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Research article

Maximal Antipodal Subgroups and Covering Homomorphisms with Odd Degree

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

Abstract

We show that all of the maximal antipodal subgroups in compact Lie groups, which are not necessarily connected, do not change through covering homomorphisms with odd degree.

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1. Maximal antipodal subgroups

It is known that a compact Lie group is a Riemannian symmetric space with respect to a bi-invariant Riemannian metric. We define an antipodal set of a compact Riemannian symmetric space and related notions.

Let M be a compact Riemannian symmetric space. A subset A in M is called an *antipodal set* if $s_x(y) = y$ holds for any $x, y \in A$, where s_x denotes the geodesic symmetry at x. Since an antipodal set is a discrete subset in a compact Hausdorff space, it is finite. The maximum of the cardinalities of antipodal sets of M is called the *2-number* of M denoted by $\#_2 M$. If the cardinality of an antipodal set A attains $\#_2 M$, A is called a *great* antipodal set. A great antipodal set is a maximal antipodal set. However, the converse is not true. Chen-Nagano introduced these notions and determined the 2-numbers of most but not all compact Riemannian symmetric spaces in [2]. The authors obtained classifications of maximal antipodal sets and great antipodal sets in many cases in [3, 4, 5, 6]. It has been shown that maximal antipodal sets are related to several important areas in mathematics in previous studies by many researchers (see [1]), and it is a natural problem to classify maximal antipodal sets. We can find that maximal antipodal subgroups of a compact Lie group G and those of the quotient groups of G are quite different in general. For example, maximal antipodal subgroups of $SU(4)$ are unique up to conjugation but those of $SU(4)/\mathbb{Z}_4$ are not unique up to conjugation (see Section 5 in [3]).

Let G be a compact Lie group equipped with bi-invariant Riemannian metric. The geodesic symmetry s_x at $x \in G$ is given by $s_x(y) = xy^{-1}x$ for $y \in G$. In particular, for the identity element e of G we have $s_e(y) = y^{-1}$.

Lemma 1.1 (Lemma 3.1 in [5]). *Let G be a compact Lie group. We have the following for any $x, y \in G$.*

(1) $s_e(x) = x$ if and only if $x^2 = e$.

(2) If $x^2 = y^2 = e$, $s_x(y) = y$ if and only if $xy = yx$.

Lemma 1.2 (Lemma 3.2 in [5]). *Let G be a compact Lie group. If a maximal antipodal set $A \subset G$ satisfies $e \in A$, then A is an abelian subgroup of G which is isomorphic to a product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of some copies of $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ denotes the cyclic group of order 2.*

According to this lemma a maximal antipodal set of G containing e is a subgroup, which we call a maximal antipodal subgroup.

Let Z be the center of a compact Lie group G and let Z' be a discrete subgroup of Z. Then the quotient group $G' := G/Z'$ is a compact Lie group locally isomorphic to G. Let $\pi : G \to G'$ be the natural projection which is a covering homomorphism whose kernel $\ker \pi = Z'$. Conversely, any covering homomorphism from G is obtained like that. We have $\pi \circ s_x = s_{\pi(x)} \circ \pi$ for any $x \in G$.

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2. Covering homomorphisms with odd degree

We discussed the relation between covering homomorphisms $\pi : G \to G'$ with odd degree and the correspondence between maximal antipodal subgroups of G and those of G' in Section 3 in [5]. We proved Proposition 3.4 in [5] under the assumption that G is connected. In this section we show that the same assertion holds even if G is not necessarily connected.

We need Theorem 2.1 and Lemma 2.1 in order to prove Theorem 2.2.

Theorem 2.1 (Sylow). *Let G be a finite group with cardinality $p^n m$ for a prime p, where p and m are mutually prime. Then G has a subgroup with cardinality p^n, which is called a p-Sylow subgroup. Any two p-Sylow subgroups of G are conjugate. Any subgroup whose cardinality is a power of p is included in a p-Sylow subgroup.*

Lemma 2.1 (Lemma 3.3 in [5]). *Let G, G' be compact Lie groups and let $\pi : G \to G'$ be a covering homomorphism whose covering degree is odd. If A' is an antipodal subgroup of G', then there exists an antipodal subgroup B of G which satisfies the following conditions.*

(1) B is a 2-Sylow subgroup of $\pi^{-1}(A')$ such that $|B| = |A'|$.

(2) The restriction of π to B is an isomorphism from B onto A'.

Theorem 2.2. *Let G, G' be compact Lie groups and let $\pi : G \to G'$ be a covering homomorphism with odd degree. We denote by G_0, G'_0 the identity components of G, G' respectively.*

(1) If A is an antipodal subgroup of G, $\pi(A)$ is an antipodal subgroup of G'. If A is a maximal antipodal subgroup of G, $\pi(A)$ is a maximal antipodal subgroup of G'. If maximal antipodal subgroups A_1, A_2 of G are conjugate, maximal antipodal subgroups $\pi(A_1), \pi(A_2)$ of G' are conjugate. If maximal antipodal subgroups A_1, A_2 of G are G_0-conjugate, maximal antipodal subgroups $\pi(A_1), \pi(A_2)$ of G' are G'_0-conjugate.

(2) If A' is an antipodal subgroup of G', there exists an antipodal subgroup A of G such that $\pi|_A : A \to A'$ is an isomorphism. If A' is a maximal antipodal subgroup of G', there exists a maximal antipodal subgroup A of G such that $\pi|_A : A \to A'$ is an isomorphism. If maximal antipodal subgroups A'_1, A'_2 of G' are conjugate, maximal antipodal subgroups A_1, A_2 of G are conjugate, where $\pi|_{A_i} : A_i \to A'_i$ is an isomorphism $(i = 1, 2)$. Furthermore, if G_0 contains $\ker \pi$, the following holds: if maximal antipodal subgroups A'_1, A'_2 of G' are G'_0-conjugate, maximal antipodal subgroups A_1, A_2 of G are G_0-conjugate, where $\pi|_{A_i} : A_i \to A'_i$ is an isomorphism $(i = 1, 2)$.

Corollary 2.1. *The statements of Theorem 2.2 hold even if we change the words "maximal antipodal subgroup" to "great antipodal subgroup".*

Proof of Theorem 2.2. (1) It is clear that the image $\pi(A)$ is an antipodal subgroup of G' if A is an antipodal subgroup of G. Next we consider the case where A is a maximal antipodal subgroup of G. We set $Z' = \ker \pi$. Since we know $\pi(A)$ is an antipodal subgroup of G', we show $\pi(A)$ is a maximal antipodal subgroup of G'. We assume that an antipodal subgroup A' of G' satisfies $\pi(A) \subset A'$. By Lemma 2.1 there exists an antipodal subgroup B of G such that $\pi|_B : B \to A'$ is an isomorphism and B is a 2-Sylow subgroup of $\tilde{A} = \pi^{-1}(A')$. Since $B \subset \tilde{A}$ and $Z' \subset \tilde{A}$, we have $BZ' \subset \tilde{A}$. Since $|B|$ is a power of 2 and $|Z'|$ is odd, $B \cap Z'$ consists of the identity element. Hence, an element of BZ' is uniquely described as a product of an element of B and an element of Z'. Thus we have

$$|BZ'| = |B| \cdot |Z'| = |A'| \cdot |Z'| = |\pi^{-1}(A')| = |\tilde{A}|.$$

Since $BZ' \subset \tilde{A}$, we obtain $BZ' = \tilde{A}$. Because of $\pi(A) \subset A'$ we have $A \subset \pi^{-1}(A') = \tilde{A}$. By Theorem 2.1 there exists $g \in \tilde{A}$ such that $gAg^{-1} \subset B$. Since A is a maximal antipodal subgroup of G, we obtain $gAg^{-1} = B$. Therefore, we have $\pi(A) \subset A' = \pi(B) = \pi(gAg^{-1})$ and we obtain $\pi(A) = A'$, since $|\pi(A)| = |\pi(gAg^{-1})|$. Hence, $\pi(A)$ is a maximal antipodal subgroup of G'.

If maximal antipodal subgroups A_1, A_2 of G are conjugate by $g \in G$, that is, $A_2 = gA_1g^{-1}$, we have $\pi(A_2) = \pi(g)\pi(A_1)\pi(g)^{-1}$. Hence, $\pi(A_1)$ and $\pi(A_2)$ are conjugate. If $g \in G_0$, $\pi(A_1)$ and $\pi(A_2)$ are G'_0-conjugate, since $\pi(g) \in \pi(G_0) = G'_0$.

(2) By Lemma 2.1 if A' is an antipodal subgroup of G', there exists an antipodal subgroup A of G such that $\pi|_A : A \to A'$ is an isomorphism and A is a 2-Sylow subgroup of $\pi^{-1}(A')$. Next we consider the case where A' is a maximal antipodal subgroup of G'. Since we know there exists an antipodal subgroup B of G such that $\pi|_B : B \to A'$ is an isomorphism, we show B is a maximal antipodal subgroup of G. Let C be an antipodal subgroup of G satisfying $B \subset C$. Since $|\ker \pi|$ is odd and $|C|$ is a power of 2, $\ker \pi \cap C$ consists of the identity

element. Thus $\pi|_C : C \to \pi(C)$ is injective, hence it is an isomorphism. Therefore $\pi(C)$ is an antipodal subgroup of G'. Because of $B \subset C$ we obtain $A' = \pi(B) \subset \pi(C)$. Since A' is a maximal antipodal subgroup of G', we get $\pi(B) = \pi(C)$, hence $B = C$. Thus B is a maximal antipodal subgroup of G.

Next, we assume that maximal antipodal subgroups A'_1, A'_2 of G' are conjugate. Then there exists a maximal antipodal subgroup A_i of G such that $\pi|_{A_i} : A_i \to A'_i$ is an isomorphism $(i = 1, 2)$. By the assumption there exists $g' \in G'$ satisfying $A'_2 = g'A'_1(g')^{-1}$. Then we have $\pi^{-1}(A'_2) = \pi^{-1}(g'A'_1(g')^{-1})$. For $g \in G$ satisfying $\pi(g) = g'$, we obtain

$$\begin{aligned}
\pi^{-1}(g'A'_1(g')^{-1}) &= \{x \in G \mid \pi(x) \in g'A'_1(g')^{-1}\} \\
&= \{x \in G \mid (g')^{-1}\pi(x)g' \in A'_1\} \\
&= \{x \in G \mid \pi(g^{-1}xg) \in A'_1\} \\
&= \{x \in G \mid g^{-1}xg \in \pi^{-1}(A'_1)\} \\
&= \{x \in G \mid x \in g\pi^{-1}(A'_1)g^{-1}\} \\
&= g\pi^{-1}(A'_1)g^{-1},
\end{aligned}$$

which implies $\pi^{-1}(A'_2) = g\pi^{-1}(A'_1)g^{-1}$. Since A_1 is a 2-Sylow subgroup of $\pi^{-1}(A'_1)$, gA_1g^{-1} is a 2-Sylow subgroup of $\pi^{-1}(A'_2) = g\pi^{-1}(A'_1)g^{-1}$. By Theorem 2.1 gA_1g^{-1} is conjugate to A_2 by an element of $\pi^{-1}(A'_2)$. In particular, A_1 is conjugate to A_2 by an element of G.

Furthermore we assume that G_0 contains $\ker \pi$. If $g' \in G'_0$, we can take $g \in G_0$ satisfying $\pi(g) = g'$. Since $|\ker \pi|$ is odd and $|A_2|$ is a power of 2, $\ker \pi \cap A_2$ consists of the identity element. An element of $\pi^{-1}(A'_2)$ is uniquely described as a product az of $a \in A_2$ and $z \in \ker \pi$. Therefore we have $azgA_1g^{-1}(az)^{-1} = A_2$. Thus we have $zgA_1g^{-1}z^{-1} = a^{-1}A_2a = A_2$. Hence, A_1 and A_2 are G_0-conjugate. □

This result is a refinement of the following result obtained by Chen-Nagano [2] in the case of compact Lie groups.

Proposition 2.1 (Proposition 3.1 in [2]). *One has $\#_2M' = \#_2M$, if there exists a k-fold covering morphism $f : M' \to M$ between compact Riemannian symmetric spaces and k is odd.*

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

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