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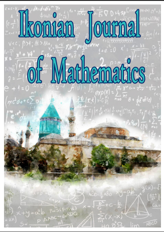

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Solvability of two-dimensional system of difference equations with constant coefficients

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Difference equations systems, Fibonacci number, Solution, Periodicity, Explicit Solutions

Abstract — In the present paper, the solutions of the following system of difference equations

$$u_n = \alpha_1 v_{n-2} + \frac{\delta_1 v_{n-2} u_{n-4}}{\beta_1 u_{n-4} + \gamma_1 v_{n-6}},\ v_n = \alpha_2 u_{n-2} + \frac{\delta_2 u_{n-2} v_{n-4}}{\beta_2 v_{n-4} + \gamma_2 u_{n-6}},\ n \in \mathbb{N}_0,$$

where the initial values u_{-l}, v_{-l}, for $l = \overline{1,6}$ and the parameters α_p, β_p, γ_p, δ_p, for $p \in \{1,2\}$ are non-zero real numbers, are investigated. In addition, the solutions of the aforementioned system of difference equations are presented by utilizing the Fibonacci sequence when the parameters are equal to 1. Finally, the periodic solutions according to some special cases of the parameters are obtained.

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1. Introduction and Preliminaries

Difference equations are one of the important topics of applied mathematics. Therefore, some mathematicians have studied in this field [1–20]. Some difference equations occur as the recurrence relation of a number sequence. For example, Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is identified by

$$F_{n+1} = F_n + F_{n-1},\ n \in \mathbb{N}, \tag{1.1}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$ in [21]. Binet's formula for equation (1.1) is

$$F_n = \frac{A^n - B^n}{A - B},\ n \in \mathbb{N}_0, \tag{1.2}$$

where $A = \frac{1+\sqrt{5}}{2}$, $B = \frac{1-\sqrt{5}}{2}$. Equation (1.2) is a solution of equation (1.1) and the general term Fibonacci sequence. In addition, there are some types of nonlinear difference equations for which their general solutions can be found. One of them is Riccati difference equation, which is in the following form:

$$z_{n+1} = \frac{\epsilon z_n + \theta}{\zeta z_n + \eta}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

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for $\zeta \neq 0$, $\epsilon\eta - \zeta\theta \neq 0$, where the parameters $\epsilon, \theta, \zeta, \eta$ and the initial condition z_0 are real numbers. The general solution of equation (1.3) can be written as follows

$$z_n = \frac{z_0 \left(\theta\zeta - \epsilon\eta\right) s_{n-1} + (\epsilon z_0 + \theta)\, s_n}{(\zeta z_0 - \epsilon)\, s_n + s_{n+1}}, \quad n \in \mathbb{N}, \tag{1.4}$$

where the sequence $(s_n)_{n\in\mathbb{N}_0}$ is satisfying

$$s_{n+1} - \left(\epsilon + \eta\right) s_n - \left(\theta\zeta - \epsilon\eta\right) s_{n-1} = 0, \quad n \in \mathbb{N},$$

where $s_0 = 0$, $s_1 = 1$, in [22].

The following higher-order difference equation,

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}},\ n \in \mathbb{N}_0, \tag{1.5}$$

where k and l are fixed natural numbers, the initial conditions x_{-j}, $j = \overline{1, k+l}$ and the parameters α, β, γ, δ are real numbers, was solved by the authors in [23]. In addition, the case $k = 2$, $l = 4$ in equation (1.5), it was obtained the exact solutions and investigated equilibria, local stability and global attractivity in [24]. Similarly, the authors of [25] studied the behavior of the solutions of the difference equation which was obtained by taking $k = 1$, $l = 3$ in equation (1.5).

There are some difference equations that are similar in shape to the difference equation in (1.5). But, they are not particular cases of equation (1.5). For example, in [26], the authors explored the qualitative behavior of the solutions of the following difference equations:

$$y_{n+1} = A y_{n-1} + \frac{\pm B y_{n-1} y_{n-3}}{C y_{n-3} \pm D y_{n-5}},\ n \in \mathbb{N}_0, \tag{1.6}$$

where the initial conditions y_{-k}, for $k = \overline{0,5}$, are arbitrary positive real numbers and the parameters A, B, C and D are positive real numbers.

Similarly, the authors studied the behaviour of the rational difference equation

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}},\ n \in \mathbb{N}_0, \tag{1.7}$$

where the initial conditions y_{-k}, for $k = \overline{0,4}$, are positive real numbers and the parameters α, β, A and B are real numbers, in [27].

In addition, in [28], Almatrafi and Alzubaidi studied the local and global stability, periodicity and solutions of the following rational difference equations

$$u_{n+1} = a u_{n-1} \pm \frac{b u_{n-1} u_{n-4}}{c u_{n-4} - d u_{n-6}}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where the parameters a,b,c and d are positive real numbers and the initial values u_{-k}, for $k = \overline{0,6}$, are non-zero real numbers.

Moreover, the authors of [29] studied the behavior of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_{n-1} + dx_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.9}$$

where the initial conditions x_{-k}, for $k = \overline{0,2}$ are arbitrary positive real numbers and the parameters a,b,c and d are positive constants. In [30], Elsayed and Al-Rakhami investigated some of the qualitative behavior of the rational difference equation

$$\Psi_{n+1} = \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3} + \delta\Psi_{n-6}}, \quad n \in \mathbb{N}_0, \tag{1.10}$$

where the parameters α, β, γ and δ are arbitrary positive real numbers.

Further, in [31] Elsayed studied the qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.11}$$

where a,b,c and d, are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers. There are some difference equations as equations in (1.6)-(1.11) in literature (see [32–35]).
In [36], the authors generalized the equation (1.5) to the following two-dimensional system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, n \in \mathbb{N}_0, \tag{1.12}$$

where k and l are positive integers, the initial conditions x_{-i}, y_{-i}, $i = \overline{1,k+l}$ and the parameters a, b, c, d, α, β, γ, δ are real numbers. They showed that system (1.12) can be solved in closed form.

A natural question is if equation (1.6) generalizes to a two-dimensional system of difference equations. Here, we give a positive answer. We expand equation (1.6) to the following two-dimensional system of difference equations

$$u_n = \alpha_1 v_{n-2} + \frac{\delta_1 v_{n-2}u_{n-4}}{\beta_1 u_{n-4} + \gamma_1 v_{n-6}}, v_n = \alpha_2 u_{n-2} + \frac{\delta_2 u_{n-2}v_{n-4}}{\beta_2 v_{n-4} + \gamma_2 u_{n-6}}, n \in \mathbb{N}_0, \tag{1.13}$$

where the initial values u_{-l}, v_{-l}, for $l = \overline{1,6}$, are positive real numbers and the parameters α_p, β_p, γ_p and δ_p, for $p \in \{1,2\}$, are positive real numbers.

Our aim to show that system (1.13) is solvable in explicit form. Also, we investigate the periodicity of the solutions depending on special cases of the parameters. Additionally, we gain the solutions for the case $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 1$ by using Fibonacci sequence.

We give the following very well-known definition which used in this paper.

Definition 1.1. [37] (Periodicity) A sequence $(x_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there exists $n_0 \geq -k$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$ then the sequence $(x_n)_{n=-k}^{\infty}$ is said to be periodic with period p.

2. Explicit Solutions of System (1.13)

The system (1.13) can be written in the following form

$$\frac{u_n}{v_{n-2}} = \frac{\left(\alpha_1\beta_1+\delta_1\right)\frac{u_{n-4}}{v_{n-6}}+\alpha_1\gamma_1}{\beta_1\frac{u_{n-4}}{v_{n-6}}+\gamma_1}, \quad \frac{v_n}{u_{n-2}} = \frac{\left(\alpha_2\beta_2+\delta_2\right)\frac{v_{n-4}}{u_{n-6}}+\alpha_2\gamma_2}{\beta_2\frac{v_{n-4}}{u_{n-6}}+\gamma_2}, \qquad n\in\mathbb{N}_0.$$

By employing the change of variables

$$x_n = \frac{u_n}{v_{n-2}}, \quad y_n = \frac{v_n}{u_{n-2}}, \qquad n\geq -4, \tag{2.1}$$

system (1.13) is transformed into the following system

$$x_n = \frac{\left(\alpha_1\beta_1+\delta_1\right)x_{n-4}+\alpha_1\gamma_1}{\beta_1 x_{n-4}+\gamma_1}, \quad y_n = \frac{\left(\alpha_2\beta_2+\delta_2\right)y_{n-4}+\alpha_2\gamma_2}{\beta_2 y_{n-4}+\gamma_2},\ n\in\mathbb{N}_0. \tag{2.2}$$

We consider the following equation

$$z_n = \frac{\left(\alpha\beta+\delta\right)z_{n-4}+\alpha\gamma}{\beta z_{n-4}+\gamma}, \qquad n\in\mathbb{N}_0, \tag{2.3}$$

instead of equations in (2.2). If we apply decomposition of indices $n\to 4\left(m+1\right)+i$, $i=\overline{-4,-1},\ m\geq -1$, in equation (2.3), then it can be written the following equation

$$z^{(i)}_{m+1} = \frac{\left(\alpha\beta+\delta\right)z^{(i)}_m+\alpha\gamma}{\beta z^{(i)}_m+\gamma}, \tag{2.4}$$

where $z^{(i)}_m = z_{4m+i}, \quad i=\overline{-4,-1}, \quad m\in\mathbb{N}_0,$
From equation (1.4), the general solutions of the equations in (2.4) as follows

$$z^{(i)}_m = \frac{-\delta\gamma z^{(i)}_0 s_{m-1}+\left(\left(\alpha\beta+\delta\right)z^{(i)}_0+\alpha\gamma\right)s_m}{\left(\beta z^{(i)}_0-\alpha\beta-\delta\right)s_m+s_{m+1}}, \qquad m\in\mathbb{N}, \tag{2.5}$$

for $i=\overline{-4,-1}$, where sequence of $(s_m)_{m\in\mathbb{N}_0}$ is satisfying

$$s_{m+1}-\left(\alpha\beta+\delta+\gamma\right)s_m+\delta\gamma s_{m-1}=0, \qquad m\in\mathbb{N}. \tag{2.6}$$

From equation (2.5), the solutions of equations in (2.2) are expressed as

$$x_{4m+i} = \frac{-\delta_1\gamma_1 x_i s_{m-1}+\left(\left(\alpha_1\beta_1+\delta_1\right)x_i+\alpha_1\gamma_1\right)s_m}{\left(\beta_1 x_i-\alpha_1\beta_1-\delta_1\right)s_m+s_{m+1}}, \qquad m\in\mathbb{N}_0, \tag{2.7}$$

$$y_{4m+i} = \frac{-\delta_2\gamma_2 y_i s_{m-1}+\left(\left(\alpha_2\beta_2+\delta_2\right)y_i+\alpha_2\gamma_2\right)s_m}{\left(\beta_2 y_i-\alpha_2\beta_2-\delta_2\right)s_m+s_{m+1}}, \qquad m\in\mathbb{N}_0, \tag{2.8}$$

for $i=\overline{-4.-1}$.
From (2.1) , we have

$$u_n = x_n v_{n-2} = x_n y_{n-2} u_{n-4},\ v_n = y_n u_{n-2} = y_n x_{n-2} v_{n-4}, \quad n\geq -2. \tag{2.9}$$

From system (2.9), we obtain

$$u_{4m+j} = x_{4m+j} y_{4m+j-2} u_{4(m-1)+j}, \quad m \in \mathbb{N}_0,$$
$$v_{4m+j} = y_{4m+j} x_{4m+j-2} v_{4(m-1)+j}, \quad m \in \mathbb{N}_0, \tag{2.10}$$

for $j = \overline{-2,1}$.

From system (2.10), we get

$$u_{4m+j} = u_{j-4} \prod_{p=0}^{m} x_{4p+j} y_{4p+j-2}, \quad m \in \mathbb{N}_0,$$
$$v_{4m+j} = v_{j-4} \prod_{p=0}^{m} y_{4p+j} x_{4p+j-2}, \quad m \in \mathbb{N}_0, \tag{2.11}$$

for $j = \overline{-2,1}$.

By putting formulas (2.7) and (2.8) back into system (2.11), we gain

$$u_{4m-2} = u_{-6} \prod_{p=0}^{m} \left(\frac{-\delta_1\gamma_1 u_{-2} s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right) u_{-2} + \alpha_1\gamma_1 v_{-4}\right) s_p}{\left(\beta_1 u_{-2} - \left(\alpha_1\beta_1+\delta_1\right) v_{-4}\right) s_p + v_{-4} s_{p+1}} \right.$$
$$\left. \times \left(\frac{-\delta_2\gamma_2 v_{-4} s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right) v_{-4} + \alpha_2\gamma_2 u_{-6}\right) s_p}{\left(\beta_2 v_{-4} - \left(\alpha_2\beta_2+\delta_2\right) u_{-6}\right) s_p + u_{-6} s_{p+1}} \right), \tag{2.12}$$

$$v_{4m-2} = v_{-6} \prod_{p=0}^{m} \left(\frac{-\delta_2\gamma_2 v_{-2} s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right) v_{-2} + \alpha_2\gamma_2 u_{-4}\right) s_p}{\left(\beta_2 v_{-2} - \left(\alpha_2\beta_2+\delta_2\right) u_{-4}\right) s_p + u_{-4} s_{p+1}} \right.$$
$$\left. \times \left(\frac{-\delta_1\gamma_1 u_{-4} s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right) u_{-4} + \alpha_1\gamma_1 v_{-6}\right) s_p}{\left(\beta_1 u_{-4} - \left(\alpha_1\beta_1+\delta_1\right) v_{-6}\right) s_p + v_{-6} s_{p+1}} \right), \tag{2.13}$$

$$u_{4m-1} = u_{-5} \prod_{p=0}^{m} \left(\frac{-\delta_1\gamma_1 u_{-1} s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right) u_{-1} + \alpha_1\gamma_1 v_{-3}\right) s_p}{\left(\beta_1 u_{-1} - \left(\alpha_1\beta_1+\delta_1\right) v_{-3}\right) s_p + v_{-3} s_{p+1}} \right.$$
$$\left. \times \left(\frac{-\delta_2\gamma_2 v_{-3} s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right) v_{-3} + \alpha_2\gamma_2 u_{-5}\right) s_p}{\left(\beta_2 v_{-3} - \left(\alpha_2\beta_2+\delta_2\right) u_{-5}\right) s_p + u_{-5} s_{p+1}} \right), \tag{2.14}$$

$$v_{4m-1} = v_{-5} \prod_{p=0}^{m} \left(\frac{-\delta_2\gamma_2 v_{-1} s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right) v_{-1} + \alpha_2\gamma_2 u_{-3}\right) s_p}{\left(\beta_2 v_{-1} - \left(\alpha_2\beta_2+\delta_2\right) u_{-3}\right) s_p + u_{-3} s_{p+1}} \right.$$
$$\left. \times \left(\frac{-\delta_1\gamma_1 u_{-3} s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right) u_{-3} + \alpha_1\gamma_1 v_{-5}\right) s_p}{\left(\beta_1 u_{-3} - \left(\alpha_1\beta_1+\delta_1\right) v_{-5}\right) s_p + v_{-5} s_{p+1}} \right), \tag{2.15}$$

$$u_{4m}=u_{-4}\prod_{p=0}^{m}\left(\frac{-\delta_1\gamma_1u_{-4}s_p+\left(\left(\alpha_1\beta_1+\delta_1\right)u_{-4}+\alpha_1\gamma_1v_{-6}\right)s_{p+1}}{\left(\beta_1u_{-4}-\left(\alpha_1\beta_1+\delta_1\right)v_{-6}\right)s_{p+1}+v_{-6}s_{p+2}}\right)\times\left(\frac{-\delta_2\gamma_2v_{-2}s_{p-1}+\left(\left(\alpha_2\beta_2+\delta_2\right)v_{-2}+\alpha_2\gamma_2u_{-4}\right)s_p}{\left(\beta_2v_{-2}-\left(\alpha_2\beta_2+\delta_2\right)u_{-4}\right)s_p+u_{-4}s_{p+1}}\right), \tag{2.16}$$

$$v_{4m}=v_{-4}\prod_{p=0}^{m}\left(\frac{-\delta_2\gamma_2v_{-4}s_p+\left(\left(\alpha_2\beta_2+\delta_2\right)v_{-4}+\alpha_2\gamma_2u_{-6}\right)s_{p+1}}{\left(\beta_2v_{-4}-\left(\alpha_2\beta_2+\delta_2\right)u_{-6}\right)s_{p+1}+u_{-6}s_{p+2}}\right)\times\left(\frac{-\delta_1\gamma_1u_{-2}s_{p-1}+\left(\left(\alpha_1\beta_1+\delta_1\right)u_{-2}+\alpha_1\gamma_1v_{-4}\right)s_p}{\left(\beta_1u_{-2}-\left(\alpha_1\beta_1+\delta_1\right)v_{-4}\right)s_p+v_{-4}s_{p+1}}\right), \tag{2.17}$$

$$u_{4m+1}=u_{-3}\prod_{p=0}^{m}\left(\frac{-\delta_1\gamma_1u_{-3}s_p+\left(\left(\alpha_1\beta_1+\delta_1\right)u_{-3}+\alpha_1\gamma_1v_{-5}\right)s_{p+1}}{\left(\beta_1u_{-3}-\left(\alpha_1\beta_1+\delta_1\right)v_{-5}\right)s_{p+1}+v_{-5}s_{p+2}}\right)\times\left(\frac{-\delta_2\gamma_2v_{-1}s_{p-1}+\left(\left(\alpha_2\beta_2+\delta_2\right)v_{-1}+\alpha_2\gamma_2u_{-3}\right)s_p}{\left(\beta_2v_{-1}-\left(\alpha_2\beta_2+\delta_2\right)u_{-3}\right)s_p+u_{-3}s_{p+1}}\right), \tag{2.18}$$

$$v_{4m+1}=v_{-3}\prod_{p=0}^{m}\left(\frac{-\delta_2\gamma_2v_{-3}s_p+\left(\left(\alpha_2\beta_2+\delta_2\right)v_{-3}+\alpha_2\gamma_2u_{-5}\right)s_{p+1}}{\left(\beta_2v_{-3}-\left(\alpha_2\beta_2+\delta_2\right)u_{-5}\right)s_{p+1}+u_{-5}s_{p+2}}\right)\times\left(\frac{-\delta_1\gamma_1u_{-1}s_{p-1}+\left(\left(\alpha_1\beta_1+\delta_1\right)u_{-1}+\alpha_1\gamma_1v_{-3}\right)s_p}{\left(\beta_1u_{-1}-\left(\alpha_1\beta_1+\delta_1\right)v_{-3}\right)s_p+v_{-3}s_{p+1}}\right), \tag{2.19}$$

for $m\in\mathbb{N}_0$.

3. Periodicity

We obtain the periodicity of the solutions of the system (1.13) depending on the parameters are equal either 1 or -1 in this section.

Theorem 3.1. Suppose that α_p, β_p, γ_p, δ_p, for $p\in\{1,2\}$ and the initial values u_{-l}, v_{-l}, for $l=\overline{1,6}$ are non-zero real numbers. Then, the following statements hold.

a) If $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, $\beta_2=1$, $\gamma_1=-1$, $\gamma_2=-1$, $\delta_1=-1$, $\delta_1=-1$, the solutions of the system (1.13) are periodic with period 12.

b) If $\alpha_1=1$, $\alpha_2=1$, $\beta_1=-1$, $\beta_2=-1$, $\gamma_1=1$, $\gamma_2=1$, $\delta_1=1$, $\delta_1=1$, the solutions of the system (1.13) are periodic with period 12.

c) If $\alpha_1=-1$, $\alpha_2=-1$, $\beta_1=1$, $\beta_2=1$, $\gamma_1=1$, $\gamma_2=1$, $\delta_1=1$, $\delta_1=1$, the solutions of the system (1.13) are periodic with period 12.

d) If $\alpha_1=-1$, $\alpha_2=-1$, $\beta_1=-1$, $\beta_2=-1$, $\gamma_1=-1$, $\gamma_2=-1$, $\delta_1=-1$, $\delta_1=-1$, the solutions of the system (1.13) are periodic with period 12.

Proof.

a) If $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, $\beta_2=1$, $\gamma_1=-1$, $\gamma_2=-1$, $\delta_1=-1$, $\delta_1=-1$, system (1.13) turns into the following system

$$u_n = v_{n-2} - \frac{v_{n-2}u_{n-4}}{u_{n-4} - v_{n-6}},\ v_n = u_{n-2} - \frac{u_{n-2}v_{n-4}}{v_{n-4} - u_{n-6}},\ n \in \mathbb{N}_0. \tag{3.1}$$

From (2.7) and (2.8), we have

$$x_{4m+i} = \frac{-x_i s_{m-1} - s_m}{x_i s_m + s_{m+1}}, \tag{3.2}$$

$$y_{4m+i} = \frac{-y_i s_{m-1} - s_m}{y_i s_m + s_{m+1}}, \tag{3.3}$$

where $m \in \mathbb{N}_0$ and $i = \overline{-4,-1}$.

From (2.6), we obtain

$$s_{m+1} + s_m + s_{m-1} = 0,$$

where $s_0 = 0$ and $s_1 = 1$.

From this, we get

$$s_{3t+b} = b, \tag{3.4}$$

for $t \in \mathbb{N}_0$ and $b = \overline{-1,1}$.

From (2.1), we have

$$\begin{aligned} u_{12m+j} =& x_{12m+j}y_{12m+j-2}x_{12m+j-4}y_{12m+j-6} \\ & \times x_{12m+j-8}y_{12m+j-10}u_{12(m-1)+j}, \\ v_{12m+j} =& y_{12m+j}x_{12m+j-2}y_{12m+j-4}x_{12m+j-6} \\ & \times y_{12m+j-8}x_{12m+j-10}v_{12(m-1)+j}, \end{aligned} \tag{3.5}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

From system (3.5), we obtain

$$\begin{aligned} u_{12m+j} = u_{j-12} \prod_{p=0}^{m} & x_{12p+j}y_{12p+j-2}x_{12p+j-4}y_{12p+j-6} \\ & \times x_{12p+j-8}y_{12p+j-10}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} v_{12m+j} = v_{j-12} \prod_{p=0}^{m} & y_{12p+j}x_{12p+j-2}y_{12p+j-4}x_{12p+j-6} \\ & \times y_{12p+j-8}x_{12p+j-10}, \end{aligned} \tag{3.7}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

By using (3.2), (3.3) and (3.4) into (3.6) and (3.7), we get

$$u_{12m+j} = u_{j-12}, \quad v_{12m+j} = v_{j-12},$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

b) If $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = -1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\delta_1 = 1$, $\delta_1 = 1$, system (1.13) turns into the system (3.1). Then, it can be proven like (a).

c) If $\alpha_1 = -1$, $\alpha_2 = -1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\delta_1 = 1$, $\delta_1 = 1$, system (1.13) turns into the following system

$$u_n = -v_{n-2} + \frac{v_{n-2}u_{n-4}}{u_{n-4}+v_{n-6}},\ v_n = -u_{n-2} + \frac{u_{n-2}v_{n-4}}{v_{n-4}+u_{n-6}},\ n \in \mathbb{N}_0. \tag{3.8}$$

From (2.7) and (2.8), we obtain

$$x_{4m+i} = \frac{-x_i s_{m-1} - s_m}{x_i s_m + s_{m+1}}, \tag{3.9}$$

$$y_{4m+i} = \frac{-y_i s_{m-1} - s_m}{y_i s_m + s_{m+1}}, \tag{3.10}$$

where $m \in \mathbb{N}_0$ and $i = \overline{-4,-1}$.

We obtain, from (2.6),

$$s_{m+1} - s_m + s_{m-1} = 0,$$

where $s_0 = 0$ and $s_1 = 1$.

From this, we get

$$s_{6t+3r+q} = \begin{cases} 0, & \text{if } 3r+q \in \{0,3\}, \\ 1, & \text{if } 3r+q \in \{1,2\}, \\ -1, & \text{if } 3r+q \in \{4,5\}, \end{cases} \tag{3.11}$$

for $t \in \mathbb{N}_0$, $r \in \{0,1\}$ and $q = \overline{0,2}$.

From (2.1), we have

$$\begin{aligned} u_{12m+j} =& x_{12m+j}y_{12m+j-2}x_{12m+j-4}y_{12m+j-6} \\ &\times x_{12m+j-8}y_{12m+j-10}u_{12(m-1)+j}, \\ v_{12m+j} =& y_{12m+j}x_{12m+j-2}y_{12m+j-4}x_{12m+j-6} \\ &\times y_{12m+j-8}x_{12m+j-10}v_{12(m-1)+j}, \end{aligned} \tag{3.12}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

From system (3.12), we obtain

$$\begin{aligned} u_{12m+j} =& u_{j-12}\prod_{p=0}^{m} x_{12p+j}y_{12p+j-2}x_{12p+j-4}y_{12p+j-6} \\ &\times x_{12p+j-8}y_{12p+j-10}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} v_{12m+j} =& v_{j-12}\prod_{p=0}^{m} y_{12p+j}x_{12p+j-2}y_{12p+j-4}x_{12p+j-6} \\ &\times y_{12p+j-8}x_{12p+j-10}, \end{aligned} \tag{3.14}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

By using (3.9)-(3.11) into (3.13) and (3.14), we get

$$u_{12m+j} = u_{j-12}, \quad v_{12m+j} = v_{j-12},$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

d) If $\alpha_1 = -1$, $\alpha_2 = -1$, $\beta_1 = -1$, $\beta_2 = -1$, $\gamma_1 = -1$, $\gamma_2 = -1$, $\delta_1 = -1$, $\delta_1 = -1$, system (1.13) turns into the system (3.8). Then, it can be proven like (c).

4. An Application

We obtain the solutions of the system (1.13) with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 1$. In this case, we have the following system

$$u_n = v_{n-2} + \frac{v_{n-2}u_{n-4}}{u_{n-4}+v_{n-6}}, \quad v_n = u_{n-2} + \frac{u_{n-2}v_{n-4}}{v_{n-4}+u_{n-6}}, \quad n \in \mathbb{N}_0. \tag{4.1}$$

From (2.6), we obtain

$$s_{m+1} - 3s_m + s_{m-1} = 0, \quad m \in \mathbb{N}, \tag{4.2}$$

where $s_0 = 0, \quad s_1 = 1$.
Binet Formula for (4.2) is

$$s_m = \frac{\left(\frac{3+\sqrt{5}}{2}\right)^m - \left(\frac{3-\sqrt{5}}{2}\right)^m}{\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right)}, \quad m \in \mathbb{N}_0. \tag{4.3}$$

Note that

$$\left(\frac{1 \mp \sqrt{5}}{2}\right)^2 = \frac{3 \mp \sqrt{5}}{2}. \tag{4.4}$$

Using (4.4) in (4.3), we have

$$s_m = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2m} - \left(\frac{1-\sqrt{5}}{2}\right)^{2m}}{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2} = F_{2m}, \quad m \in \mathbb{N}_0. \tag{4.5}$$

Using (4.5) into (2.12)-(2.19), we get

$$u_{4m-2} = u_{-6} \prod_{p=0}^{m} \frac{\left(u_{-2}F_{2p+1} + v_{-4}F_{2p}\right)\left(v_{-4}F_{2p+1} + u_{-6}F_{2p}\right)}{\left(v_{-4}F_{2p-1} + u_{-2}F_{2p}\right)\left(u_{-6}F_{2p-1} + v_{-4}F_{2p}\right)}, \tag{4.6}$$

$$v_{4m-2} = v_{-6} \prod_{p=0}^{m} \frac{\left(v_{-2}F_{2p+1} + u_{-4}F_{2p}\right)\left(u_{-4}F_{2p+1} + v_{-6}F_{2p}\right)}{\left(u_{-4}F_{2p-1} + v_{-2}F_{2p}\right)\left(v_{-6}F_{2p-1} + u_{-4}F_{2p}\right)}, \tag{4.7}$$

$$u_{4m-1} = u_{-5} \prod_{p=0}^{m} \frac{\left(u_{-1}F_{2p+1} + v_{-3}F_{2p}\right)\left(v_{-3}F_{2p+1} + u_{-5}F_{2p}\right)}{\left(v_{-3}F_{2p-1} + u_{-1}F_{2p}\right)\left(u_{-5}F_{2p-1} + v_{-3}F_{2p}\right)}, \tag{4.8}$$

$$v_{4m-1} = v_{-5} \prod_{p=0}^{m} \frac{\left(v_{-1}F_{2p+1} + u_{-3}F_{2p}\right)\left(u_{-3}F_{2p+1} + v_{-5}F_{2p}\right)}{\left(u_{-3}F_{2p-1} + v_{-1}F_{2p}\right)\left(v_{-5}F_{2p-1} + u_{-3}F_{2p}\right)}, \tag{4.9}$$

$$u_{4m} = u_{-4} \prod_{p=0}^{m} \frac{\left(u_{-4}F_{2p+3} + v_{-6}F_{2p+2}\right)\left(v_{-2}F_{2p+1} + u_{-4}F_{2p}\right)}{\left(v_{-6}F_{2p+1} + u_{-4}F_{2p+2}\right)\left(u_{-4}F_{2p-1} + v_{-2}F_{2p}\right)}, \tag{4.10}$$

$$v_{4m} = v_{-4} \prod_{p=0}^{m} \frac{\left(v_{-4}F_{2p+3} + u_{-6}F_{2p+2}\right)\left(u_{-2}F_{2p+1} + v_{-4}F_{2p}\right)}{\left(u_{-6}F_{2p+1} + v_{-4}F_{2p+2}\right)\left(v_{-4}F_{2p-1} + u_{-2}F_{2p}\right)}, \tag{4.11}$$

$$u_{4m+1} = u_{-3} \prod_{p=0}^{m} \frac{\left(u_{-3}F_{2p+3} + v_{-5}F_{2p+2}\right)\left(v_{-1}F_{2p+1} + u_{-3}F_{2p}\right)}{\left(v_{-5}F_{2p+1} + u_{-3}F_{2p+2}\right)\left(u_{-3}F_{2p-1} + v_{-1}F_{2p}\right)}, \tag{4.12}$$

$$v_{4m+1} = v_{-3} \prod_{p=0}^{m} \frac{\left(v_{-3}F_{2p+3} + u_{-5}F_{2p+2}\right)\left(u_{-1}F_{2p+1} + v_{-3}F_{2p}\right)}{\left(u_{-5}F_{2p+1} + v_{-3}F_{2p+2}\right)\left(v_{-3}F_{2p-1} + u_{-1}F_{2p}\right)}, \tag{4.13}$$

for $m \in \mathbb{N}_0$.

5. Conclusion

In this paper, we have obtained the solutions of two-dimensional system of difference equations in explicit form by using convenient transformation. In addition, we have investigated the periodic solutions of aforementioned system of difference equations when the parameters are equal to 1 or equal to −1. Finally, an application was given to show that the solutions of the mentioned system are related to Fibanacci numbers when all parameters are equal to 1.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] N. Akgunes, A. S. Kurbanli, *On the system of rational difference equations* $x_n = f\left(x_{n-a_1}, y_{n-b_1}\right)$, $y_n = g\left(y_{n-b_2}, z_{n-c_1}\right)$, $z_n = g\left(z_{n-c_2}, x_{n-a_2}\right)$, Selcuk Journal of Applied Mathematics, 15(1), (2014) 1–8.

[2] Y. Halim, A. Khelifa, M. Berkal, A. Bounchair, *On a solvable system of p difference equations of higher order*, Periodica Mathematica Hungarica, 85(1), (2022) 109–127.

[3] A. Ghezal, *Note on a rational system of* $4k+4-$*order difference equations: periodic solution and convergence*, Journal of Applied Mathematics & Computing, 69(2), (2023) 2207–2215.

[4] A. Ghezal, M. Balegh, I. Zemmouri, *Solutions and local stability of the jacobsthal system of difference equations*, AIMS Mathematics, 9(2), (2024) 3576–3591.

[5] M. Gümüş, R. Abo-Zeid, O. Ocalan, *Dynamical behavior of a third-order difference equation with arbitrary powers*, Kyungpook Mathematical Journal, 57(2), (2017) 251–263.

[6] M. Gümüş, R. Abo-Zeid, *Qualitative study of a third order rational system of difference equations*, Mathematica Moravica, 25(1), (2021) 81–97.

[7] M. Gümüş, *Global asymptotic behavior of a discrete system of difference equations with delays*, Filomat, 37(1), (2023) 251–264.

[8] M. Gümüş, Ş. I. Eğilmez, *The qualitative analysis of some difference equations using homogeneous functions*, Fundamental Journal of Mathematics and Applications, 6(4), 218–231.

[9] M. Kara, Y. Yazlik, *Solvable three-dimensional system of higher-order nonlinear difference equations*, Filomat, 36(10), (2022) 3449–3469.

[10] M. Kara, Y. Yazlik, *On a solvable system of rational difference equations of higher order*, Turkish Journal of Mathematics, 46(2), (2022) 587–611.

[11] M. Kara, Y. Yazlik, *On the solutions of three-dimensional system of difference equations via recursive relations of order two and applications*, Journal of Applied Analysis & Computation, 12(2), (2022) 736–753.

[12] M. Kara, *Investigation of the global dynamics of two exponential-form difference equations systems*, Electronic Research Archive, 31(11), (2023) 6697–6724.

[13] A. Khelifa, Y. Halim, *General solutions to systems of difference equations and some of their representations*, Journal of Applied Mathematics and Computing, 67, (2021) 439–453.

[14] N. Taskara, D. T. Tollu, Y. Yazlik, *Solutions of rational difference system of order three in terms of padovan numbers*, Journal of Advanced Research in Applied Mathematics, 7(3), (2015) 18–29.

[15] N. Taskara, D. T. Tollu, N. Touafek, Y. Yazlik, *A solvable system of difference equations*, Communications of the Korean Mathematical Society, 35(1), (2020) 301–319.

[16] D. T. Tollu, I. Yalcinkaya, *On solvability of a three-dimensional system of nonlinear difference equations*, Dynamics of Continuous, Discrete & Impulsive Systems Series B: Applications & Algorithms, 29(1), (2022) 35–47.

[17] I. Yalcinkaya, C. Cinar, D. Simsek, *Global asymptotic stability of a system of difference equations*, Applicable Analysis, 87, (2008) 677–687.

[18] I. Yalcinkaya, D. T. Tollu, *Global behavior of a second order system of difference equations*, Advanced Studies in Contemporary Mathematics, 26(4), (2016) 653–667.

[19] I. Yalcinkaya, H. Ahmad, D. T. Tollu, Y. M. Li, *On a system of k-difference equations of order three*, Mathematical Problems in Engineering, (2020) 1–11.

[20] Y. Yazlik, D. T. Tollu, N. Taskara, *On the solutions of a three-dimensional system of difference equations*, Kuwait Journal of Science, 43(1), (2016) 95–111.

[21] T. Koshy, Fibonacci and lucas numbers with applications, Wiley, New York, 2001.

[22] S. Stević, *Representation of solutions of bilinear difference equations in terms of generalized fibonacci sequences*, Electronic Journal of Qualitative Theory of Differential Equation, 2014(67), (2014) 1–15.

[23] D. T. Tollu, Y. Yazlik, N. Taskara, *On a solvable nonlinear difference equation of higher order*, Turkish Journal of Mathematics, 42, (2018) 1765–1778.

[24] M. B. Almatrafi, *Exact solution and stability of sixth order difference equations*, Electronic Journal of Mathematical Analysis and Applications, 10(1), (2022) 209–225.

[25] R. P. Agarwal, E. M. Elsayed, *On the solution of fourth-order rational recursive sequence*, Advanced Studies in Contemporary Mathematics, 20(4), (2010) 525–545.

[26] H. S. Alayachi, M. S. M. Noorani, A. Q. Khan, M. B. Almatrafi, *Analytic solutions and stability of sixth order difference equations*, Mathematical Problems in Engineering, Article ID 1230979, (2020) 1–12.

[27] M. M. El-Dessoky, E. M. Elabbasy, A. Asiri, *Dynamics and solutions of a fifth-order nonlinear difference*

equation, Discrete Dynamics in Nature and Society, Article ID 9129354, (2018) 1–21.

[28] M. B. Almatrafi, M. M. Alzubaidi, *Qualitative analysis for two fractional difference equations*, Nonlinear Engineering, 9(1), (2020) 265–272.

[29] E. M. Elabbasy, F. Alzahrani, I. Abbas, N. H. Alotaibi, *Dynamical behaviour and solution of nonlinear difference equation via fibonacci sequence*, Journal of Applied Analysis and Computation, 10(1), (2020) 282–296.

[30] E. M. Elabbasy, F. A. Al-Rakhami, *On dynamics and solutions expressions of higher-order rational difference equations*, Ikonion Journal of Mathematics, 5(1), (2023) 39–61.

[31] E. M. Elabbasy, *Qualitative behavior of difference equation of order two*, Mathematical and Computer Modellings, 50(2009), (2009) 1130–1141.

[32] H. S. Alayachi, M. S. M. Noorani, E. M. Elsayed, *Qualitative analysis of a fourth order difference equation*, Journal of Applied Analysis and Computation, 10(4), (2020) 1343–1354.

[33] T. D. Alharbi, E. M. Elsayed, *Forms of solution and qualitative behavior of twelfth-order rational difference equation*, International Journal of Difference Equations, 17(2), (2022) 281–292.

[34] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *On the difference equation* $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$, Advances in Difference Equations, Article ID 82579, (2006) 1–10.

[35] N. Touafek, E. M. Elsayed, *On a second order rational systems of difference equations*, Hokkaido Mathematical Journal, 44(1), (2015) 29–45.

[36] M. Kara, Y. Yazlik, D. T. Tollu, *Solvability of a system of higher order nonlinear difference equations*, Hacettepe Journal of Mathematics and Statistics, 49(5), (2020) 1566–1593.

[37] S. Elaydi, An introduction to difference equations, Springer, New York, 1996.