^2\left(\frac{3}{5}\right) \longrightarrow S^3\left(\frac{3}{5}\right) \longrightarrow \cdots$

is an infinite.

Proof. We conclude the proof as in Theorem 2.1.

Theorem 2.6 Let $S := \begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix} \epsilon \Gamma_0(5)$ and $a, b \epsilon \mathbb{N}$ such that $\frac{3}{5} \le \frac{a}{5b} < \frac{5+\sqrt{5}}{10}$. Then,

i) $\dfrac{a}{5b} < S\left(\dfrac{a}{5b}\right) < \dfrac{5+\sqrt{5}}{10}$,

ii) $\dfrac{a}{5b} \longrightarrow S\left(\dfrac{a}{5b}\right)$ is an edge in $F_{1,5}$ if and only if $a = \dfrac{5b+\sqrt{5b^2-4}}{2}$ and

there exists some $r \epsilon \mathbb{N}$ such that $5b^2-4=r^2$.

Proof. Given $\dfrac{a}{5b} < \dfrac{5+\sqrt{5}}{10}$, we have $2a-5b < \sqrt{5}b \Rightarrow (2a-5b)^2 < 5b^2$.

From this, we get $4a^2-20ab+20b^2<0,\ a^2-5ab+5b^2<0$ and $5a^2-25ab+25b^2<0$. Then,

$$\frac{a}{5b} < \frac{4a-5b}{5a-5b} = S\left(\frac{a}{5b}\right) \tag{6}$$

Since S is increasing on $\left[\frac{3}{5}, \frac{5+\sqrt{5}}{10}\right) \cap \mathbb{Q}$, and $S\left(\frac{5+\sqrt{5}}{10}\right) = \frac{5+\sqrt{5}}{10}$ from (1), we have

$$S\left(\frac{a}{5b}\right) < \frac{5+\sqrt{5}}{10} \tag{7}$$

From (6) and (7), we get $\dfrac{a}{5b} < S\left(\dfrac{a}{5b}\right) < \dfrac{5+\sqrt{5}}{10}$.

ii) Given $\dfrac{a}{5b} \longrightarrow S\left(\dfrac{a}{5b}\right)$ we have that $a^2-5ab+5b^2<0$ and by Theorem 1.1, we have

$$a^2-5ab+5b^2=-1 .$$

Then, $-4a^2+20ab-20b^2=4,\ \ -4a^2+20ab-20b^2-5b^2=4-5b^2$ and $(2a-5b)^2=-4+5b^2$

so $|2a-5b| = \sqrt{5b^2-4}$. Since $\dfrac{3}{5} \le \dfrac{a}{5b} < \dfrac{5+\sqrt{5}}{10}$ we have $2a-5b>0$ and

$2a-5b=\sqrt{5b^2-4}$, and

$$a=\frac{5b+\sqrt{5b^2-4}}{2}$$

Furthermore, since $\sqrt{5b^2-4}\ \epsilon\ \mathbb{N}$, there exists some $r\epsilon\mathbb{N}$ such that $5b^2-4=r^2$.

Conversely, let $a=\dfrac{5b+\sqrt{5b^2-4}}{2}$ and $5b^2-4=r^2$ for some $r\epsilon\mathbb{N}$.

Then,

$$\frac{a}{5b}=\frac{\frac{5b+\sqrt{5b^2-4}}{2}}{5b}\quad,\qquad S\left(\frac{a}{5b}\right)=\frac{\frac{10b+4\sqrt{5b^2-4}}{2}}{5\left(\frac{3b+\sqrt{5b^2-4}}{2}\right)}\qquad \text{and}$$

$$\frac{5b+\sqrt{5b^2-4}}{2}\cdot\frac{3b+\sqrt{5b^2-4}}{2}-b\cdot\frac{10b+4\sqrt{5b^2-4}}{2}$$
$$=\frac{15b^2+5b\sqrt{5b^2-4}+3b\sqrt{5b^2-4}+5b^2-4-20b^2-8b\sqrt{5b^2-4}}{4}$$
$$=-1$$

Since

$$\frac{10b+4\sqrt{5b^2-4}}{2}+\frac{5b+\sqrt{5b^2-4}}{2}=5\left(\frac{3b+\sqrt{5b^2-4}}{2}\right)\ \equiv 0\ (mod5\,),$$

we get that

$$\frac{10b+4\sqrt{5b^2-4}}{2}\equiv-\left(\frac{5b+\sqrt{5b^2-4}}{2}\right)(mod5)\,,$$

and

$$\frac{a}{5b}\longrightarrow S\left(\frac{a}{5b}\right)\text{ is an edge in } F_{1,5}\,.$$

We get the following corollaries.

Corollary 2.7

i) $$\frac{4}{5}-\frac{1}{5.1}\longrightarrow\frac{4}{5}-\frac{1}{5.2}\longrightarrow\frac{4}{5}-\frac{2}{5.5}\longrightarrow\cdots\longrightarrow\frac{4}{5}-\frac{a_k}{5b_k}\longrightarrow\frac{4}{5}-\frac{b_k}{5.(3b_k-a_k).}\longrightarrow\cdots$$

is an infinite path in $F_{1,5}$,

ii) Every vertex in (i) is less than $\dfrac{5-\sqrt{5}}{10}$,

iii) The numbers $5a_k^2-4,\ 5b_k^2-4$ are perfect squares for the natural numbers $a_k,b_k\in\mathbb{N}$ in (i) .

Proof. Theorem 2.5 and Theorem 2.6 conclude the proof.

Corollary 2.8 The numbers $k\in\mathbb{Z}^+$ making $5k^2-4$ perfect squares are

$$1,2,5,13,\dots,x,y,3y-x \tag{8}$$

Proof. Corollary 2.7 concludes the proof.

In view of Corollary 2.4 and Corollary 2.8, we have the main result as follows:

Corollary 2.9 Let the sequences $\{k_n\}_{n\in\mathbb{N}}$, $\{m_n\}_{n\in\mathbb{N}}$ be satisfying the conditions (5) and (8), respectively. Then, the sequence $\{F_n\}_{n\in\mathbb{N}}$ defined by

$$F_n = \begin{cases} k_{\frac{n+1}{2}}, & if\ n\ is\ an\ odd \\ m_{\frac{n}{2}}, & if\ n\ is\ an\ even \end{cases}$$

for all $n \in \mathbb{N}$, that is $(k_1, m_1, \ldots, k_n, m_n, k_{n+1}, m_{n+1}, \ldots)$, is the Fibonacci sequence.

III. CONCLUSIONS

In this work, we have found a connection between the suborbital graph of the Modular group and the Fibonacci sequence. Fibonacci sequence and suborbital graphs are widely studied by several authors [6-13]. These studies using suborbital graphs are not so old. So we believe that our contributions open avenues to some more advanced studies. It might be that the Fibonacci sequence may be generalized by the theory of graphs, namely suborbital graphs by taking some groups other than the Modular group, using the method in the study or different methods.

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