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Bayesian Parameter Estimation for Geometric Process with Rayleigh Distribution

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Abstract
The main purpose of this study is to deal with the parameter estimation problem for the geometric process (GP) when the distribution of the first occurrence time of an event is assumed to be Rayleigh. For this purpose, maximum likelihood and Bayesian parameter estimation methods are discussed. Lindley and Markov chain Monte Carlo (MCMC) approximation methods are used in Bayesian calculations. Additionally, a novel method called the Modified-Lindley approximation has been proposed as an alternative to the Lindley approximation. An extensive simulation study was conducted to compare the performances of the prediction methods. Finally, a real data set is analyzed for illustrative purposes.

1. Introduction

The counting process is an appropriate and frequently employed method for the statistical analysis of the times at which successive events occur. Let us consider a set of data with successive arrival times. If successive arrival times are independently and identically distributed (iid), the renewal process (RP) can be utilized to analyze this data. Although this method appears to be theoretically easy, real-world situations frequently have a monotone trend in the data set because of the effect of aging and accumulated wear [1], meaning that the successive arrival times may be independently distributed but not identically distributed. Non-homogeneous Poisson process and geometric process (GP) are two more procedures that can be used in the literature to analyze a set of successive arrival times with a trend. The GP was first introduced by Lam [1-2], as a generalization of a renewal process. See the following definition to understand it, [3].

Definition 1: A set of nonnegative random variables $\{X_i\ , i = 1,2, \dots\}$ is said to be a GP, If $a^{i-1}X_i, i = 1,2, \dots$ are iid random variables, where $a > 0$ is the ratio of GP. The GP is stochastically decreasing when $a > 1$, increasing when $a < 1$. It will become a RP if $a = 1$. In other words, it is simple to determine the density function of X_i from Definition 1, if $\{X_i\ , i = 1,2, \dots\}$ is a GP and the density function of X_1 is f.

$$f_{x_i}(x) = a^{i-1}f\left(a^{i-1}x_i\right) \qquad (1)$$

Furthermore, with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$, the expected value (EV) and the variance (Var) of X_i,are given as follows:

$$E(X_i) = \frac{\mu}{a^{i-1}} \text{ and } Var(X_i) = \frac{\sigma^2}{a^{2(i-1)}},$$

where μ and σ^2 are EV and Var of the first occurrence time X_1, respectively.

Lam introduced and applied the GP to maintenance and repair problems, see [1-2]. Several researchers have researched the basic properties of GP, such as [3-4]. Furthermore, the parameter estimation problems for the GP have recently been presented based on the assumption that the random variable X_1 follows particular distributions, for example, the lognormal distribution Yeh and Chan [5], gamma distribution Kara et al.[6], Weibull distribution Aydoğdu et al. [7], the inverse Rayleigh distribution Usta [8], generalized Rayleigh distribution Biçer et al. [9] and Rayleigh distribution Biçer et al.[10].

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In studies on the GP, different classical parameter estimation methods, including maximum likelihood estimators (MLEs), have been used to estimate the parameters of the process. Since GP was introduced, it has been studied by many researchers, see Kara et al. [11] and also the references cited therein. In these studies on the GP, classical (maximum likelihood and modified maximum likelihood) methods were used to estimate the parameters of the process. However, there are not many studies on the Bayesian parameter estimation problem in GP. Recently, the Bayesian estimators for GP with Lindley and Weibull distributions, respectively, are developed by Yılmaz et al. [12] and Usta [13].

This scenario has motivated us to investigate the Bayesian parameter estimation problem in the GP. On the other hand, Bayesian inference is an alternative framework in estimation problems and received a great deal of attention in recent years.

One of the main advantages of Bayesian statistics is that it allows us to use prior information to analyze unknown parameters. Thus, stronger inferences are obtained. Additionally, Bayesian models outperform classical models, particularly for small sample sizes.

Therefore, in this paper, we have discussed Bayesian inference in GP with Rayleigh distribution.

Here, we assume that the first inter-arrival time X_1 distribution in GP follows a Rayleigh distribution with parameter λ. The remainder of the article is organized as follows: In section 2, the Rayleigh distribution is briefly given. The MLEs of the parameters a and λ are obtained. The limiting distributions of the MLEs are investigated. The Bayes estimators of the unknown parameters under square error loss function (SELF) are constructed. For the Bayesian computation, Lindley's, Modified Lindley (M-Lindley), and Gibbs sampling methods are used. In Section 3, a Monte Carlo Simulation study is carried out to compare the performance of the various estimation methods developed in the previous sections. A real-life data set is presented in Section 4. Finally, some concluding remarks are provided in Section 5.

2. Material and Method

In this section of the study, Rayleigh Distribution, maximum likelihood method and Bayesian parameter estimation methods are investigated.

2.1 Rayleigh Distribution

The Rayleigh distribution is one of the most widely used distributions for modeling positive data in reliability, health, and engineering. Let the distribution of the first occurrence time X_1 in GP has a Rayleigh distribution. The probability density function (pdf) and the cumulative density function (cdf) of the Rayleigh distribution are given by

$$f(x) = \frac{x}{\lambda^2} e^{\frac{-x^2}{2\lambda^2}}, \quad x > 0, \lambda > 0 \qquad (2)$$

and

$$F(x) = 1 - e^{\frac{-x^2}{2\lambda^2}} \qquad x > 0, \lambda > 0, \qquad (3)$$

respectively. In Equations (2)-(3), λ represents the scale parameter of the distribution.

Note that, the EV and the Var of the Rayleigh distribution are

$$E(X) = \frac{\lambda\sqrt{\pi}}{\sqrt{2}} \text{ and } Var(X) = \frac{(4-\pi)\lambda^2}{2}, \quad (4)$$

respectively. The MLEs and Bayesian methods are given as follows.

2.2 Maximum Likelihood Estimators

Assume that $X_1, X_2, \ldots, X_n$ is a set of data that follows a GP with ratio a and X_1 has the Rayleigh distribution with parameter λ. Afterward, using Equation (1), and the likelihood function for X_i, $i = 1,2, \ldots n$ is then obtained

$$L(a, \lambda; x) = \prod_{i=1}^{n} a^{i-1} f(a^{i-1} x_i) = \frac{a^{n(n-1)}}{\lambda^{2n}} \prod_{i=1}^{n} x_i e^{\frac{-\left(a^{i-1}x_i\right)^2}{2\lambda^2}} . \qquad (5)$$

The MLEs of the parameters a and λ are obtained by taking logarithms of Equation (5), differentiating with respect to a and λ, and equating the normal equations to zero as follows:

$$lnL(a, \lambda; x) = n(n-1)lna - 2nln\lambda + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} \frac{\left(a^{i-1}x_i\right)^2}{2\lambda^2} . \qquad (6)$$

Then, differentiating Equation (6) with respect to a and λ, and equating the normal equations to zero given as

$$\frac{\partial lnL(a,\lambda)}{\partial a} = \frac{n(n-1)}{a} - \frac{1}{a\lambda^2} \sum_{i=1}^{n} \left(a^{i-1}x_i\right)^2 (i-1) \qquad (7)$$

and

$$\frac{\partial lnL(a,\lambda)}{\partial\lambda}=\frac{-2n}{\lambda}+\frac{1}{\lambda^3}\sum_{i=1}^{n}\left(a^{i-1}x_i\right)^2 \qquad (8)$$

Then, from Equations (7) and (8), the parameter λ is found as follows:

$$\lambda=\left(\frac{1}{2n}\sum_{i=1}^{n}\left(a^{i-1}x_i\right)^2\right)^{\frac{1}{2}}. \qquad (9)$$

By substituting Equation (9) into (7), the resulting equation in a becomes

$$\frac{n(n-1)}{a}-\left(2n\sum_{i=1}^{n}(i-1)x_i^2a^{2i-3}\right)\left(\sum_{i=1}^{n}\left(a^{i-1}x_i\right)^2\right)^{-1}=0. \qquad (10)$$

These equations are simultaneously solved to yield the MLEs for $\hat{a}$ and $\hat{\lambda}$. Equation (10) must instead be solved iteratively because there are no explicit solutions to the equation. In this study, the Newton Rapson method is used.

Now, we built the asymptotically distribution of the MLEs. With a mean vector (a,λ),and covariance $(I^{-1})$the joint distribution of $\hat{a}$ and $\hat{\lambda}$ is asymptotically normal (AN), thus,

$$\binom{\hat{a}}{\hat{\lambda}}\sim AN\left(\binom{a}{\lambda},I^{-1}(\mathrm{a},\lambda)\right). \qquad (11)$$

Here $I(a,\lambda)=\left[I_{ij}\right]_{2x2}$ where is defined as the Fisher information matrix I and is obtained as shown below:

$$I^{-1}(\mathrm{a},\lambda)=\begin{bmatrix} I_{11} & I_{12}\\ I_{21} & I_{22}\end{bmatrix}=\begin{bmatrix}-E\left(\frac{\partial^2 logL}{\partial^2 a}\right) & -E\left(\frac{\partial^2 logL}{\partial a\partial\lambda}\right)\\ -E\left(\frac{\partial^2 logL}{\partial\lambda\,\partial \mathrm{a}}\right) & -E\left(\frac{\partial^2 logL}{\partial^2\lambda}\right)\end{bmatrix}. \qquad (12)$$

Since $E\left(a^{i-1}X_i\right)^2=2\lambda^2$, the expected values of the second derivatives are calculated as follows:

$$I_{11}=\frac{-n(n-1)}{a^2}+\frac{1}{a^2\lambda^2}\sum_{i=1}^{n}(i-1)E\left(a^{i-1}X_i\right)^2$$
$$\frac{-2}{a^2\lambda^2}\sum_{i=1}^{n}(i-1)^2E\left(a^{i-1}X_i\right)^2\approx\frac{4n^3}{3a^2},$$

$$I_{12}=\frac{-2}{a\lambda^3}\sum_{i=1}^{n}(i-1)E\left(a^{i-1}X_i\right)^2\approx\frac{-2n^2}{a\lambda},$$

and

$$I_{22}=\frac{-2n}{\lambda^2}+\frac{3}{\lambda^4}\sum_{i=1}^{n}E\left(a^{i-1}X_i\right)^2\approx\frac{4n}{\lambda^2}.$$

The symbol ≈ stands for "asymptotically equivalent". Therefore, the asymptotic variance-covariance matrix of $\hat{a}$ and $\hat{\lambda}$ is obtained by $I^{-1}(\mathrm{a},\lambda)$,

$$I^{-1}(\mathrm{a},\lambda)=\begin{bmatrix}\frac{3a^2}{n^3} & \frac{3a\lambda}{2n^2}\\ \frac{3a\lambda}{2n^2} & \frac{\lambda^2}{n}\end{bmatrix}. \qquad (13)$$

Then, by using equation (11), the marginal asymptotic distribution of $\hat{a}$ and $\hat{\lambda}$ can be seen as

$$\hat{a}\sim AN\left(a,\frac{3a^2}{n^3}\right) \text{ and } \hat{\lambda}\sim \mathrm{AN}\left(\lambda,\frac{\lambda^2}{n}\right).$$

Thus, the all the proposed estimators are asymptotically unbiased. Furthermore, these estimators are also consistent because the asymptotic variance of each estimator goes to zero as n goes to infinity.

2.3. Bayesian Inference

The Bayesian estimation under the Lindley, M-Lindley, and Markov chain Monte Carlo (MCMC) approximation techniques based on the SELF of parameters a and λ in GP with Rayleigh distribution is discussed in this section. The Bayesian estimation framework has drawn a lot of interest recently. A suitable loss function and prior distribution play a significant role in making the best decision in Bayesian parameter estimation. In line with this purpose, SELF is one of the widely used loss functions.

This loss function is obtained by the following

$$L_{SELF}(\hat{\lambda},\ \lambda)=(\hat{\lambda}-\lambda),^2 \qquad (14)$$

where $\hat{\lambda}$ is the estimate of the parameter λ, see [14].

Before seeing the new data, what was known or thought to be true is expressed by the prior distribution. It is assumed that λ has independent Gamma prior distribution in this study. The independent Gamma prior distribution is reasonable since it is flexible and appropriate. When the values of the hyper-parameters in the gamma prior are taken to be zero, the prior is non-informative. Several writers have employed independent gamma priors on the shape and scale parameters for lifespan distributions, see [15]-[16].

The prior distribution of the ratio parameter a would be the uniform distribution since the parameter has bounded support. Consequently, the prior pdfs of parameters a and λ, independent uniform, and gamma priors may be given as

$$\pi_1(a)=(\Phi_2-\Phi_1)^{-1}$$

and

$$\pi_2(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}.$$

Then, the joint prior pdfs of a and λ is given as

$$\pi(a,\lambda) = \pi_1(a)\pi_2(\lambda) = (\Phi_2 - \Phi_1)^{-1}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}. \quad (15)$$

Here, it is assumed that the hyper-parameters (α, β) and (Φ_1, Φ_2) are known and positive.

Combining (5) with (15) and using Bayes theorem, the joint posterior density function of a and λ is given by:

$$\pi(a,\lambda|x) = \frac{L(a,\lambda;\mathrm{x})\pi(a,\lambda)}{\int\int(a,\lambda;\mathrm{x})\pi(a,\lambda)dad\lambda} \propto \frac{a^{n(n-1)}}{(\Phi_2-\Phi_1)}\frac{\lambda^{\alpha-1-2n}}{e^{\beta\lambda}}\prod_{i=1}^{n}x_i e^{\frac{-\left(a^{i-1}x_i\right)^2}{2\lambda^2}}. \quad (16)$$

The marginal conditional posterior pdfs of a and λ are thus provided in Equation (16), $\pi_a(a|\lambda,\mathrm{x}) \propto$

$$\frac{a^{n(n-1)}}{(\Phi_2-\Phi_1)}\prod_{i=1}^{n}x_i e^{\frac{-\left(a^{i-1}x_i\right)^2}{2\lambda^2}}, \quad (17)$$

and

$$\pi_2(\lambda|a,\mathrm{x}) \propto \lambda^{\alpha-1-2n}e^{-\beta\lambda}\prod_{i=1}^{n}x_i e^{\frac{-\left(a^{i-1}x_i\right)^2}{2\lambda^2}}, \quad (18)$$

respectively. The EV of the conditional posterior pdfs shown in (17) and (18) provides the Bayes estimators for the parameters a and λ. However, an explicit solution for the estimates does not yield by expected values of the conditional posterior pdfs. Therefore, we are taking into consideration the Lindley, M-Lindley, and MCMC techniques to compute the Bayes estimators of the parameters a and λ for GP with Rayleigh distribution. Summaries of these methods are given below.

2.4. Lindley Approximation

For calculating the ratio of the two integrals, Lindley [17] proposed an approximation method. This method may be used for calculating the posterior mean of an arbitrary function $u(a,\ \lambda)$, as shown below:

$$\hat{u} = E\big(u(a,\lambda|x)\big) = \frac{\int_0^\infty\int_0^\infty u(a,\lambda)\pi(a,\lambda)L(a,\lambda;\mathrm{x})dad\lambda}{\int_0^\infty\int_0^\infty \pi(a,\lambda)L(a,\lambda;\mathrm{x})dad\lambda}. \quad (19)$$

Here, $u(a,\lambda)$ is a function of a and λ only, $\pi(a,\lambda)$is the joint prior density function, $L(a,\lambda;\mathrm{x})$and is the likelihood function.

By using Lindley approximation, Equation (19) can be written by the following formula:

$$u \approx u(\hat{a},\hat{\lambda}) + 0.5[u_{11}\sigma_{11} + u_{22}\sigma_{22} + 2u_{12}\sigma_{12} + 2u_1(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + 2u_2(\sigma_{12}\rho_1 + \sigma_{22}\rho_2)] + 0.5\Big[L_{111}(u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}) + L_{112}\left(3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\right) + L_{122}(u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}) + L_{222}(u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2)\Big]. \quad (20)$$

Here, $\hat{a}$ and $\hat{\lambda}$ are the MLEs of a and λ respectively.

$u_1 = \frac{\partial u(a,\lambda)}{\partial a}, u_{11} = \frac{\partial^2 u(a,\lambda)}{\partial a^2}, \quad u_2\frac{\partial u(a,\lambda)}{\partial\lambda}, u_{22} = \frac{\partial^2 u(a,\lambda)}{\partial^2\lambda}, u_{12} = \frac{\partial^2 u(a,\lambda)}{\partial \mathrm{a}\partial\lambda}, \rho_1 = \frac{\partial In\pi(a,\lambda)}{\partial a},$

$\rho_2 = \frac{\partial In\pi(a,\lambda)}{\partial\lambda}, L_{111} = \frac{\partial^3 InL}{\partial^3 a}, L_{112} = \frac{\partial^3 InL}{\partial^2 a\partial\lambda}, L_{122} = \frac{\partial^3 InL}{\partial^2\lambda\partial\mathrm{a}},\ L_{222} = \frac{\partial^3 InL}{\partial^3\lambda},$

and $\sigma_{ij}, i,j = 1{,}2$ are the elements of the inverse Fisher information matrix.

Hence, it follows from Equation (20) that the Bayes estimators of the parameters a and λ, say $\hat{a}_L$ and $\hat{\lambda}_L$,are given:

If $u(a,\lambda) = a$,$\hat{u}_1 = 1, \hat{u}_{11} = \hat{u}_{12} = \hat{u}_2 = \hat{u}_{22} = 0$, then

$$\hat{a}_L = \hat{a} + \hat{\sigma}_{21}\hat{\rho}_2 + 0.5\big[\hat{L}_{111}\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\left((\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\sigma_{12}^2)\right) + \hat{L}_{222}\hat{\sigma}_{12}\hat{\sigma}_{22}\big]. \quad (21)$$

If,$(a,\lambda) = \lambda, \hat{u}_2 = 1, \hat{u}_{22} = \hat{u}_{12} = \hat{u}_2 = \hat{u}_{11} = 0$, then

$$\hat{\lambda}_L = \hat{\lambda} + \hat{\sigma}_{22}\hat{\rho}_2 + 0.5\big[\hat{L}_{111}\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{11}\hat{\sigma}_{22} + \hat{L}_{222}\hat{\sigma}_{22}^2\big], \quad (22)$$

respectively.

Here,

$$\hat{L}_{111} = \frac{2n(n-1)}{\hat{a}^3} - \frac{2}{\hat{\lambda}^2\hat{a}^3}\sum_{i=1}^{n}\left(\hat{a}^{i-1}x_i\right)^2(i-1)(i-2)(2i-3)$$

$\hat{L}_{222} = \frac{-4n}{\hat{\lambda}^3} + \frac{12}{\hat{\lambda}^5}\sum_{i=1}^{n}\left(\hat{a}^{i-1}x_i\right)^2,$

$\hat{L}_{112} = \frac{2}{\hat{\lambda}^2\hat{a}^3}\sum_{i=1}^{n}\left(\hat{a}^{i-1}x_i\right)^2(i-1)(2i-3),$

$\hat{L}_{122} = \frac{-6}{\hat{a}\hat{\lambda}^4}\sum_{i=1}^{n}\left(\hat{a}^{i-1}x_i\right)^2(i-1),$

$\hat{\rho}_1 = 0,\ \hat{\rho}_2 = \frac{\alpha-1}{\hat{\lambda}} - \hat{\beta},$

and $\sigma_{ij}, i,j = 1{,}2$ are the elements of the variance-covariance matrix defined in Equation (13).

2.5 Modified-Lindley Approximation

In Lindley approximation, the Bayes estimators of a and λ are obtained by incorporating all (L_{ijk})into (21) and (22), respectively. However, in our case, it will be quite complicated. Therefore, we present the M-Lindley approximation as a novel approximation technique. The EVs of the terms in the expression (L_{ijk}) form the basis of this approximation technique. In other words, $E(L_{ijk})$ $(i,j,k=1,2,3)$ exists, it can be obtained under the M-Lindley approximation.

Therefore, if all of the L terms are available, Equation (20) can be estimated as follows:

$$\hat{u} \approx u(\hat{a},\hat{\lambda}) + 0.5[u_{11}\sigma_{11} + u_{22}\sigma_{22} + 2u_{12}\sigma_{12} + 2u_1(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + 2u_2(\sigma_{12}\rho_1 + \sigma_{22}\rho_2)] + 0.5\Big[E(L_{111})(u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}) + E(L_{112})\Big(3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\Big) + E(L_{122})(u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}) + E(L_{222})(u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2)\Big],$$
(23)

where all the remaining terms will be the same as the Lindley approximation.

Hence, it follows from Equation (23) that the Bayes estimators of the parameters a and λ, say $\hat{a}_{M-L}$ and $\hat{\lambda}_{M-L}$are given as follows:

If $u(a,\lambda) = a$, $\hat{u}_1 = 1, \hat{u}_{11} = \hat{u}_{12} = \hat{u}_2 = \hat{u}_{22} = 0$, then

$$\hat{a}_{M-L} = \hat{a} + \hat{\sigma}_{21}\hat{\rho}_2 + 0.5\big[E(\hat{L}_{111})\hat{\sigma}_{11}^2 \; 3E(\hat{L}_{112})\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\Big((\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\sigma_{12}^2)\Big) + E(\hat{L}_{222})\hat{\sigma}_{12}\hat{\sigma}_{22}\Big],$$

If,$(a,\lambda) = \lambda, \hat{u}_2 = 1, \hat{u}_{22} = \hat{u}_{12} = \hat{u}_2 = \hat{u}_{11} = 0$, then

$$\hat{\lambda}_{M-L} = \hat{\lambda} + \hat{\sigma}_{22}\hat{\rho}_2 + 0.5\big[E(\hat{L}_{111})\hat{\sigma}_{11}\hat{\sigma}_{12} + E(\hat{L}_{112})(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3E(\hat{L}_{122})\hat{\sigma}_{11}\hat{\sigma}_{22} + E(\hat{L}_{222})\hat{\sigma}_{22}^2\big].$$

In our case, the Bayesian estimators of a and λ using the M-Lindley approximation are derived as:

$\hat{a}_{M-L} = \hat{a} + \hat{\sigma}_{12}\hat{\rho}_2 + \frac{3\hat{a}}{2n^2}$ and

$\hat{\lambda}_{M-L} = \hat{\lambda} + \hat{\sigma}_{22}\hat{\rho}_2 + \frac{2\hat{\lambda}}{n}$.

Here,

$E(L_{111}) \approx \frac{-2n^4}{a^3}, E(L_{222}) = \frac{20n}{\lambda^3}$,

$E(L_{122}) \approx \frac{-6n^2}{a\lambda^2}$ and $E(L_{112}) \approx \frac{8n^3}{\lambda a^2}$,

$\sigma_{11}, \sigma_{12}, \sigma_{22}$ and ρ_1, ρ_2 will be the same as the Lindley approximation.

2.6 Markov Chain Monte Carlo Method

Here, the Gibbs sampling method is used to produce samples from the posterior distributions. It is a subclass of the MCMC, see [18]. We know that the posterior conditional density function of a and λ is given as follows in Equations (17)-(18),

$$\pi_a(a|\lambda,\mathrm{x}) \propto \frac{a^{n(n-1)}}{(\Phi_2-\Phi_1)}\prod_{i=1}^n x_i e^{\frac{-\left(a^{i-1}x_i\right)^2}{2\lambda^2}} \text{ and}$$

$$\pi_\lambda(\lambda|\mathrm{a},\mathrm{x}) \propto \frac{\lambda^{\alpha-1-2n}}{e^{\beta\lambda}}\prod_{i=1}^n x_i e^{\frac{-\left(a^{i-1}x_i\right)^2}{2\lambda^2}} ,$$

respectively. It is clear from these equations that the conditional density function of a and λ cannot be found in the form the well-known density functions. So, we can use the Metropolis- Hasting (M-H) algorithm, introduced by Metropolis et al. [19], with normal proposal distribution to generate random samples from these distributions.

The steps of the Gibbs sampling method are as follows:

Step1: Start with an initial guess $a_0 = \hat{a}$ and $\lambda_0 = \hat{\lambda}$.

Step2: Set $j = 1$.

Step 3: Using the M-H algorithm, generate a posterior sample for $a_0^{(j)}$and $\lambda_0^{(j)}$from Equations (17) and (18), respectively.

Step 4: Set $j = j + 1$

Step 5: Repeat Step 3-4 N times and obtain MCMC sample as$(a_1,\lambda_1), \ldots, , (a_N,\lambda_N)$.

So, the Bayes estimator of the parameters a and λ under SELF, say $\hat{a}_{MCMC}$ and, $\hat{\lambda}_{MCMC}$ are computed as follows:

$\hat{a}_{MCMC} = \frac{1}{N}\sum_{i=1}^N a_j$and $\hat{\lambda}_{MCMC} = \frac{1}{N}\sum_{i=1}^N \lambda$.

3. Simulation Study

In this Section, we carried out an extensive Monte Carlo simulation study to compare the performances of Bayesian and classical estimators with respect to mean, bias and mean square error (MSE) values for the different sample size. In the context of the Bayesian parameter estimation, two different informative priors are used. Firstly, we take $\alpha = 2, \beta = 1$ and $\Phi_2 = 1.5, \Phi_1 = 0.5$ and, call them Prior-I. Then we chose $\alpha = \beta = 0$ and $\Phi_2 = 1.5, \Phi_1 = 0.5$ and, call them Prior-II. The Bayesian estimators of the parameters a and λ are calculated under SELF using the Lindley, M-Lindley, and MCMC approximation methods based on these priors. The ratio parameters are $a = 0.90, 0.95, 1.05$ and 1.10 and the sample $n = 20, 30, 50, 100, 190$ are used in the simulation study. A sequence of random variables $\{Y_i,\ i = 1,2, \dots\}$ each having Rayleigh distribution with the parameter λ is generated by using MATLAB2013. The data set $\{X_i,\ i = 1,2, \dots\}$ then becomes a realization of the GP with ratio a by the transformation $X_i = \frac{Y_i}{a^{i-1}}$. Since the simulation results are similar, the results are summarized only for $a = 0.90, 1.10$ and $\lambda = 0.5,\ 0.8, 1$ and 2. Additionally, the results obtained for n=190 are not added to the study because they are similar to the results obtained for n=100 in terms of bias and MSE. The mean, bias and MSE values of the estimators based on 2000 replications are given in Table 1-4.

The bias and MSE values are given as follows:

$$Bias = \frac{1}{2000}\sum_{1=1}^{2000}\left(\hat{\theta}_i - \theta\right) \text{ and}$$

$$MSE = \frac{1}{2000}\sum_{1=1}^{2000}\left(\hat{\theta}_i - \theta\right).^2$$

Here $\hat{\theta}_i$ is the ith simulated estimate of the parameter interest and θ is the true parameter value.

Table1. The mean and MSE values for the estimators of parameters a and λ when a=0.90,1.10 and λ=0.5.

			Prior-I						Prior-II					
			a			λ			a			λ		
N	a	Method	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
20	$a = 0.9$	MLE	0.90007	0.0007	$3.1x10^{-4}$	0.5030	-0.0030	0.0131	0.8999	-0.0001	$3.29x10^{-4}$	0.4973	-0.0027	0.0124
		Lindley	0.8957	-0.0043	$6.3x10^{-4}$	0.4980	-0.0020	0.0125	0.8985	-0.0005	$6.55x10^{-4}$	0.4967	-0.0033	0.0116
		M-Lindley	0.8952	-0.0048	$7.2x10^{-4}$	0.4975	-0.0035	0.0124	0.8987	-0.0003	$7.96x10^{-4}$	0.4961	-0.0039	0.0116
		MCMC	0.9005	0.0005	$1.1x10^{-4}$	0.5022	0.0022	0.0032	0.9003	0.0003	$8.80x10^{-5}$	0.5021	0.0021	0.0085
30		MLE	0.8998	-0.0002	$9.63x10^{-5}$	0.4951	-0.0049	0.0076	0.8999	-0.0001	$9.92x10^{-5}$	0.4951	-0.0049	0.0076
		Lindley	0.8996	-0.0004	$1.27x10^{-4}$	0.4981	-0.0019	0.0079	0.8997	-0.0003	$1.26x10^{-4}$	0.4964	-0.0036	0.0074
		M-Lindley	0.8997	-0.0003	$1.42x10^{-4}$	0.4983	-0.0017	0.0077	0.8998	-0.0002	$1.73x10^{-4}$	0.4966	-0.0034	0.0074
		MCMC	0.9000	0.0000	$2.45x10^{-5}$	0.5032	0.0032	0.0056	0.8999	-0.0001	$2.43x10^{-5}$	0.5065	0.0065	0.0022
50		MLE	0.8999	-0.0001	$1.88x10^{-5}$	0.4983	-0.0017	0.0046	0.8999	-0.0001	$2.14x10^{-5}$	0.4981	-0.0019	0.0046
		Lindley	0.8998	-0.0002	$2.47x10^{-5}$	0.4989	-0.0011	0.0047	0.8999	-0.0001	$3.41x10^{-5}$	0.4976	-0.0024	0.0044
		M-Lindley	0.8998	-0.0002	$2.61x10^{-5}$	0.4987	-0.0013	0.0045	0.8999	-0.0001	$3.59x10^{-5}$	0.4974	-0.0026	0.0045
		MCMC	0.9000	0.0000	$5.08x10^{-6}$	0.5018	0.0018	0.0033	0.9000	0.0000	$5.08x10^{-6}$	0.5033	0.0033	0.0033
100		MLE	0.9000	0.0000	$2.33x10^{-6}$	0.4999	-0.0001	0.0025	0.9001	0.0001	$2.58x10^{-6}$	0.5024	0.0024	0.0026
		Lindley	0.8999	-0.0001	$2.77x10^{-6}$	0.4997	-0.0003	0.0020	0.9000	0.0000	$3.36x10^{-6}$	0.5021	0.0021	0.0020
		M-Lindley	0.8999	-0.0001	$2.81x10^{-6}$	0.4995	-0.0005	0.0018	0.9000	0.0000	$3.42x10^{-6}$	0.5021	0.0021	0.0018
		MCMC	0.9000	0.0000	$8.41x10^{-6}$	0.5011	0.0011	0.0012	0.9000	0.0000	$8.03x10^{-6}$	0.5026	0.0026	0.0007
20	$a = 1.1$	MLE	1.1004	0.0004	$5.13x10^{-4}$	0.4989	-0.0011	0.0116	1.1005	0.0005	$5.01x10^{-4}$	0.4994	-0.0006	0.0117
		Lindley	1.0989	-0.0011	$5.87x10^{-4}$	0.4948	-0.0042	0.0110	1.0987	-0.0013	$8.49x10^{-4}$	0.4894	0.0116	0.0116
		M-Lindley	1.0987	-0.0013	$6.93x10^{-4}$	0.4934	-0.0066	0.0111	1.0983	-0.0017	$1.17x10^{-4}$	0.4899	0.0101	0.0117
		MCMC	1.1009	0.0009	$1.34x10^{-4}$	0.5031	0.0031	0.0096	1.1004	0.0004	$1.32x10^{-4}$	0.5075	0.0075	0.0083
30		MLE	1.0999	-0.0001	$1.14x10^{-4}$	0.4993	-0.0007	0.0088	1.1002	0.0002	$1.42x10^{-4}$	0.5010	0.0010	0.0078
		Lindley	1.0996	-0.0004	$1.82x10^{-4}$	0.4994	-0.0006	0.0085	1.0997	-0.0003	$2.49x10^{-4}$	0.5011	0.0011	0.0074
		M-Lindley	1.0997	-0.0003	$2.02x10^{-4}$	0.4996	-0.0004	0.0080	1.0997	-0.0003	$2.78x10^{-4}$	0.5008	0.0008	0.0070
		MCMC	1.0998	-0.0002	$3.67x10^{-5}$	0.5010	0.0010	0.0062	1.0998	-0.0002	$3.65x10^{-5}$	0.5005	0.0005	0.0052
50		MLE	1.1002	0.0002	$3.11x10^{-5}$	0.4995	-0.0005	0.0051	1.1002	0.0002	$3.25x10^{-5}$	0.5017	0.0017	0.0050
		Lindley	1.0998	-0.0002	$3.82x10^{-5}$	0.4996	-0.0004	0.0049	1.0998	-0.0002	$4.66x10^{-5}$	0.5008	0.0008	0.0048
		M-Lindley	1.0999	-0.0001	$4.00x10^{-5}$	0.4997	-0.0003	0.0048	1.0999	-0.0001	$4.89x10^{-5}$	0.5006	0.0006	0.0045
		MCMC	1.1002	0.0002	$7.69x10^{-6}$	0.4998	-0.0002	0.0032	1.1003	0.0003	$8.10x10^{-6}$	0.5018	0.0018	0.0034
100		MLE	1.1000	0.0000	$3.69x10^{-6}$	0.5002	0.0002	0.0024	1.1000	0.0000	$3.92x10^{-6}$	0.5002	0.0002	0.0024
		Lindley	1.0998	-0.0002	$4.54x10^{-6}$	0.5006	0.0006	0.0020	1.0998	-0.0002	$5.18x10^{-6}$	0.5001	0.0001	0.0020
		M-Lindley	1.0997	-0.0003	$4.61x10^{-6}$	0.5005	0.0005	0.0018	1.0999	-0.0001	$5.27x10^{-6}$	0.5000	0.0000	0.0020
		MCMC	1.1001	0.0001	$5.91x10^{-6}$	0.5008	0.0008	0.0013	1.1000	0.0000	$6.83x10^{-6}$	0.5001	0.0001	0.0017

Table2. The mean and MSE values for the estimators of parameters a and λ when a=0.90,1.10 and λ=0.8.

			Prior-I						Prior-II					
			a			λ			a			λ		
n	a	Method	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
20	$a = 0.9$	MLE	0.90007	0.0007	$3.17x10^{-4}$	0.7984	-0.0016	0.0286	0.9002	0.0002	$3.26x10^{-4}$	0.7972	-0.0028	0.0304
		Lindley	0.8950	-0.0050	$5.47x10^{-4}$	0.7965	-0.0035	0.0270	0.8988	-0.0012	$5.25x10^{-4}$	0.7941	-0.0059	0.0297
		M-Lindley	0.8951	-0.0049	$7.24x10^{-4}$	0.7956	-0.0044	0.0270	0.8981	-0.0019	$5.56x10^{-4}$	0.7955	-0.0045	0.0298
		MCMC	0.9004	0.0004	$9.27x10^{-5}$	0.8030	0.0030	0.0194	0.9002	0.0002	$7.70x10^{-5}$	0.8037	0.0037	0.0190
30		MLE	0.9001	0.0001	$9.28x10^{-5}$	0.7964	-0.0036	0.0199	0.9001	0.0001	$8.93x10^{-5}$	0.7985	-0.0015	0.0200
		Lindley	0.8982	-0.0008	$1.38x10^{-4}$	07962	-0.0038	0.0196	0.8993	-0.0007	$1.74x10^{-4}$	0.7983	-0.0017	0.0199
		M-Lindley	0.8989	-0.0001	$1.57x10^{-4}$	0.7962	-0.0038	0.0196	0.8991	-0.0009	$1.74x10^{-4}$	0.7975	-0.0025	0.0199
		MCMC	0.9000	0.0000	$2.51x10^{-5}$	0.8018	0.0018	0.0159	0.9003	0.0003	$2.44x10^{-5}$	0.8034	0.0034	0.0157
50		MLE	0.8998	-0.0002	$2.06x10^{-5}$	0.8058	0.0058	0.0136	0.8999	-0.0001	$2.15x10^{-5}$	0.7976	-0.0024	0.0131
		Lindley	0.8995	-0.0005	$2.62x10^{-5}$	0.8048	0.0048	0.0135	0.8996	-0.0004	3.4510^{-5}	0.7963	-0.0037	0.0130
		M-Lindley	0.8994	-0.0006	$2.76x10^{-5}$	0.8046	0.0046	0.0135	0.8994	-0.0006	$3.64x10^{-5}$	0.7961	-0.0039	0.0130
		MCMC	0.8998	-0.0002	$5.45x10^{-6}$	0.8029	0.0029	0.0128	0.9000	0.0000	$5.06x10^{-6}$	0.8011	0.0011	0.0120
100		MLE	0.8999	-0.0001	$2.45x10^{-6}$	0.8001	0.0001	0.0059	0.8999	-0.0001	$2.40x10^{-6}$	0.7973	-0.0027	0.0064
		Lindley	0.8999	-0.0001	$3.05x10^{-6}$	08004	0.0004	0.0058	0.8997	-0.0003	$3.37x10^{-6}$	0.7979	-0.0021	0.0060
		M-Lindley	0.8999	-0.0001	$3.08x10^{-6}$	07995	0.0005	0.0058	0.8997	-0.0003	$3.44x10^{-6}$	0.7981	-0.0019	0.0060
		MCMC	0.9000	0.0000	$8.03x10^{-6}$	0.8005	0.0005	0.0037	0.9000	0.0000	$8.13x10^{-6}$	0.8004	0.0004	0.0038
20		MLE	1.1003	0.0003	$4.82x10^{-4}$	0.7979	-0.0021	0.0300	1.0990	-0.0010	$5.52x10^{-4}$	0.7915	-0.0085	0.0328
		Lindley	1.0953	-0.0047	$6.92x10^{-4}$	0.7936	-0.0064	0.0290	1.0987	-0.0013	$8.31x10^{-4}$	0.7923	-0.0077	0.0314

		M-Lindley	1.0955	-0.0045	$8.64x10^{-4}$	0.7901	-0.0099	0.0290	1.0989	-0.0011	$9.73x10^{-4}$	0.7916	-0.0084	0.0317
		MCMC	1.0994	0.0006	$1.29x10^{-4}$	0.8104	0.0104	0.0189	1.0988	-0.0012	$1.73x10^{-4}$	0.7923	-0.0067	0.0104
30		MLE	1.0997	-0.0003	$1.41x10^{-4}$	0.7997	-0.0023	0.0211	1.1005	0.0005	$1.47x10^{-4}$	0.8039	0.0039	0.0215
		Lindley	1.0979	-0.0021	$2.04x10^{-4}$	0.7980	-0.0020	0.0206	1.0998	-0.0002	$2.44x10^{-4}$	0.7992	-0.0008	0.0213
		M-Lindley	1.0974	-0.0026	$2.31x10^{-4}$	0.7970	-0.0030	0.0206	1.0997	-0.0003	$2.72x10^{-4}$	0.7982	-0.0018	0.0213
		MCMC	1.0999	-0.0001	$3.79x10^{-5}$	0.8080	0.0080	0.0161	1.1001	0.0001	$3.70x10^{-5}$	0.8053	0.0053	0.0055
50	$a = 1.1$	MLE	1.0999	-0.0001	$3.19x10^{-5}$	0.7982	-0.0018	0.0128	1.0998	-0.0002	$3.13x10^{-5}$	0.7986	- 0.0014	0.0133
		Lindley	1.0997	-0.0003	$4.21x10^{-5}$	0.7975	-0.0025	0.0127	1.0998	-0.0002	$5.44x10^{-5}$	0.7973	-0.0027	0.0132
		M-Lindley	1.0994	-0.0006	$4.43x10^{-5}$	0.7977	-0.0023	0.0127	1.0997	-0.0003	$5.75x10^{-5}$	0.7981	-0.0019	0.0132
		MCMC	1.1002	0.0002	$7.85x10^{-6}$	0.8030	0.0030	0.0083	1.1000	0.0000	$7.80x10^{-6}$	0.8019	0.0019	0.0088
100		MLE	1.1000	0.0000	$3.60x10^{-6}$	0.7990	0.0063	0.0024	1.1000	0.0000	$3.87x10^{-6}$	0.7998	-0.0002	0.0064
		Lindley	1.0998	-0.0002	$4.37x10^{-6}$	0.7988	0.0063	0.0020	1.0999	-0.0001	$5.14x10^{-6}$	0.7994	-0.0006	0.0060
		M-Lindley	1.0997	-0.0003	$4.46x10^{-6}$	0.7987	0.0062	0.0018	1.0999	-0.0001	$5.23x10^{-6}$	0.7993	-0.0007	0.0059
		MCMC	1.1001	0.0001	$6.43x10^{-6}$	0.8017	0.0047	0.0013	1.1000	0.0000	$6.08x10^{-6}$	0.7995	-0.0005	0.0035

Table3. The mean and MSE values for the estimators of parameters a and λ when a=0.90,1.10 and λ=1.

			Prior-I						Prior-II					
			a			λ			a			λ		
n	a	Method	Bias	Mean	MSE	Bias	Mean	MSE	Bias	Mean	MSE	Bias	Mean	MSE
20	a=0.9	MLE	0.8998	-0.0002	$3.42x10^{-4}$	0.9902	0.0098	0.0513	0.9005	0.0005	$3.46x10^{-4}$	1.0058	0.0058	0.0507
		Lindley	0.8996	-0.0004	$5.71x10^{-4}$	0.9866	-0.0134	0.0499	0.8988	-0.0002	$7.51x10^{-4}$	1.0029	0.0029	0.0494
		M-Lindley	0.8997	-0.0003	$7.33x10^{-4}$	0.9895	0.0115	0.0499	0.8986	-0.0004	$9.22x10^{-4}$	0.9982	-0.0018	0.0493
		MCMC	0.9003	0.0003	$8.91x10^{-5}$	1.0122	0.0122	0.0343	0.9907	0.0007	$9.27x10^{-5}$	1.0039	0.0039	0.0343
30		MLE	0.9007	0.0007	$9.58x10^{-5}$	1.0060	0.0060	0.0298	0.8999	-0.0001	$1.51x10^{-4}$	0.9956	-0.0044	0.0314
		Lindley	0.8998	-0.0002	$1.55x10^{-4}$	1.0025	0.0025	0.0295	0.8995	-0.0005	$2.62x10^{-4}$	0.9967	-0.0033	0.0309
		M-Lindley	0.8996	-0.0004	$1.77x10^{-4}$	1.0016	0.0016	0.0293	0.8994	-0.0006	$2.94x10^{-4}$	0.9968	-0.0032	0.0309
		MCMC	0.9005	0.0005	$3.14x10^{-5}$	1.0013	0.0013	0.0207	0.8999	-0.0001	$4.64x10^{-4}$	1.0109	0.0109	0.0278
50		MLE	0.9001	0.0001	$1.70x10^{-5}$	1.0046	0.0046	0.0190	0.8998	-0.0002	$1.84x10^{-5}$	0.9960	-0.0040	0.0192
		Lindley	0.8999	-0.0001	$2.33x10^{-5}$	1.0028	0.0028	0.0189	0.8997	-0.0003	$2.71x10^{-5}$	0.9970	-0.0030	0.0180
		M-Lindley	0.8998	-0.0002	$2.35x10^{-5}$	1.0026	0.0026	0.0182	0.8997	-0.0003	$2.90x10^{-5}$	0.9973	-0.0027	0.0171
		MCMC	0.9000	0.0000	$8.58x10^{-6}$	1.0028	0.0028	0.0150	0.8998	-0.0002	$4.81x10^{-5}$	0.9986	-0.0014	0.0146
100		MLE	0.9000	0.0000	$2.41x10^{-6}$	0.9974	-0.0026	0.0102	0.8999	-0.0001	$3.84x10^{-6}$	0.9989	-0.0011	0.0097
		Lindley	0.8999	-0.0001	$2.53x10^{-6}$	0.9970	-0.0030	0.0080	0.8999	-0.0001	$3.95x10^{-6}$	0.9986	-0.0024	0.0095
		M-Lindley	0.8999	-0.0001	$2.57x10^{-6}$	0.9973	-0.0027	0.0075	0.8999	-0.0001	$4.01x10^{-6}$	0.9988	-0.0012	0.0093
		MCMC	0.9001	0.0001	$7.29x10^{-6}$	1.0010	0.0010	0.0066	0.9001	0.0001	$7.52x10^{-6}$	0.9990	-0.0010	0.0065
20	a=1.1	MLE	1.0995	0.0005	$5.69x10^{-4}$	0.9864	-0.0136	0.0473	1.1001	0.0001	$4.87x10^{-5}$	0.9853	-0.0147	0.0457
		Lindley	1.0988	-0.0012	$8.49x10^{-4}$	0.9829	0.0171	0.0457	1.0996	-0.0004	$9.56x10^{-4}$	0.9886	-0.0114	0.0445
		M-Lindley	1.0984	-0.0016	$7.37x10^{-4}$	0.9793	-0.0207	0.0457	1.0994	-0.0006	$9.22x10^{-4}$	0.9854	-0.0116	0.0445
		MCMC	1.0992	-0.0008	$1.28x10^{-4}$	1.0127	0.0127	0.0320	1.0102	0.0002	$1.42x10^{-5}$	1.0101	0.0102	0.0340
30		MLE	1.0998	-0.0002	$9.67x10^{-5}$	0.9982	-0.0018	0.0324	1.0998	-0.0002	$3.68x10^{-5}$	0.9969	-0.0031	0.0319
		Lindley	1.0997	-0.0003	$1.75x10^{-4}$	0.9989	-0.0011	0.0290	1.0997	-0.0003	$5.18x10^{-5}$	0.9967	-0.0033	0.0315
		M-Lindley	1.0997	-0.0003	$2.00x10^{-4}$	0.9980	-0.0020	0.0285	1.0996	-0.0004	$5.38x10^{-5}$	0.9964	-0.0036	0.0312
		MCMC	1.0996	-0.0004	$3.76x10^{-5}$	0.9983	-0.0017	0.0184	1.0998	-0.0002	$1.08x10^{-5}$	1.0034	0.0034	0.0285
50		MLE	1.1002	0.0002	$2.10x10^{-5}$	1.0060	0.0060	0.0212	1.0999	-0.0001	$3.57x10^{-5}$	1.0036	0.0036	0.0194
		Lindley	1.0986	-0.0014	$3.42x10^{-5}$	1.0046	0.0046	0.0180	1.0998	-0.0002	$5.27x10^{-5}$	0.9985	-0.0015	0.0190
		M-Lindley	1.0987	-0.0013	$3.68x10^{-5}$	1.0044	0.0044	0.0176	1.0998	-0.0002	$5.43x10^{-5}$	0.9987	-0.0013	0.0188
		MCMC	1.1001	0.0001	$6.27x10^{-6}$	1.0033	0.0033	0.0157	1.0999	-0.0001	$1.03x10^{-5}$	1.0030	0.0030	0.0155
100		MLE	1.1000	0.0000	$3.46x10^{-6}$	0.9969	-0.0031	0.0100	1.1001	0.0001	$3.80x10^{-6}$	1.0027	0.0027	0.0107
		Lindley	1.0998	-0.0002	$3.53x10^{-6}$	0.9966	-0.0034	0.0096	1.0999	-0.0001	$4.87x10^{-6}$	1.0024	0.0024	0.0096
		M-Lindley	1.0998	-0.0002	$3.60x10^{-6}$	0.9966	-0.0034	0.0094	1.0999	-0.0001	$4.96x10^{-6}$	1.0022	0.0022	0.0094
		MCMC	1.1000	0.0000	$8.53x10^{-6}$	0.9970	-0.0030	0.0086	1.1000	0.0000	$6.47x10^{-6}$	1.0018	0.0018	0.0069

Table 4. The mean and MSE values for the estimators of parameters a and λ when a=0.90,1.10 and λ=2.

			Prior-I						Prior-II					
			a			λ			a			λ		
n		Method	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
20	$a = 0.9$	MLE	0.8995	-0.0005	$3.52x10^{-4}$	1.9794	-0.0206	0.2006	0.9010	0.0010	$2.91x10^{-4}$	2.0126	0.0126	0.1760
		Lindley	0.8997	-0.0003	$7.99x10^{-4}$	1.9712	-0.0288	0.1990	0.8995	-0.0005	$7.32x10^{-4}$	2.0063	0.0063	0.1743
		M-Lindley	0.8998	-0.0002	$6.94x10^{-4}$	1.9680	-0.0320	0.1991	0.8989	-0.0011	$9.42x10^{-4}$	2.0033	0.0033	0.1740
		MCMC	0.8999	-0.0001	$8.40x10^{-5}$	2.0169	0.0169	0.1473	0.9009	0.0009	$9.29x10^{-5}$	2.0024	0.0024	0.1530
30		MLE	0.8998	-0.0002	$8.86x10^{-5}$	1.9886	-0.0114	0.1291	0.9005	0.0005	$8.48x10^{-5}$	2.0114	0.0114	0.1251
		Lindley	0.8997	-0.0003	$1.44x10^{-4}$	1.9847	-0.0153	0.1288	0.8997	-0.0003	$1.91x10^{-5}$	2.0050	0.0050	0.1244
		M-Lindley	0.8994	-0.0006	$1.65x10^{-4}$	1.9839	-0.0161	0.1275	0.8997	-0.0003	$2.33x10^{-5}$	2.0041	0.0041	0.1243
		MCMC	0.8999	-0.0001	$2.36x10^{-5}$	2.0153	0.0153	0.1153	0.9006	0.0006	$2.70x10^{-5}$	2.0029	0.0029	0.1129
50		MLE	0.9001	0.0001	$2.26x10^{-5}$	2.0047	0.0047	0.0808	0.8999	-0.0001	$2.15x10^{-5}$	1.9945	-0.0055	0.0754
		Lindley	0.8998	-0.0002	$3.09x10^{-5}$	2.0034	0.0034	0.0806	0.8999	-0.0001	$3.42x10^{-5}$	1.9920	-0.0080	0.0752
		M-Lindley	0.8998	-0.0002	$3.29x10^{-5}$	2.0033	0.0033	0.0805	0.8999	-0.0001	$3.59x10^{-5}$	19919	-0.0081	0.0752
		MCMC	0.9000	0.0000	$6.45x10^{-6}$	2.0056	0.0056	0.0607	0.9000	0.0000	$4.72x10^{-5}$	1.9970	-0.0030	0.0686
100		MLE	0.9000	0.0000	$1.96x10^{-6}$	1.9989	-0.0011	0.0404	0.9001	0.0001	$3.08x10^{-6}$	2.0020	0.0020	0.0408
		Lindley	0.8999	-0.0001	$3.02x10^{-6}$	1.9992	-0.0008	0.0400	0.9000	0.0000	$3.83x10^{-6}$	2.0014	0.0014	0.0405
		M-Lindley	0.8999	-0.0001	$3.08x10^{-6}$	1.9993	-0.0007	0.0398	0.9000	0.0000	$3.89x10^{-6}$	2.0013	0.0013	0.0405
		MCMC	0.9000	0.0000	$9.77x10^{-6}$	1.9990	-0.0010	0.0299	0.9000	0.0000	$9.70x10^{-6}$	2.0008	0.0008	0.0389
20	$a = 1.1$	MLE	1.1002	0.0002	$2.56x10^{-4}$	1.9889	-0.0111	0.1844	1.0995	-0.0005	$5.52x10^{-4}$	1.9848	-0.0152	0.2161
		Lindley	1.0990	-0.0010	$6.52x10^{-4}$	1.9790	-0.0210	0.1825	1.0994	-0.0006	$2.56x10^{-3}$	1.9822	-0.0178	0.2115
		M-Lindley	1.0988	-0.0009	$8.80x10^{-4}$	1.9773	-0.0227	0.1826	1.0993	-0.0007	$1.26x10^{-3}$	1.9794	-0.0206	0.2112
		MCMC	1.0993	-0.0007	$8.74x10^{-4}$	2.0121	0.0121	0.1479	1.0999	-0.0001	$1.17x10^{-4}$	2.0163	0.0163	0.1488
30		MLE	1.1002	0.0002	$4.66x10^{-5}$	1.9963	-0.0037	0.1309	1.1003	0.0003	$1.10x10^{-4}$	1.9843	-0.0152	0.1214
		Lindley	1.0997	-0.0003	$2.10x10^{-4}$	1.9966	-0.0034	0.1307	1.0996	-0.0004	$1.52x10^{-4}$	1.9802	-0.0198	0.1204
		M-Lindley	1.0996	-0.0004	$2.75x10^{-4}$	1.9964	-0.0036	0.1306	1.0997	-0.0003	$1.68x10^{-4}$	1.9799	-0.0201	0.1196
		MCMC	1.1002	0.0002	$4.66x10^{-5}$	2.0030	0.0030	0.1250	1.1002	0.0002	$3.30x10^{-4}$	2.0130	0.0130	0.1080
50		MLE	1.1000	0.0000	$3.08x10^{-5}$	2.0011	0.0011	0.0798	1.1000	0.0000	$1.80x10^{-5}$	1.9972	-0.0028	0.0799
		Lindley	1.0954	-0.0046	$5.10x10^{-5}$	1.9984	-0.0016	0.0797	1.0999	-0.0001	$2.94x10^{-5}$	1.9978	-0.0022	0.0796
		M-Lindley	1.0950	-0.0050	$5.36x10^{-5}$	1.9982	-0.0018	0.0795	1.0998	-0.0002	$3.14x10^{-5}$	1.9976	-0.0024	0.0789
		MCMC	1.1000	0.0000	$1.06x10^{-5}$	2.0019	0.0019	0.0609	1.1001	0.0001	$4.96x10^{-5}$	2.0020	0.0020	0.0693

100		MLE	1.1000	0.0000	$6.16x10^{-6}$	1.9976	-0.0024	0.0383	1.1001	0.0001	$3.89x10^{-6}$	2.0053	0.0053	0.0411
		Lindley	1.0999	-0.0001	$8.15x10^{-6}$	1.9971	-0.0029	0.0380	1.0999	-0.0001	$3.37x10^{-6}$	2.0049	0.0049	0.0393
		M-Lindley	1.0998	-0.0002	$8.31x10^{-6}$	1.9970	-0.0030	0.0377	1.0999	-0.0001	$3.43x10^{-6}$	2.0043	0.0043	0.0390
		MCMC	1.1002	0.0002	$6.06x10^{-6}$	1.9988	-0.0012	0.0298	1.1000	0.0000	$6.87x10^{-6}$	2.0040	0.0040	0.0300

The following conclusions can be drawn from the Monte Carlo simulation.

- As the sample size increases, bias and MSE values decrease in most of the cases. This is an expected case because all estimators are asymptotically unbiased and consistent as given in Section 2.
- When the proposed Bayesian Methods and the MLEs are compared, Bayesianmethods have smaller bias and MSE values for estimating parameters a and λ the most of the cases.
- Considering Bayesian methods, the performance of the MCMC approximation method is generally demonstrated slightly better than Lindley and M-Lindley approximation methods in the case. Also, the performances of Lindley and M-Lindley approximation methods under both Prior-I and Prior-II are more or less the same with respect to bias and MSE values in all cases. When Prior-I and Prior-II are compared, Prior-II is somewhat more efficient.

4. Application

In this Section, we take a real data set from the literature to apply the proposed methods. This data set is about coal mining disaster data. It is found in Andrews and Herzberg [20]. The 190 observations in the data set demonstrate the days between successive disasters in Great Britain, see [21]. Moreover, Biçer et al. [10] analyzed the same data set for the Rayleigh distribution. They obtained that the a ration parameter is less than 1 and the data set consists with a GP. They showed with the Z^* test statistic ($Z^* = 1.0049, p - value$ =0.8938) that this data set is appropriate with the Rayleigh distribution. For more details about Z^* test statistic, see Tiku [22]. Additionally we use Anderson-Darling (A-D) test statistic ($A - D = 0.2414, p - value = 0.7368$) to test whether the data set is fit with Rayleigh distribution. These test results on whether the coal mining disaster data set is suitable for the Rayleigh geometric process. Therefore, this data set can be modeled with GP. We have the same result as other authors who analyzed this data set. However, they are only considered classical parameter estimators. In our study, in addition to the classical parameter estimation, the proposed Bayesian parameter estimators are also taken consideration into for the Rayleigh distribution. The MLEs and Bayesian estimators of the parameters a and λ by using MLE, Lindley, M-Lindley, and MCMC approximation methods are given in Table 5.

Table 5. Estimation of parameters the time between failure times of a coal mining disaster data

	Prior-I		Prior-II	
Method	$\hat{a}$	$\hat{\lambda}$	$\hat{a}$	$\hat{\lambda}$
MLE	0.9916	91.2931	0. 9916	91.2931
Lindley	0.9923	92.3196	0.9887	93.3755
M-Lindley	0.9920	91.7736	0.9880	94.8301
MCMC	0.9936	91.0412	1.0102	92.4682

There are some differences among the estimators even though they are close to one another. We select the most appropriate estimators using the simulation results provided in Section 3. We observed in simulation that the estimators obtained by MCMC method outperform the others. For this reason, in these examples, we recommend using MCMC estimators.

5. Results and Discussion

Here, MLEs and Bayesian parameter estimation methods for GP are discussed, assuming that the distribution of the first occurrence time is Rayleigh with the scale parameter λ. As far as we know, this is the first study to compare Bayesian and classical parameter estimation methods. The asymptotic distributions of the MLEs are also constructed. These features are very useful for practitioners. Bayesian estimators under Prior-I and Prior-II based on SELF are considered. In Bayesian computation, Lindley and MCMC approximation methods are used. We also proposed the M-Lindley approximation as an alternative to the Lindley approximation. We conducted a simulation study to evaluate the performance of these methods. It is clear from the simulation study that the M-Lindley approximation is very close to the Lindley approximation in terms of bias and MSE values. Additionally, the MCMC approximation has slightly better performance than the Lindley and M-Lindley approximations in terms of bias and

MSE values. Therefore, considering the computational difficulties, we propose to use the M-Lindley and MCMC approximations instead of the Lindley and MLEs methods.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics

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