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RESEARCH ARTICLE

K-Ricci-Bourguignon Almost Solitons

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In the current article, we introduce and characterize a K-Ricci-Bourguignon almost solitons in perfect fluid spacetimes and generalized Robertson-Walker spacetimes. First, we demonstrate that if a perfect fluid spacetime admits a K-Ricci-Bourguignon almost soliton, then the integral curves produced by the velocity vector field are geodesics and the acceleration vector vanishes. Then we establish that if perfect fluid spacetimes permit a gradient K-Ricci-Bourguignon soliton with Killing velocity vector field, then either state equation of the perfect fluid spacetime is presented by $p = \frac{3-n}{n-1}\sigma$, or the gradient K-Ricci-Bourguignon soliton is shrinking or expanding under some condition. Moreover, we illustrate that the spacetime represents a perfect fluid spacetime and the divergence of the Weyl tensor vanishes if a generalized Robertson-Walker spacetime admits a K-Ricci-Bourguignon almost soliton.

Keywords: *Perfect fluid spacetimes, GRW spacetimes, K Ricci-Bourguignon solitons.*
AMS Subject Classification (2020): *Primary: 83C05; Secondary: 53C50; 53E20; 53Z05.*

1. Introduction

Einstein's "General Relativity" (GR) theory is commonly known as the gravitation theory of geometry. The basic connection between the geometry of spacetime and physics has been established by GR which is a significant physics theories of the last century. Over the past century, GR has been the field of greatest interest in both mathematics and physics. Currently, one of the fascinating problems is attempting to solve Einstein's field equation (EFE) using a variety of methods.

The Minkowski spacetime, or Euclidean space R^4 (4-dimension) with a Lorentzian metric, provides the simplest solution to the aforementioned problem. The Kerr, de-Sitter, and Schwartzchild solutions are additional non-trivial solutions. Warped product Lorentzian manifolds were extensively modified in GR theory in order to obtain a general solution to EFEs. Two notable examples are standard static spacetime and generalized Robertson-Walker spacetime (GRW).

According to GR, a spacetime is a Lorentzian manifold $\mathcal{M}^4$ with the signature $(+, +, +, -)$ for the metric g and admits a globally time-oriented vector. The idea of GRW spacetimes was created by Alias et al. [1]. The Lorentzian manifold $\{\mathcal{M}^n\ , n \geq 3\}$ is called a GRW spacetime if it is constructed as $\mathcal{M} = -I \times \varrho^2 \mathcal{M}^*$. Here, $I \subset \mathbb{R}$, $\mathcal{M}^*$ represents a Riemannian manifold of dimension $(n-1)$, and $\rho > 0$ is a scalar. It is claimed that ρ is a scale factor or warping function. In the case where $\mathcal{M}^*$ has dimension three and of constant curvature, the GRW spacetime reduces to a Robertson-Walker (RW) spacetime. The features of GRW spacetimes are found in ([9], [10], [24]). The following theorem has been demonstrated by Mantica and Molinari [24].

Theorem A.([24]) A Lorentzian manifold $\{\mathcal{M}^n\ , n \geq 3\}$ is a GRW spacetime if and only if it admits a unit time-like torse-forming vector field: $\nabla_F u = \Psi[F + A(F)u]$, the one-form A is defined by $g(F, u) = A(F)$ for any F which is also an eigenvector of the Ricci tensor.

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In a perfect fluid (PF) spacetime the non-vanishing Ricci tensor S has the shape

$$S = a_1 g + b_1 \eta \otimes \eta, \tag{1.1}$$

where a_1 and b_1 represent scalars, ϱ is described by $g(F,\varrho) = \eta(F)$ for any F. Also, ϱ is a unit time-like vector field of the PF spacetime that is, $g(\varrho,\varrho) = -1$.

Conversely, the Weyl tensor is important in relativity theory and geometry. Weyl tensors have been used by several researchers to characterise spacetimes. Weyl or, conformal curvature tensor C is defined as

$$\begin{aligned} C(F,G)H &= R(F,G)H \\ &\quad -\frac{1}{n-2}[g(QG,H)F - g(QF,H)G \\ &\quad +g(G,H)QF - g(F,H)QG] \\ &\quad +\frac{r}{(n-1)(n-2)}[g(G,H)F - g(F,H)G], \end{aligned}$$

in which R stands for the Riemann curvature tensor, r indicates the scalar curvature and the Ricci operator Q is presented by $g(QF,G) = S(F,G)$.

Furthermore, we see that

$$\begin{aligned} (divC)(F,G)H &= \frac{n-3}{n-2}[\{(\nabla_F S)(G,H) - (\nabla_G S)(F,H)\} \\ &\quad -\frac{1}{2(n-1)}\{(Fr)g(G,H) - (Gr)g(F,H)\}], \end{aligned} \tag{1.2}$$

'div' denotes the divergence.

Now, we state some theorems which are used in our article later on.

Theorem([20]) For a warped product $\mathcal{M} = -I \times \rho^2 \mathcal{M}^*$, the fibers are Einstein if and only if $div\, C = 0$.

Theorem([26]) A GRW spacetime $\mathcal{M}$ is a PF spacetime if and only if $\mathcal{M}^*$ is an Einstein manifold.

Previous two theorems jointly reveal that

Theorem B. A GRW spacetime is a PF spacetime if and only if $div\, C = 0$.

Theorem C.([25]) In a GRW spacetime $(div\, C)(F,G)H = 0$ if and only if

$$C(F,G)\varrho = 0$$

.

To determine EFEs, spacetime symmetries must be studied. Symmetry is a defining feature of geometry that makes the physics clear. There are various symmetries in the geometry of spacetime and matter. In GR, they are primarily used to classify solutions to EFEs. One kind of symmetry that incorporates the geometric flow is called a soliton. In order to address the challenge of identifying a canonical metric on a smooth manifold, Hamilton [21] introduces the Ricci and Yamabe solitons, which are subsequently examined in ([11], [12]).

A Riemannian manifold $(\mathcal{M}^n, g)$ is said to evolve by the Ricci-Bourguignon (RB) flow if g(t), a family of metrices obeys the subsequent equation

$$\frac{\partial g}{\partial t} = -2S - 2\beta_1 r g, \tag{1.3}$$

in which S and r denote the Ricci tensor and scalar curvature respectively, and β_1 is a real constant. Jean-Pierre Bourguignon [5], depending on some unpublished work by Lichnerowicz and a publication by Aubin [2], was the first to define the flow in the previous equation.

We define the following, analogous to the Ricci flow scenario:

Definition 1.1. A RB soliton [7] on a semi-Riemannian manifold $(\mathcal{M}, g)$ is described by

$$\pounds_V g + 2S + 2(\lambda_1 + \beta_1 r)g = 0, \tag{1.4}$$

in which $\pounds$ stands for Lie-derivative and $\lambda_1 \in \mathbb{R}$.

Here, we introduce a novel soliton, named K-Ricci-Bourguignon almost (K-RBA) soliton which is a generalization of RB soliton, described by

$$K\pounds_V g + 2S + 2\lambda_1 g + 2\beta_1 r g = 0, \tag{1.5}$$

where K, λ_1 and β_1 are scalars.

We notice that, the equation (1.5) is of specific importance for numerous values of β_1, for instance

- K-Ricci almost soliton, for $\beta_1 = 0$.
- K-Einstein almost soliton, for $\beta_1 = \frac{1}{2}$.
- K-Schouten almost soliton, for $\beta_1 = \frac{1}{2(n-1)}$.

If $V = Df$, then the above stated notion is named gradient K-RBA soliton and equation (1.5) takes the shape

$$K\nabla^2 f + S + \lambda_1 g + \beta_1 r g = 0. \tag{1.6}$$

For $\lambda_1 < 0$, $\lambda_1 = 0$ or $\lambda_1 > 0$ the K-RBA soliton (or, gradient K-RBA soliton) is called shrinking, steady or expanding, respectively.

It is fascinating to record that for a particular f it is feasible to transform any K-RBA soliton to an m-quasi Einstein generalized metric. For example, if we choose $\beta_1 = 0$, $u = e^{\frac{f}{m}}$ and $K = -\frac{m}{u}$, then equation (1.6) turns into

$$\nabla^2 f + S = \frac{1}{m} df \otimes df - \lambda_1 g. \tag{1.7}$$

Therefore, K-RBA soliton generalizes m-quasi Einstein generalized metric and also it covers Ricci solitons and almost Ricci solitons of gradient type.

Because of their connection to GR, there was a notable increase of quest in researching Ricci solitons and related generalizations in a variety of geometrical contexts. Many researchers have investigated many sorts of solitons in PF spacetimes including Ricci and gradient type Ricci solitons ([17], [18]), η-Ricci solitons [4], Yamabe and gradient type Yamabe solitons [17], k-almost Yamabe solitons [15], η-Einstein solitons of gradient type [18], gradient ϱ-Einstein solitons [13], m-quasi Einstein solitons of gradient type [17], gradient Schouten solitons [18], Ricci-Yamabe solitons [14], respectively.

The research mentioned above motivate us to introduce K-RBS solitons and explore K-RBA solitons in PF spacetimes and GRW spacetimes. Specifically, we arrive at the following conclusions:

Theorem 1.1. *If a PF spacetime admits a K-RBA soliton, then the acceleration vector vanishes and the integral curves produced by the velocity vector field ϱ are geodesics.*

Theorem 1.2. *If a PF spacetime admits a gradient K-RBA soliton with Killing velocity vector field ϱ and K and λ_1 are invariant under ϱ, then either the equation of state of the PF spacetime is represented by $p = \frac{3-n}{n-1}\sigma$, or the gradient K-RBA soliton is shrinking or expanding for $b_1 < a_1$ or $b_1 > a_1$, respectively.*

Theorem 1.3. *Let a GRW spacetime admit a K-RBA soliton. Then the spacetime becomes a PF-spacetime and the divergence of the Weyl tensor vanishes. Also, the soliton is expanding for $r < \frac{(1-n)\mu_1}{\beta_1}$, steady if $r = \frac{(1-n)\mu_1}{\beta_1}$ and shrinking for $r > \frac{(1-n)\mu_1}{\beta_1}$.*

Corollary 1.1. *In dimension 4, a GRW spacetime admitting a K-RBA soliton is of Petrov type I, D or O and the spacetime reduces to a RW spacetime.*

2. Perfect fluid spacetimes and GRW spacetimes

The PF equation (1.1) provides

$$QF = a_1 F + b_1 \eta(F)\varrho, \tag{2.1}$$

in which Q indicates the Ricci operator described by $g(QF, G) = S(F, G)$ and contracting the above equation gives

$$r = \sum_j \epsilon_i Q e_j = n a_1 - b_1, \tag{2.2}$$

in which at every point of the spacetime, $\{e_j\}$ denotes the orthonormal basis of the tangent space and $\epsilon_j = g(e_j, e_j) = \pm 1$. The covariant differentiation of equation (2.1) yields

$$(\nabla_F Q)(G) = (Fa_1)G + (Fb_1)\eta(G)\varrho + b_1(\nabla_F \eta)(G)\varrho + b_1\eta(G)\nabla_F \varrho. \tag{2.3}$$

For a gravitational constant k, the EFEs without a cosmological constant is given by

$$S - \frac{r}{2}g = kT, \tag{2.4}$$

in which the energy momentum tensor is denoted by T.

For a PF spacetime if p denotes the isotropic pressure and σ indicates the energy density, then T is described by

$$T = pg + (p+\sigma)\eta \otimes \eta. \tag{2.5}$$

The equations (1.1), (2.4) and (2.5) together provide

$$a_1 = \frac{k(p-\sigma)}{2-n}, \quad b_1 = \kappa(p+\sigma). \tag{2.6}$$

Also, an equation of state (EOS) with the form $p = p(\sigma)$ is known as stiff matter and $p = \frac{\sigma}{3}$ is known as the radiation era. If $p = 0$ and $p + \sigma = 0$, then the PF spacetime represents the dust matter fluid and the dark energy era [8]. Moreover, it includes the phantom era when $\frac{p}{\sigma} < -1$.

Let the potential vector field ϱ be a unit torse-forming vector field. Hence, making use of Theorem A, we infer

$$\nabla_F \varrho = \Psi[F + \eta(F)\varrho] \tag{2.7}$$

and

$$S(F, \varrho) = \phi\eta(F), \tag{2.8}$$

where ϕ indicates a non-zero eigenvalue and Ψ stands for a scalar.

Proposition 2.1. *For any GRW spacetime, we can write [16]*

$$R(F, G)\varrho = \mu_1[\eta(G)F - \eta(F)G] \tag{2.9}$$

and

$$S(F, \varrho) = (n-1)\mu_1\eta(F), \tag{2.10}$$

where we set $\mu_1 = (\varrho\Psi + \Psi^2)$.

3. Proof of the Main Results

Proof of the Theorem 1.1. Let the PF spacetime admit a K-RBA soliton. Then equation (1.5) infers

$$K(\pounds_V g)(F, G) + 2S(F, G) + 2(\lambda_1 + \beta_1 r)g(F, G) = 0, \tag{3.1}$$

which implies

$$S(F, G) = -\frac{K}{2}[g(\nabla_F \varrho, G) + g(F, \nabla_G \varrho)] - (\lambda_1 + \beta_1 r)g(F, G). \tag{3.2}$$

Contracting the previous equation yields

$$r = -K div\varrho - 4(\lambda_1 + \beta_1 r). \tag{3.3}$$

Again, contracting the equation (1.1) gives

$$r = -a_1 + 4b_1. \tag{3.4}$$

From equations (3.3) and (3.4), we provide

$$(1 + 4\beta_1)(-a_1 + 4b_1) = -K div\varrho - 4\lambda_1. \tag{3.5}$$

Also, from equations (1.1) and (3.2), we infer

$$a_1 g(F,G) + b_1\eta(F)\eta(G) = -\frac{K}{2}[g(\nabla_F \varrho, G) + g(F, \nabla_G \varrho)] - (\lambda_1 + \beta_1 r)g(F,G). \tag{3.6}$$

Putting $F = G = \varrho$ in equation (3.6), we get

$$a_1 - b_1 = -(\lambda_1 + \beta_1 r). \tag{3.7}$$

Using equation (3.7), in equation (3.6), we acquire

$$a_1 g(F,G) + b_1\eta(F)\eta(G) = -\frac{K}{2}[g(\nabla_F \varrho, G) + g(F, \nabla_G \varrho)] + (a_1 - b_1)g(F,G). \tag{3.8}$$

Putting $F = \varrho$, the foregoing equation yields $\nabla_\varrho \varrho = 0$. Also, $\nabla_\varrho \varrho = 0$ means that the integral curves of the velocity vector are geodesics. Further, the acceleration vector is represented by $\nabla_\varrho \varrho$ [3].

Thus the proof is finished.

If $div\varrho = 0$, then the equations (3.5) and (3.7) together reveal

$$\lambda_1 = -\frac{(1+4\beta_1)}{(4\beta_1 - 5)}[3b_1 + \beta_1 r]. \tag{3.9}$$

Corollary 3.1. *If a PF spacetime admits a K-RBA soliton, then the soliton is shrinking for $r < \frac{3b_1}{\beta_1}$, steady if $r = \frac{3b_1}{\beta_1}$ and expanding for $r > \frac{3b_1}{\beta_1}$, provided that the velocity vector is divergence-free.*

Corollary 3.2. *If a PF spacetime admits a K-Ricci almost soliton, then the soliton is expanding for $b_1 < a_1$, steady if $b_1 = a_1$ and shrinking for $b_1 > a_1$.*

Proof. In particular, if we take $\beta_1 = 0$, then the equation (3.7) implies

$$\lambda_1 = a_1 - b_1.$$

Thus, the soliton is expanding for $b_1 < a_1$, steady if $b_1 = a_1$ and shrinking for $b_1 > a_1$.

Hence the result follows.

□

Remark 3.1. The foregoing corollary has been established in [18] by considering a Ricci soliton.

Proof of the Theorem 1.2. Let the PF spacetime admit a RB soliton of gradient type and therefore from the equation (1.6), we acquire

$$K\nabla_F Df + QF = -(\lambda_1 + \beta_1 r)F. \tag{3.10}$$

Differentiating the equation (3.10), we get

$$\begin{aligned} K\nabla_G\nabla_F Df &= -\frac{1}{K}(GK)\{(\lambda_1 + \beta_1 r)F + QF\} \\ &\quad -\nabla_G QF - (\lambda_1 + \beta_1 r)\nabla_G F \\ &\quad -\beta_1 (Gr)F - (G\lambda_1)F. \end{aligned} \tag{3.11}$$

Interchanging F and G, we provide

$$\begin{aligned} K\nabla_F\nabla_G Df &= -\frac{1}{K}(FK)\{(\lambda_1 + \beta_1 r)G + QG\} \\ &\quad -\nabla_F QG - (\lambda_1 + \beta_1 r)\nabla_F G \\ &\quad -\beta_1 (Fr)G - (F\lambda_1)G. \end{aligned} \tag{3.12}$$

Again, from the equation (3.10), we infer

$$K\nabla_{[F,G]}Df = -Q([F,G]) - (\lambda_1 + \beta_1 r)[F,G]. \tag{3.13}$$

From the equations (3.11), (3.12) and (3.13), we reveal

$$\begin{aligned} KR(F,G)Df &= -\frac{1}{K}(FK)\{(\lambda_1+\beta_1 r)G+QG\} \\ &+\frac{1}{K}(GK)\{(\lambda_1+\beta_1 r)F+QF\} \\ &-(\nabla_F Q)G+(\nabla_G Q)F-\beta_1[(Fr)G-(Gr)F] \\ &-\{(F\lambda_1)G-(G\lambda_1)F\}. \end{aligned} \tag{3.14}$$

The covariant differentiation of the equation (2.1) yields

$$(\nabla_F Q)(G)=(Fa_1)G+(Fb_1)\eta(G)\varrho+b_1(\nabla_F\eta)(G)\varrho+b_1\eta(G)\nabla_F\varrho. \tag{3.15}$$

Using the equation (3.15) in (3.14), we acquire

$$\begin{aligned} KR(F,G)Df &= -\frac{1}{K}(FK)\{(\lambda_1+\beta_1 r)G+QG\} \\ &+\frac{1}{K}(GK)\{(\lambda_1+\beta_1 r)F+QF\} \\ &-(Fa_1)G+(Ga_1)F-\{(Fb_1)\eta(G)-(Gb_1)\eta(F) \\ &+b_1(\nabla_F\eta)(G)-b_1(\nabla_G\eta)(F)\}\varrho+b_1\{\eta(G)\nabla_F\varrho-\eta(F)\nabla_G\varrho\} \\ &-\beta_1[(Fr)G-(Gr)F]-\{(F\lambda_1)G-(G\lambda_1)F\}. \end{aligned} \tag{3.16}$$

Now contracting the equation (3.16), we provide

$$\begin{aligned} S(G,Df) &= -\frac{1}{K}(1-n)(\lambda_1+\beta_1 r)(GK)-\frac{1}{K}a_1(GK)-\frac{1}{K}b_1\eta(G)(\varrho K) \\ &-(1-n)(Ga_1)-(Gb_1)-(\varrho b_1)\eta(G) \\ &-b_1[(\nabla_\varrho\eta)(G)-(\nabla_G\eta)(\varrho)+\eta(G)\,div\varrho] \\ &-\beta_1(1-n)(Gr)-(1-n)(G\lambda_1). \end{aligned} \tag{3.17}$$

Also the PF equation (1.1) gives

$$S(G,Df)=a_1(Gf)+b_1\eta(G)(\varrho f). \tag{3.18}$$

Putting $G=\varrho$ in equations (3.17) and (3.18) and then comparing, we reveal

$$\begin{aligned} K(a_1-b_1)(\varrho f) &= -\frac{1}{K}\{(1-n)(\lambda_1+\beta_1 r)+a_1-b_1\}(\varrho K) \\ &-(1-n)(\varrho a_1)+b_1\,div\varrho-\beta_1(1-n)(\varrho r) \\ &-(1-n)(\varrho\lambda_1). \end{aligned} \tag{3.19}$$

Let K and λ_1 are invariant under ϱ and ϱ be Killing, therefore we acquire (see, [19], p. 89), $\pounds_\varrho p=0$ and $\pounds_\varrho\sigma=0$. It is known that $a_1=\frac{k(p-\sigma)}{n-2}$ and $b_1=k(p+\sigma)$. Thus, we infer

$$(\varrho a_1)=(\varrho b_1)=0.$$

Again, from (2.2) we obtain

$$r=na_1-b_1.$$

Hence, we get $(\varrho r)=0$. Because of the hypothesis ϱ is Killing, then $\mathrm{div}\varrho=0$.

Thus, using the foregoing result the equation (3.19) yields

$$(a_1-b_1)(\varrho f)=0, \tag{3.20}$$

since $K\neq 0$.

This reflects that either $a_1=b_1$ or $(\varrho f)=0$ on a PF spacetime with the gradient K-RBA soliton. Here, we consider the following two cases:

Case (i): Let $a_1=b_1$ and $(\varrho f)\neq 0$ and hence the equation (1.6) gives

$$p=\frac{3-n}{n-1}\sigma,$$

which provides the EOS in a PF spacetime equipped with a gradient K-RBA soliton.

Case (ii): Let $(\varrho f) = 0$ and $a_1 \neq b_1$. The covariant differentiation of $g(\varrho, Df) = 0$ produces

$$g(\nabla_F \varrho, Df) = -[\lambda_1 + (a_1 - b_1)]\eta(F), \tag{3.21}$$

in which we have used equations (2.1) and (3.10). Since here ϱ is Killing, we infer $g(\nabla_F \varrho, G) + g(F, \nabla_G \varrho) = 0$. Now putting $G = \varrho$ in the last relation, we acquire that $g(F, \nabla_\varrho \varrho) = 0$, since $g(\nabla_F \varrho, \varrho) = 0$. Therefore, we state that $\nabla_\varrho \varrho = 0$. Using the previous relation, putting $F = \varrho$ in equation (3.21), we find that

$$\lambda_1 = b_1 - a_1, \tag{3.22}$$

which shows that the K-RBA soliton of gradient type in a PF spacetime is expanding or shrinking if $b_1 > a_1$ or $b_1 < a_1$, respectively.

This finishes the proof.

For $n = 4$, the EOS is $3p + \sigma = 0$, which entails that the PF spacetime represents phantom era.

Hence, we write

Corollary 3.3. *In dimension 4, if the PF spacetimes permit a gradient K-RBA soliton with Killing velocity vector field ϱ and K and λ_1 are invariant under ϱ, then either the PF spacetime represents phantom era or the gradient K-RBA soliton is shrinking or expanding for $b_1 < a_1$ or $b_1 > a_1$, respectively.*

Proof of the Theorem 1.3.

Let the GRW spacetime permit a K-RBA soliton and hence the equation (1.5) infers

$$K(\pounds_V g)(F, G) + 2S(F, G) + 2(\lambda_1 + \beta_1 r)g(F, G) = 0, \tag{3.23}$$

which implies

$$\begin{aligned} &K\{g(\nabla_F \varrho, G) + g(F, \nabla_G \varrho)\} \\ &+2S(F, G) + 2(\lambda_1 + \beta_1 r)g(F, G) = 0. \end{aligned} \tag{3.24}$$

Using the equation (2.7) in equation (3.24), we acquire

$$S(F, G) = -\{(\lambda_1 + \beta_1 r + K\Psi\}g(F, G) - K\Psi\eta(F)\eta(G), \tag{3.25}$$

which represents PF spacetime.

Hence, Theorem B infers that $div\, C = 0$.

Putting $F = G = \varrho$ in the equation (3.25) yields

$$\lambda_1 = -(n-1)\mu_1 - \beta_1 r.$$

Therefore, the soliton is expanding for $r < \frac{(1-n)\mu_1}{\beta_1}$, steady if $r = \frac{(1-n)\mu_1}{\beta_1}$ and shrinking for $r > \frac{(1-n)\mu_1}{\beta_1}$.

This completes the proof.

Proof of the Corollary 1.1. With the help of Theorem C, we see that in a GRW spacetime, $(div\, C)(F, G)H = 0$ if and only if $C(F, G)\varrho = 0$. Moreover, $C(F, G)\varrho = 0$ entails that the Weyl tensor is purely electric [22]. In dimension 4, the spacetimes are of Petrov types I, D or O if C is purely electric ([27], p. 73).

For dimension 4 ([23], p. 128), $C(F, G)\varrho = 0$ is identical to

$$\begin{aligned} &\eta(U)C(F, G, H, E) + \eta(F)C(G, U, H, E) \\ &+\eta(G)C(U, F, H, E) = 0, \end{aligned} \tag{3.26}$$

in which $\eta(F) = g(F, \varrho)$ and $C(F, G, H, E) = g(C(F, G)H, E)$ for any F, G, H, E, U.

Now, replacing U by ϱ yields

$$C(F, G, H, E) = 0. \tag{3.27}$$

Therefore, it represents a conformally flat spacetime.

A GRW spacetime has been found to be conformally flat if and only it is a RW spacetime [6].

Thus, we have the proof.

By considering a special case, we acquire:

Corollary 3.4. *The GRW spacetime permitting a K-Ricci almost soliton becomes a PF spacetime and the Weyl tensor is divergence-free. Also, the soliton is expanding for $\mu_1 < 0$, steady if $\mu_1 = 0$ and shrinking for $\mu_1 > 0$.*

Proof. In particular, if we take $\beta_1 = 0$, then the equation (3.25) implies

$$S(F, G) = -\{(\lambda_1 + K\Psi\}g(F, G) - K\Psi\eta(F)\eta(G),$$

which tells us that it is a PF spacetime and hence from Theorem B, we conclude that $div\, C = 0$.

Setting $F = G = \varrho$, the above equation produces

$$\lambda_1 = -(n-1)\mu_1.$$

Thus, the soliton is expanding for $\mu_1 < 0$, steady if $\mu_1 = 0$ and shrinking for $\mu_1 > 0$.

Therefore, the corollary follows. □

Discussions

Currently, spacetime, a torsion-free globally time-oriented Lorentzian manifold, is the stage of forecasting models used for the physical world. According to GR theory, the matter content of the Universe can be found by applying the appropriate energy-momentum tensor, which is recognised to behave in cosmological models like a PF spacetime. A PF is the most basic type of fluid, lacking of the ability to transfer heat. Since a perfect fluid has no viscosity, it is unable to resist a tangential force even when it is flowing. Perfect fluids are used in general relativity to simulate idealized matter distributions, like those found inside stars or in an isotropic universe. Modelling large-scale cosmology, GRW spacetimes are an intrinsic and natural extension of RW spacetimes.

In this research, we demonstrate that if the metric of a PF spacetime admits a K-RBA soliton, then the acceleration vector vanishes and the integral curves produced by the velocity vector field are geodesics. We also find the circumstances in which the K-RBA solitons and gradient K-RBA solitons in a PF spacetime are expanding, stable, or shrinking. Furthermore, we deduce that the spacetime represents a PF spacetime and the Weyl tensor is divergence-free if a GRW spacetime admits a K-RBA soliton.

In future, we or perhaps other geometers will further examine the characteristic of these solitons in cosmology and GR theory.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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