Ordu Üniversitesi Bilim ve Teknoloji Dergisi
Ordu Üniv. Bil. Tek. Derg.

Ordu University Journal of Science and Technology
Ordu Univ. J. Sci. Tech.

E-ISSN: 2146-6459 | 2024, 14(1), 154-175 | Araştırma Makalesi / Research Article

https://doi.org/10.54370/ordubtd.1438188

Some Applications on Spherical Indicatrices of the Helix Curve

Süleyman Şenyurt[1], Sümeyye Gür Mazlum[2]

[1]Ordu University, Faculty of Arts and Sciences, Department of Mathematics, Ordu
[2] Gümüşhane University, Kelkit Aydın Doğan Vocational School, Computer Technology, Gümüşhane

Geliş Tarihi / Received Date: 16.02.2024 Kabul Tarihi / Accepted Date: 14.06.2024

Abstract

In this study, the Frenet elements of the curves that are drawn on the unit sphere by the unit vectors obtained from linear combinations of Frenet vectors of the helix curve are calculated. Moreover, Sabban frames of these curves are created and Smarandache curves are defined. Finally, the geodesic curvatures of each Smarandache curve are calculated.

Keywords: geodesic curvature, helix curve, sabban frame, smarandache curve.

Helis Eğrisinin Küresel Göstergeleri Üzerine Bazı Uygulamalar

Öz

Bu çalışmada, helis eğrisinin Frenet vektörlerinin lineer birleşimden elde edilen birim vektörlerin birim küre üzerinde çizdikleri eğrilerin Frenet elemanları hesaplanmıştır. Dahası bu eğrilere ait Sabban çatıları oluşturularak Smarandache eğrileri tanımlanmıştır. Son olarak bu Smarandache eğrilerinin geodezik eğrilikleri hesaplanmıştır.

Anahtar Kelimeler: Geodezik eğrilik, helis eğrisi, Sabban çatısı, Smarandache eğrisi.

Sorumlu Yazar / Corresponding Author: Sümeyye Gür Mazlum sumeyyegur@gumushane.edu.tr

Introduction

When the Frenet vectors of a differentiable curve are taken as position vectors, the regular curves that are drawn by these vectors are called Smarandache curves (Taşköprü & Tosun, 2014). Some properties of Smarandache curves obtained by using different frames and different curves were examined (Alıç & Yılmaz, 2021; Ali, 2010; Bektaş &Yüce, 2013; Çetin et al., 2014; Çetin &Kocayiğit, 2013; Şenyurt, 2018; Şenyurt & Canlı, 2023; Şenyurt & Çalışkan, 2015; Şenyurt & Öztürk, 2018; Şenyurt & Sivas, 2013; Şenyurt et al., 2019; Şenyurt et. al, 2020; Şenyurt et. al, 2022; Şenyurt et al. 2023a; Şenyurt et al. 2023b; Turgut & Yılmaz, 2008). Sabban frame and geodesic curvature of spherical indicatrix curves were defined by Koenderink (1990). Later, Smarandache curves obtained from Sabban frames were defined and the geodesic curvatures of these curves were calculated. Frenet vectors and curvatures of a differentiable curve are respectively (Abbena et al., 2017; Carmo, 1976)

$$T=\frac{\alpha'}{\|\alpha'\|},\quad N=B\wedge T,\quad B=\frac{\alpha'\wedge\alpha''}{\|\alpha'\wedge\alpha''\|}, \tag{1}$$

$$\kappa=\frac{\|\alpha'\wedge\alpha''\|}{\|\alpha'\|^3},\quad \tau=\frac{\det(\alpha',\alpha'',\alpha''')}{\|\alpha'\wedge\alpha''\|^2}. \tag{2}$$

Let $\Gamma=\gamma'$ be the tangent vector of the unit speed spherical curve $\gamma=\gamma(s)$. The orthonormal system $\{\gamma,\Gamma,D\}$ is called Sabban frame, where $D=\gamma\wedge\Gamma$ (Koenderink, 1990). According to this frame, the Sabban formulas and geodesic curvature of the curve are as follows (Koenderink, 1990; Taşköprü & Tosun, 2014):

$$\begin{gathered}\gamma'=\Gamma,\quad \Gamma'=-\gamma+K_gD,\quad D'=-K_g\Gamma,\\ K_g=\langle\Gamma',D\rangle\end{gathered}. \tag{3}$$

If the velocity vector of a curve makes a constant angle with a fixed direction, the curve is called a helix curve, and the constant direction is called the axis of the helix. For any curve to be a helix, a necessary and sufficient condition is that the ratio of its curvatures is constant. For example, Frenet vectors, curvatures and Frenet formulas of the helix curve $\alpha(t)=(\cos t,\sin t,t)$ are as follows (Gür Mazlum, 2023; Şenyurt & Gür Mazlum, 2023):

$$\begin{gathered}T(t)=\frac{(-\sin t,\cos t,1)}{\sqrt{2}},\quad N(t)=-(\cos t,\sin t,0),\quad B(t)=\frac{(\sin t,-\cos t,1)}{\sqrt{2}},\\ \kappa=\tau=\frac{1}{2},\end{gathered} \tag{4}$$

$$T'(t)=\frac{1}{\sqrt{2}}(-\cos t,-\sin t,0),\quad N'(t)=(\sin t,-\cos t,0),\quad B'(t)=\frac{1}{\sqrt{2}}(\cos t,\sin t,0)\cdot$$

Some Applications on Spherical Indicatrices of the Helix Curve

The unit vector written as the linear combination of the Frenet vectors of the helix curve is

$$\Gamma=\frac{xT+yN+zB}{\sqrt{x^2+y^2+z^2}},\qquad x,y,z\in\mathbb{R}. \tag{5}$$

When the vector Γ in (5) is taken as the position vector, let the resulting curve be denoted by β.

i. If $y = z = 0,\ x = 1$ is taken, $\Gamma = T$ is gotten. In this case, the curve β is the $T-$tangent indicatrix curve. This situation has been examined before (Şenyurt and Gür Mazlum, 2023).

ii. If $x = z = 0,\ y = 1$ is taken, $\Gamma = N$ is gotten. In this case, the curve β is the $N-$principal normal indicatrix curve.

iii. If $x = y = 0,\ z = 1$ is taken, $\Gamma = B$ is gotten. In this case, the curve β is the $B-$binormal indicatrix curve.

iv. If $x = y = 1,\ z = 0$ is taken, $\Gamma = \dfrac{T+N}{\sqrt{2}}$ is gotten. In this case, the curve β is the $TN-$ indicatrix curve.

v. If $x = z = 1,\ y = 0$ is taken, $\Gamma = \dfrac{T+B}{\sqrt{2}}$ is gotten. In this case, the curve β is a fixed point.

vi. If $y = z = 1,\ x = 0$ is taken, $\Gamma = \dfrac{N+B}{\sqrt{2}}$ is gotten. In this case, the curve β is the $NB-$ indicatrix curve.

vii. If $x = y = z = 1$ is taken, $\Gamma = \dfrac{T+N+B}{\sqrt{3}}$ is gotten. In this case, the curve β is the $TNB-$ indicatrix curve.

$N-$ Principal Normal Indicatrix Curve

Theorem 1. The Frenet vectors T_N, N_N, B_N and curvatures κ_N, τ_N of the principal normal indicatrix curve β are as follows:

$$T_N = \frac{-T+B}{\sqrt{2}},\ \ N_N = -N,\ \ B_N = \frac{T+B}{\sqrt{2}},\ \ \kappa_N = 1,\ \ \tau_N = 0. \tag{6}$$

Proof: If the first, second and third derivatives of the curve β are taken and the necessary operations are performed,

$$\beta' = \frac{-T+B}{\sqrt{2}},\ \ \beta'' = -N,\ \ \beta''' = \frac{T-B}{2\sqrt{2}},\ \ \ \beta' \wedge \beta''_T = \frac{T+B}{2\sqrt{2}},$$

$$\|\beta'\| = 1,\ \ \ \ \|\beta' \wedge \beta''\| = \frac{1}{2},\ \ \det(\beta', \beta'', \beta''') = 0$$

are obtained. From (1) and (2), the vectors in (6) are obtained.

Let $\{\beta, T_N, D_N = \beta \wedge T_N\}$ be the Sabban frame of the principal normal indicatrix curve β. So, these vectors and their derivatives are obtained as follows:

$$\beta = N,\ \ T_N = \frac{-T+B}{\sqrt{2}},\ \ D_N = \frac{T+B}{\sqrt{2}}, \tag{7}$$

$$\beta' = \frac{-T+B}{\sqrt{2}},\ \ T'_N = -N,\ \ D'_N = 0.$$

Definition 2. Let $\{\beta, T_N, D_N\}$ be the Sabban frame of the spherical curve that are drawn by the principal normal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\Delta_1 = \frac{1}{\sqrt{2}}\left(\beta + T_N\right)$$

is called the $\Delta_1 -$ Smarandache curve, (Figure 1). From (7),

$$\Delta_1 = \frac{-T + \sqrt{2}N + B}{2} \tag{8}$$

is gotten.

Theorem 3. The geodesic curvature $K_g^{\Delta_1}$ of the $\Delta_1 -$ Smarandache curve is as follows:

$$K_g^{\Delta_1} = 0.$$

Proof: From (1) and (8), the tangent vector T_{Δ_1} of $\Delta_1 -$ Smarandache curve is as follows:

$$T_{\Delta_1} = \frac{-T - \sqrt{2}N + B}{2}. \tag{9}$$

So, by the cross product of the vectors Δ_1 and T_{Δ_1} in (8) and (9),

$$\Delta_1 \wedge T_{\Delta_1} = \frac{T + B}{\sqrt{2}}$$

is gotten. If the derivative of the vector T_{Δ_1} is taken,

$$T_{\Delta_1}' = \frac{T - \sqrt{2}N - B}{2}$$

is obtained. From (3) and (4), the proof is completed.

Definition 4. Let $\{\beta, T_N, D_N\}$ be the Sabban frame of the spherical curve that are drawn by the principal normal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\Delta_2 = \frac{1}{\sqrt{2}}\left(\beta + D_N\right)$$

is called the $\Delta_2 -$ Smarandache curve, (Figure 1). From (7),

$$\Delta_2 = \frac{T + \sqrt{2}N + B}{2} \tag{10}$$

is gotten.

Theorem 5. The geodesic curvature $K_g^{\Delta_2}$ of the $\Delta_2 -$ Smarandache curve is as follows:

$$K_g^{\Delta_2} = \frac{1}{\sqrt{2}}\cdot$$

Proof: From (1) and (10), the tangent vector T_{Δ_2} of $\Delta_2 -$ Smarandache curve is as follows:

$$T_{\Delta_2} = \frac{-T + B}{\sqrt{2}}. \tag{11}$$

So, by the cross product of the vectors Δ_2 and T_{Δ_2} in (10) and (11),

$$\Delta_2 \wedge T_{\Delta_2} = \frac{T - \sqrt{2}N - B}{2}$$

is gotten. If the derivative of the vector T_{Δ_2} is taken,

$$T'_{\Delta_2} = -N$$

is obtained. From (3) and (4), the proof is completed.

Definition 6. Let $\{\beta, T_N, D_N\}$ be the Sabban frame of the spherical curve that are drawn by the principal normal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\Delta_3 = \frac{1}{\sqrt{2}}(T_N + D_N)$$

is called the Δ_3-Smarandache curve, (Figure 1). From (7),

$$\Delta_3 = B \tag{12}$$

is gotten.

Theorem 7. The geodesic curvature $K_g^{\Delta_3}$ of the Δ_3-Smarandache curve is as follows:

$$K_g^{\Delta_3} = \frac{1}{\sqrt{2}}\cdot$$

Proof: From (1) and (12), the tangent vector T_{Δ_3} of Δ_3-Smarandache curve is as follows:

$$T_{\Delta_3} = -N\,. \tag{13}$$

So, by the cross product of the vectors Δ_3 and T_{Δ_3} in (12) and (13),

$$\Delta_3 \wedge T_{\Delta_3} = T$$

is gotten. If the derivative of the vector T_{Δ_3} is taken,

$$T'_{\Delta_3} = \frac{T - B}{\sqrt{2}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 8. Let $\{\beta, T_N, D_N\}$ be the Sabban frame of the spherical curve that are drawn by the principal normal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\Delta_4 = \frac{1}{\sqrt{3}}(\beta + T_N + D_N)$$

is called the Δ_4- Smarandache curve, (Figure 1). From (7),

$$\Delta_4 = \frac{\sqrt{2}N + B}{\sqrt{3}} \tag{14}$$

is gotten.

Theorem 9. The geodesic curvature $K_g^{\Delta_4}$ of the Δ_4-Smarandache curve is as follows:

$$K_g^{\Delta_4} = \frac{\sqrt{2}+1}{\sqrt{3}}.$$

Proof: From (1) and (14), the tangent vector T_{Δ_4} of Δ_4-Smarandache curve is as follows:

$$T_{\Delta_4} = \frac{-T-\sqrt{2}N+B}{2}. \tag{15}$$

So, by the cross product of the vectors Δ_4 and T_{Δ_4} in (14) and (15),

$$\Delta_4 \wedge T_{\Delta_4} = \frac{3\sqrt{2}T-2N+\sqrt{2}B}{2\sqrt{3}}$$

is gotten. If the derivative of the vector T_{Δ_4} is taken,

$$T'_{\Delta_4} = \frac{T-\sqrt{2}N-B}{2}$$

is obtained. From (3) and (4), the proof is completed.

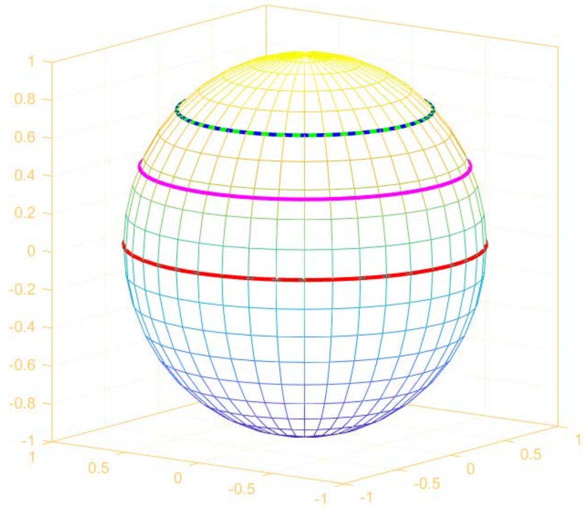

Figure 1. Δ_1 (red), Δ_2 (blue), Δ_3 (green) and Δ_4 (magenta) $-$Smarandache Curves on the Unit Sphere

B - Binormal Indicatrix Curve

Theorem 10. The Frenet vectors T_B, N_B, B_B and curvatures κ_B, τ_B of the binormal indicatrix curve β are as follows:

$$T_B = -N, \quad N_B = \frac{-T+B}{\sqrt{2}}, \quad B_B = \frac{T+B}{\sqrt{2}}, \quad \kappa_B = \sqrt{2}, \quad \tau_B = 0.$$

Proof: If the first, second and third derivatives of the curve β are taken and the necessary operations are performed,

$$\beta' = \frac{-N}{\sqrt{2}},\ \beta'' = \frac{T-B}{2},\ \beta''' = \frac{N}{\sqrt{2}}, \beta' \wedge \beta''_T = \frac{T+B}{2\sqrt{2}},$$

$$\|\beta'\| = \frac{1}{\sqrt{2}}, \quad \|\beta' \wedge \beta''\| = \frac{1}{2}, \quad \det(\beta', \beta'', \beta''') = 0,$$

are obtained. From (1) and (2), the proof is completed.

Let $\{\beta, T_B, D_B = \beta \wedge T_B\}$ be the Sabban frame of the binormal indicatrix curve β. So, these vectors and their derivatives are obtained as follows:

$$\beta = B,\ \ T_B = -N,\ \ D_B = T, \tag{16}$$

$$\beta' = \frac{-N}{\sqrt{2}},\ \ T_B' = \frac{T-B}{\sqrt{2}},\ \ D_B' = \frac{N}{\sqrt{2}}.$$

Definition 11. Let $\{\beta, T_B, D_B\}$ be the Sabban frame of the spherical curve that are drawn by the binormal indicatrix curve β on the unit sphere. The regular curve that are drawn by the vector

$$\zeta_1 = \frac{1}{\sqrt{2}}(\beta + T_B)$$

is called the ζ_1-Smarandache curve, (Figure 2). From (16),

$$\zeta_1 = \frac{-N+B}{\sqrt{2}} \tag{17}$$

is gotten.

Theorem 12. The geodesic curvature $K_g^{\zeta_1}$ of the ζ_1-Smarandache curve is as follows:

$$K_g^{\zeta_1} = \frac{3}{\sqrt{10}}.$$

Proof: From (1) and (17), the tangent vector T_{ζ_1} of ζ_1-Smarandache curve is as follows:

$$T_{\zeta_1} = \frac{-T+2B}{\sqrt{5}}. \tag{18}$$

So, by the cross product of the vectors ζ_1 and T_{ζ_1} in (17) and (18),

$$\zeta_1 \wedge T_{\zeta_1} = \frac{-2T-N-B}{\sqrt{10}}$$

is gotten. If the derivative of the vector T_{ζ_1} is taken,

$$T_{\zeta_1}' = \frac{-3N}{\sqrt{10}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 13. Let $\{\beta, T_B, D_B\}$ be the Sabban frame of the spherical curve that are drawn by the binormal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\zeta_2 = \frac{1}{\sqrt{2}}(\beta + D_B)$$

is called the ζ_2- Smarandache curve, (Figure 2). From (16),

$$\zeta_2 = \frac{T+B}{\sqrt{2}} \tag{19}$$

is gotten.

Theorem 14. The geodesic curvature $K_g{}^{\zeta_2}$ of the $\zeta_2 -$Smarandache curve is as follows:

$K_g{}^{\zeta_2} = 0.$

Proof: From (1) and (19), the tangent vector T_{ζ_2} of $\zeta_2 -$Smarandache curve is as follows:

$$T_{\zeta_2} = 0\,. \tag{20}$$

So, by the cross product of the vectors ζ_2 and T_{ζ_2} in (19) and (20),

$$\zeta_2 \wedge T_{\zeta_2} = 0$$

is gotten. From (3) and (4), the proof is completed.

Definition 15. Let $\{\beta, T_B, D_B\}$ be the Sabban frame of the spherical curve that are drawn by the binormal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\zeta_3 = \frac{1}{\sqrt{2}}(T_B + D_B)$$

is called the $\zeta_3 -$Smarandache curve, (Figure 2). From (16),

$$\zeta_3 = \frac{T - N}{\sqrt{2}} \tag{21}$$

is gotten.

Theorem 16. The geodesic curvature $K_g{}^{\zeta_3}$ of the $\zeta_3 -$Smarandache curve is as follows:

$K_g{}^{\zeta_3} = \frac{1}{3}\cdot$

Proof: From (1) and (21), the tangent vector T_{ζ_3} of $\zeta_3 -$Smarandache curve is as follows:

$$T_{\zeta_3} = \frac{T + N - B}{\sqrt{3}}\,. \tag{22}$$

So, by the cross product of the vectors ζ_3 and T_{ζ_3} in (21) and (22),

$$\zeta_3 \wedge T_{\zeta_3} = \frac{T + N + 2B}{\sqrt{6}}$$

is gotten. If the derivative of the vector T_{ζ_3} is taken,

$$T'_{\zeta_3} = \frac{-T + N + B}{\sqrt{6}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 17. Let $\{\beta, T_B, D_B\}$ be the Sabban frame of the spherical curve that are drawn by the binormal indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\zeta_4 = \frac{1}{\sqrt{3}}(\beta + T_B + D_B)$$

is called the $\zeta_4 -$ Smarandache curve, (Figure 2). From (16),

$$\zeta_4 = \frac{T - N + B}{\sqrt{3}} \tag{23}$$

is gotten.

Theorem 18. The geodesic curvature $K_g^{\zeta_4}$ of the ζ_4 – Smarandache curve is as follows:

$$K_g^{\zeta_4} = \frac{1}{3}\cdot$$

Proof: From (1) and (23), the tangent vector T_{ζ_4} of ζ_4 – Smarandache curve is as follows:

$$T_{\zeta_4} = \frac{T - B}{\sqrt{6}}. \tag{24}$$

So, by the cross product of the vectors ζ_4 and T_{ζ_4} in (23) and (24),

$$\zeta_4 \wedge T_{\zeta_4} = \frac{T - 2N + B}{3\sqrt{2}}$$

is gotten. If the derivative of the vector T_{ζ_4} is taken,

$$T'_{\zeta_4} = \frac{N}{\sqrt{3}}$$

is obtained. From (3) and (4), the proof is completed.

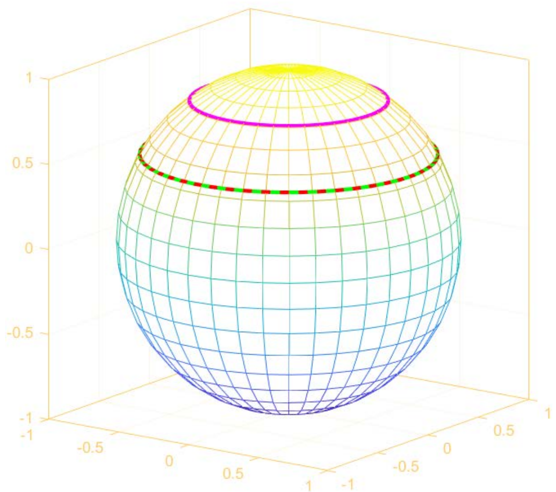

Figure 2. ζ_1 (red), ζ_3 (green) and ζ_4 (magenta) – Smarandache Curves on the Unit Sphere.

TN – Indicatrix Curve

Theorem 19. The Frenet vectors T_{TN}, N_{TN}, B_{TN} and curvatures κ_{TN}, τ_{TN} of the TN – indicatrix curve β are as follows:

$$T_{TN} = \frac{-T + N + B}{\sqrt{3}}, \quad N_{TN} = \frac{-T - 2N + B}{\sqrt{6}}, \quad B_{TN} = \frac{T + B}{\sqrt{2}}, \quad \kappa_{TN} = \frac{4}{3}, \quad \tau_{TN} = 0.$$

Proof: If the first, second and third derivatives of the curve β are taken and the necessary operations are performed,

$$\beta' = \frac{-T+N+B}{2},\ \beta'' = \frac{-T-2N+B}{2\sqrt{2}},\ \beta''' = \frac{T-N-B}{2},\ \beta' \wedge \beta'' = \frac{3}{4\sqrt{2}}(T+B),$$

$$\|\beta'\| = \frac{\sqrt{3}}{2},\quad \|\beta' \wedge \beta''\| = \frac{3}{4},\quad \det(\beta',\beta'',\beta''') = 0,$$

are obtained. From (1) and (2), the proof is completed.

Let $\{\beta, T_{TN}, D_{TN} = \beta \wedge T_{TN}\}$ be the Sabban frame of the $TN-$ indicatrix curve β. So, these vectors and their derivatives are obtained as follows:

$$\beta = \frac{T+N}{\sqrt{2}},\quad T_{TN} = \frac{-T+N+B}{\sqrt{3}},\quad D_{TN} = \frac{T-N+2B}{\sqrt{6}} \tag{25}$$

$$\beta' = \frac{-T+N+B}{2},\ T'_{TN} = \frac{-T-2N+B}{\sqrt{6}},\ D'_{TN} = \frac{T-N-B}{2\sqrt{3}}.$$

Definition 20. Let $\{\beta, T_{TN}, D_{TN}\}$ be the Sabban frame of the spherical curve that are drawn by the $TN-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\partial_1 = \frac{1}{\sqrt{2}}(\beta + T_{TN})$$

is called the ∂_1-Smarandache curve, (Figure 3). From (25),

$$\partial_1 = \frac{\left(\sqrt{3}-\sqrt{2}\right)T + \left(\sqrt{3}+\sqrt{2}\right)N + \sqrt{2}B}{2\sqrt{3}} \tag{26}$$

is gotten.

Theorem 21. The geodesic curvature $K_g^{\partial_1}$ of the ∂_1-Smarandache curve is as follows:

$$K_g^{\partial_1} = \frac{\sqrt{3}+14\sqrt{2}}{42}.$$

Proof: From (1) and (26), the tangent vector T_{∂_1} of ∂_1-Smarandache curve is as follows:

$$T_{\partial_1} = \frac{-\left(\sqrt{6}+2\right)T + \left(\sqrt{6}-4\right)N + \left(\sqrt{6}+2\right)B}{\sqrt{42}}. \tag{27}$$

So, by the cross product of the vectors ∂_1 and T_{∂_1} in (26) and (27),

$$\partial_1 \wedge T_{\partial_1} = \frac{\left(4\sqrt{3}+6\sqrt{2}\right)T - \left(3\sqrt{2}+2\sqrt{3}\right)N + \left(12\sqrt{2}-2\sqrt{3}\right)B}{6\sqrt{14}}$$

is gotten. If the derivative of the vector T_{∂_1} is taken,

$$T_{\partial_1}' = \frac{-\left(\sqrt{6}-4\right)T - 2\left(\sqrt{6}+2\right)N + \left(\sqrt{6}-4\right)B}{2\sqrt{21}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 22. Let $\{\beta, T_{TN}, D_{TN}\}$ be the Sabban frame of the spherical curve that are drawn by the $TN-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\partial_2 = \frac{1}{\sqrt{2}}(\beta + D_{TN})$$

is called the ∂_2-Smarandache curve, (Figure 3). From (25),

$$\partial_2 = \frac{(\sqrt{3}+1)T + (\sqrt{3}-1)N + 2B}{2\sqrt{3}} \tag{28}$$

is gotten.

Theorem 23. The geodesic curvature $K_g^{\partial_2}$ of the ∂_2-Smarandache curve is as follows:

$$K_g^{\partial_2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Proof: From (1) and (28), the tangent vector T_{∂_2} of ∂_2-Smarandache curve is as follows:

$$T_{\partial_2} = \frac{-T+N+B}{\sqrt{3}}. \tag{29}$$

So, by the cross product of the vectors ∂_2 and T_{∂_2} in (28) and (29),

$$\partial_2 \wedge T_{\partial_2} = \frac{(\sqrt{3}-3)T - (\sqrt{3}+3)N + 2\sqrt{3}B}{6}$$

is gotten. If the derivative of the vector T_{∂_2} is taken,

$$T'_{\partial_2} = \frac{-T-2N+B}{\sqrt{6}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 24. Let $\{\beta, T_{TN}, D_{TN}\}$ be the Sabban frame of the spherical curve that are drawn by the $TN-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\partial_3 = \frac{1}{\sqrt{2}}(T_{TN} + D_{TN})$$

is called the ∂_3-Smarandache curve, (Figure 3). From (25),

$$\partial_3 = \frac{(1-\sqrt{2})T + (\sqrt{2}-1)N + (\sqrt{2}+2)B}{2\sqrt{3}} \tag{30}$$

is gotten.

Theorem 25. The geodesic curvature $K_g^{\partial_3}$ of the ∂_3-Smarandache curve is as follows:

$$K_g^{\partial_3} = \frac{4\sqrt{3}+\sqrt{6}}{10}.$$

Proof: From (1) and (30), the tangent vector T_{∂_3} of ∂_3-Smarandache curve is as follows:

$$T_{\partial_3} = \frac{\left(1-\sqrt{2}\right)T - \left(2\sqrt{2}+1\right)N + \left(\sqrt{2}-1\right)B}{\sqrt{15}}. \tag{31}$$

So, by the cross product of the vectors ∂_3 and T_{∂_3} in (30) and (31),

$$\partial_3 \wedge T_{\partial_3} = \frac{\left(3-\sqrt{2}\right)T + \left(1-\sqrt{2}\right)N + \left(2-\sqrt{2}\right)B}{2\sqrt{5}}$$

is gotten. If the derivative of the vector T_{∂_3} is taken,

$$T'_{\partial_3} = \frac{\left(4+\sqrt{2}\right)T + \left(2\sqrt{2}-4\right)N - \left(4+\sqrt{2}\right)B}{2\sqrt{15}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 26. Let $\{\beta, T_{TN}, D_{TN}\}$ be the Sabban frame of the spherical curve that are drawn by the $TN-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\partial_4 = \frac{1}{\sqrt{3}}\left(\beta + T_{TN} + D_{TN}\right)$$

is called the ∂_4-Smarandache curve, (Figure 3). From (25),

$$\partial_4 = \frac{\left(\sqrt{3}-\sqrt{2}+1\right)T + \left(\sqrt{3}+\sqrt{2}-1\right)N + \left(\sqrt{2}+2\right)B}{3\sqrt{2}} \tag{32}$$

is gotten.

Theorem 27. The geodesic curvature $K_g^{\partial_4}$ of the ∂_4-Smarandache curve is as follows:

$$K_g^{\partial_4} = \frac{4\sqrt{3}+\sqrt{6}}{10}.$$

Proof: From (1) and (32), the tangent vector T_{∂_4} of ∂_4-Smarandache curve is as follows:

$$T_{\partial_4} = \frac{\left(-\sqrt{3}-\sqrt{2}+1\right)T + \left(\sqrt{3}-2\sqrt{2}-1\right)N + \left(\sqrt{3}+\sqrt{2}-1\right)B}{\sqrt{24-6\sqrt{3}}}. \tag{33}$$

So, by the cross product of the vectors ∂_4 and T_{∂_4} in (32) and (33),

$$\partial_4 \wedge T_{\partial_4} = \frac{\left(12+\sqrt{6}-4\sqrt{3}+\sqrt{2}\right)T + \left(3+\sqrt{3}\right)\left(-\sqrt{3}-\sqrt{2}+1\right)N + 2\left(-\sqrt{2}+1\right)\left(\sqrt{3}-2\sqrt{2}-1\right)B}{6\sqrt{12-3\sqrt{3}}}$$

is gotten. If the derivative of the vector T_{∂_4} is taken,

$$T'_{\partial_4} = \frac{-\left(\sqrt{3}-2\sqrt{2}-1\right)T + \left(-2\sqrt{3}-2\sqrt{2}+2\right)N + \left(\sqrt{3}-2\sqrt{2}-1\right)B}{2\sqrt{3}\sqrt{4-\sqrt{3}}}$$

is obtained. From (3) and (4), the proof is completed.

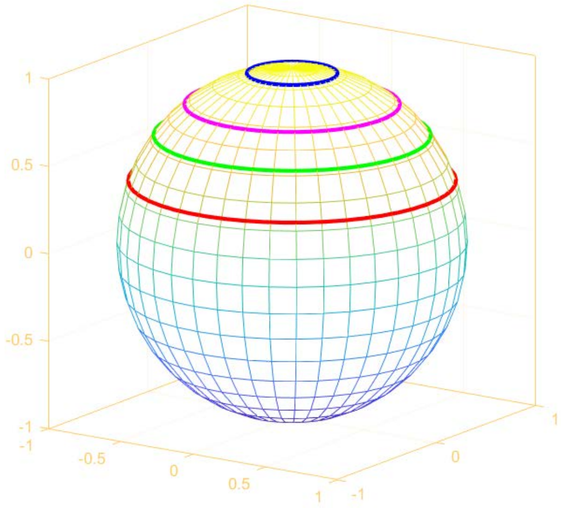

Figure 3. ∂_1 (red), ∂_2 (blue), ∂_3 (green) and ∂_4 (magenta) $-$ Smarandache Curves on the Unit Sphere

NB - Indicatrix Curve

Theorem 28. The Frenet vectors T_{NB}, N_{NB}, B_{NB} and curvatures κ_{NB}, τ_{NB} of the $NB-$indicatrix curve β are as follows:

$$T_{NB} = \frac{-T-N+B}{\sqrt{3}}, \quad N_{NB} = \frac{5T-4N+B}{\sqrt{42}}, \quad B_{NB} = \frac{T+2N+3B}{\sqrt{14}}, \quad \kappa_{NB} = \frac{2\sqrt{21}}{9}, \quad \tau_{NB} = 0.$$

Proof: If the first, second and third derivatives of the curve β are taken and the necessary operations are performed,

$$\beta' = \frac{-T-N+B}{2}, \quad \beta'' = \frac{T-2N+B}{2\sqrt{2}}, \quad \beta''' = \frac{T+N-B}{2}, \quad \beta' \wedge \beta'' = \frac{1}{4\sqrt{2}}(T+2N+3B),$$

$$\|\beta'\| = \frac{\sqrt{3}}{2}, \quad \|\beta' \wedge \beta''\| = \frac{\sqrt{7}}{4}, \quad \det(\beta', \beta'', \beta''') = 0,$$

are obtained. From (1) and (2), the proof is completed.

Let $\{\beta, T_{NB}, D_{NB} = \beta \wedge T_{NB}\}$ be the Sabban frame of the $NB-$ indicatrix curve β. So, these vectors and their derivatives are obtained as follows:

$$\beta = \frac{N+B}{\sqrt{2}}, \quad T_{NB} = \frac{-T-N+B}{\sqrt{3}}, \quad D_{NB} = \frac{2T-N+B}{\sqrt{6}} \tag{34}$$

$$\beta' = \frac{-T-N+B}{2}, \quad T'_{NB} = \frac{T-2N-B}{\sqrt{6}}, \quad D'_{NB} = \frac{T+N-B}{2\sqrt{3}}.$$

Definition 29. Let $\{\beta, T_{NB}, D_{NB}\}$ be the Sabban frame of the spherical curve that are drawn by the $NB-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\pi_1 = \frac{1}{\sqrt{2}}(\beta + T_{NB})$$

is called the π_1-Smarandache curve, (Figure 4). From (34),

$$\pi_1 = \frac{\sqrt{2}T + \left(\sqrt{3}-\sqrt{2}\right)N + \left(\sqrt{3}+\sqrt{2}\right)B}{2\sqrt{3}}$$

(35)

is gotten.

Theorem 30. The geodesic curvature $K_g^{\pi_1}$ of the π_1-Smarandache curve is as follows:

$$K_g^{\pi_1} = \frac{3+2\sqrt{6}}{6}\cdot$$

Proof: From (1) and (35), the tangent vector T_{π_1} of π_1-Smarandache curve is as follows:

$$T_{\pi_1} = \frac{\left(2-\sqrt{6}\right)T - \left(\sqrt{6}+4\right)N - \left(2-\sqrt{6}\right)B}{\sqrt{42}}. \tag{36}$$

So, by the cross product of the vectors π_1 and T_{π_1} in (35) and (36),

$$\pi_1 \wedge T_{\pi_1} = \frac{\left(12\sqrt{2}+2\sqrt{3}\right)T + \left(\sqrt{2}-2\sqrt{3}\right)N + \left(\sqrt{2}-6\sqrt{3}\right)B}{6\sqrt{7}}$$

is gotten. If the derivative of the vector T_{π_1} is taken,

$$T_{\pi_1}' = \frac{\left(4+\sqrt{6}\right)T + \left(4-2\sqrt{6}\right)N - \left(4+\sqrt{6}\right)B}{2\sqrt{21}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 31. Let $\{\beta, T_{NB}, D_{NB}\}$ be the Sabban frame of the spherical curve that are drawn by the $NB-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\pi_2 = \frac{1}{\sqrt{2}}\left(\beta + D_{NB}\right)$$

is called the π_2-Smarandache curve, (Figure 4). From (34),

$$\pi_2 = \frac{2T + \left(\sqrt{3}-1\right)N + \left(\sqrt{3}+1\right)B}{2\sqrt{3}} \tag{37}$$

is gotten.

Theorem 32. The geodesic curvature $K_g^{\pi_2}$ of the π_2-Smarandache curve is as follows:

$$K_g^{\pi_2} = \frac{1+\sqrt{3}}{12}.$$

Proof: From (1) and (37), the tangent vector T_{π_2} of π_2-Smarandache curve is as follows:

$$T_{\pi_2} = \frac{T-N+B}{\sqrt{3}}. \tag{38}$$

So, by the cross product of the vectors π_2 and T_{π_2} in (37) and (38),

$$\pi_2 \wedge T_{\pi_2} = \frac{2T + \left(\sqrt{3}+3\right)N - \left(\sqrt{3}+1\right)B}{6}$$

is gotten. If the derivative of the vector T_{π_2} is taken,

$$T_{\pi_2}' = \frac{T - B}{2\sqrt{3}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 33. Let $\{\beta, T_{NB}, D_{NB}\}$ be the Sabban frame of the spherical curve that are drawn by the $NB-$indicatrix curve β on the unit sphere. The regular curve drawn that are by the vector

$$\pi_3 = \frac{1}{\sqrt{2}}\left(T_{NB} + D_{NB}\right)$$

is called the π_3-Smarandache curve, (Figure 4). From (34),

$$\pi_3 = \frac{\left(2-\sqrt{2}\right)T - \left(1+\sqrt{2}\right)N + \left(1+\sqrt{2}\right)B}{2\sqrt{3}} \tag{39}$$

is gotten.

Theorem 34. The geodesic curvature $K_g^{\pi_3}$ of the π_3-Smarandache curve is as follows:

$$K_g^{\pi_3} = \frac{19-13\sqrt{2}}{60\sqrt{3}}.$$

Proof: From (1) and (39), the tangent vector T_{π_3} of π_3-Smarandache curve is as follows:

$$T_{\pi_3} = \frac{\left(1+\sqrt{2}\right)T + \left(1-2\sqrt{2}\right)N - \left(1+\sqrt{2}\right)B}{\sqrt{15}}. \tag{40}$$

So, by the cross product of the vectors π_3 and T_{π_3} in (39) and (40),

$$\pi_3 \wedge T_{\pi_3} = \frac{2\left(3+\sqrt{2}\right)T - \sqrt{2}N + 3\left(3-\sqrt{2}\right)B}{6\sqrt{5}}$$

is gotten. If the derivative of the vector T_{π_3} is taken,

$$T_{\pi_3}' = \frac{\left(2\sqrt{2}-1\right)T + 2\left(1+\sqrt{2}\right)N + \left(1-2\sqrt{2}\right)B}{2\sqrt{15}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 35. Let $\{\beta, T_{NB}, D_{NB}\}$ be the Sabban frame of the spherical curve that are drawn by the $NB-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\pi_4 = \frac{1}{\sqrt{3}}\left(\beta + T_{NB} + D_{NB}\right)$$

is called the π_4-Smarandache curve, (Figure 4). From (34),

$$\pi_4 = \frac{\left(2-\sqrt{2}\right)T + \left(\sqrt{3}-\sqrt{2}-1\right)N + \left(\sqrt{3}+\sqrt{2}+1\right)B}{3\sqrt{2}} \tag{41}$$

is gotten.

Theorem 36. The geodesic curvature $K_g^{\pi_4}$ of the π_4-Smarandache curve is as follows:

$$K_g^{\pi_4} = \frac{19+3\sqrt{3}+2\sqrt{2}}{18\left(4-\sqrt{3}\right)}.$$

Proof: From (1) and (41), the tangent vector T_{π_4} of π_4-Smarandache curve is as follows:

$$T_{\pi_4} = \frac{\left(1-\sqrt{3}+\sqrt{2}\right)T + \left(1-2\sqrt{2}-\sqrt{3}\right)N + \left(\sqrt{3}-\sqrt{2}-1\right)B}{\sqrt{24-6\sqrt{3}}}. \tag{42}$$

So, by the cross product of the vectors π_4 and T_{π_4} in (41) and (42),

$$\pi_4 \wedge T_{\pi_4} = \frac{\left(12+\sqrt{6}-2\sqrt{3}-\sqrt{2}\right)T + \left(\sqrt{6}-2\sqrt{3}+3\sqrt{2}\right)N + \left(12-\sqrt{6}-4\sqrt{3}-3\sqrt{2}\right)B}{6\sqrt{12-3\sqrt{3}}}$$

is gotten. If the derivative of the vector T_{π_4} is taken,

$$T_{\pi_4}' = \frac{\left(2\sqrt{2}+\sqrt{3}-1\right)T + 2\left(1-\sqrt{3}+\sqrt{2}\right)N + \left(1-2\sqrt{2}-\sqrt{3}\right)B}{2\sqrt{12-3\sqrt{3}}}$$

is obtained. From (3) and (4), the proof is completed.

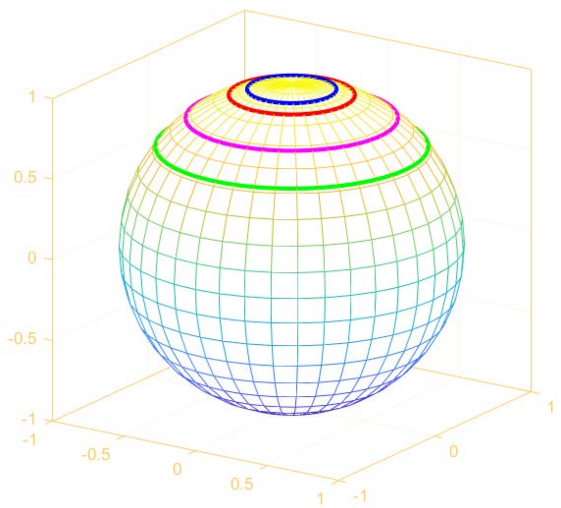

Figure 4. π_1 (red), π_2 (blue), π_3 (green) and π_4 (magenta) $-$Smarandache Curves on the Unit Sphere

TNB - Indicatrix Curve

Theorem 37. The Frenet vectors $T_{TNB}, N_{TNB}, B_{TNB}$ and curvatures κ_{TNB}, τ_{TNB} of the $TNB-$indicatrix curve β are as follows:

$$T_{TNB}=\frac{-T+B}{\sqrt{3}},\quad N_{TNB}=-\frac{2N}{\sqrt{6}},\quad B_{TNB}=\frac{T+B}{\sqrt{2}},\quad \kappa_{TNB}=\sqrt{3},\quad \tau_{TNB}=0.$$

Proof: If the first, second and third derivatives of the curve β are taken and the necessary operations are performed,

$$\beta'=\frac{-T+B}{\sqrt{6}},\quad \beta''=\frac{-N}{\sqrt{3}},\quad \beta'''=\frac{T-B}{\sqrt{6}},\quad \beta'\wedge\beta''=\frac{1}{3\sqrt{2}}(T+B),$$
$$\|\beta'\|=\frac{1}{\sqrt{3}},\quad \|\beta'\wedge\beta''\|=\frac{1}{3},\quad \det(\beta',\beta'',\beta''')=0$$

are obtained. From (1) and (2), the proof is completed.

Let $\{\beta,T_{TNB},D_{TNB}=\beta\wedge T_{TNB}\}$ be the Sabban frame of the $TNB-$ indicatrix curve β. So, these vectors and their derivatives are obtained as follows:

$$\beta=\frac{T+N+B}{\sqrt{3}},\quad T_{TNB}=\frac{-T+B}{\sqrt{2}},\quad D_{NB}=\frac{T-2N+B}{\sqrt{6}} \tag{43}$$

$$\beta'=\frac{-T+B}{\sqrt{6}},\quad T'_{TNB}=-N,\quad D'_{NB}=\frac{-T+B}{\sqrt{3}}.$$

Definition 38. Let $\{\beta,T_{TNB},D_{TNB}\}$ be the Sabban frame of the spherical curve that are drawn by the $TNB-$ indicatrix curve β on the unit sphere. The regular curve that are drawn by the vector

$$\chi_1=\frac{1}{\sqrt{2}}(\beta+T_{TNB})$$

is called the χ_1-Smarandache curve, (Figure 5). From (43),

$$\chi_1=\frac{(\sqrt{2}-\sqrt{3})T+\sqrt{2}N+(\sqrt{2}+\sqrt{3})B}{2\sqrt{3}}$$
(44)

is gotten.

Theorem 39. The geodesic curvature $K_g^{\chi_1}$ of the χ_1-Smarandache curve is as follows:

$$K_g^{\chi_1}=\frac{\sqrt{3}}{3}.$$

Proof: From (1) and (44), the tangent vector T_{χ_1} of χ_1-Smarandache curve is as follows:

$$T_{\chi_1}=\frac{-\sqrt{2}T-2\sqrt{3}N+\sqrt{2}B}{4}. \tag{45}$$

So, by the cross product of the vectors χ_1 and T_{χ_1} in (44) and (45),

$$\chi_1\wedge T_{\chi_1}=\frac{(\sqrt{6}+4)T-2N-(\sqrt{6}-4)B}{4\sqrt{3}}$$

is gotten. If the derivative of the vector T_{χ_1} is taken,

$$T_{\chi_1}{}' = \frac{\sqrt{6}T - 2N - \sqrt{6}B}{4}$$

is obtained. From (3) and (4), the proof is completed.

Definition 40. Let $\{\beta, T_{TNB}, D_{TNB}\}$ be the Sabban frame of the spherical curve that are drawn by the $TNB-$indicatrix curve β on the unit sphere. The regular curve that are drawn by the vector

$$\chi_2 = \frac{1}{\sqrt{2}}(\beta + D_{TNB})$$

is called the χ_2-Smarandache curve, (Figure 5). From (43),

$$\chi_2 = \frac{(\sqrt{2}+1)T + (\sqrt{2}-2)N + (\sqrt{2}+1)B}{2\sqrt{3}} \tag{46}$$

is gotten.

Theorem 41. The geodesic curvature $K_g{}^{\chi_2}$ of the χ_2-Smarandache curve is as follows:

$$K_g{}^{\chi_2} = \frac{\sqrt{2}+2}{\sqrt{3}}.$$

Proof: From (1) and (46), the tangent vector T_{χ_2} of χ_2-Smarandache curve is as follows:

$$T_{\chi_2} = \frac{-T + B}{\sqrt{2}}. \tag{47}$$

So, by the cross product of the vectors χ_2 and T_{χ_2} in (46) and (47),

$$\chi_2 \wedge T_{\chi_2} = \frac{(\sqrt{2}-2)T - 2(\sqrt{2}+1)N + (\sqrt{2}-2)B}{2\sqrt{6}}$$

is gotten. If the derivative of the vector T_{χ_2} is taken,

$$T_{\chi_2}{}' = -N$$

is obtained. From (3) and (4), the proof is completed.

Definition 42. Let $\{\beta, T_{TNB}, D_{TNB}\}$ be the Sabban frame of the spherical curve that are drawn by the $TNB-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\chi_3 = \frac{1}{\sqrt{2}}(T_{TNB} + D_{TNB})$$

is called the χ_3-Smarandache curve, (Figure 5). From (43),

$$\chi_3 = \frac{(1-\sqrt{3})T - 2N + (1+\sqrt{3})B}{2\sqrt{3}} \tag{48}$$

is gotten.

Theorem 43. The geodesic curvature $K_g{}^{\chi_3}$ of the χ_3-Smarandache curve is as follows:

$$K_g^{\chi_3} = \frac{5\sqrt{6}}{24}.$$

Proof: From (1) and (48), the tangent vector T_{χ_3} of χ_3-Smarandache curve is as follows:

$$T_{\chi_3} = \frac{T-\sqrt{3}N-B}{2}. \tag{49}$$

So, by the cross product of the vectors χ_3 and T_{χ_3} in (48) and (49),

$$\chi_3 \wedge T_{\chi_3} = \frac{\left(5+\sqrt{3}\right)T+2N+\left(5-\sqrt{3}\right)B}{4\sqrt{3}}$$

is gotten. If the derivative of the vector T_{χ_3} is taken,

$$T_{\chi_3}{}' = \frac{\sqrt{3}T+2N-\sqrt{3}B}{2\sqrt{2}}$$

is obtained. From (3) and (4), the proof is completed.

Definition 44. Let $\{\beta, T_{TNB}, D_{TNB}\}$ be the Sabban frame of the spherical curve that are drawn by the $TNB-$indicatrix curve β on unit sphere. The regular curve that are drawn by the vector

$$\chi_4 = \frac{1}{\sqrt{2}}\left(\beta + T_{TNB} + D_{TNB}\right)$$

is called the χ_4-Smarandache curve, (Figure 5). From (43),

$$\chi_4 = \frac{\left(1+\sqrt{2}-\sqrt{3}\right)T+\left(\sqrt{2}-2\right)N+\left(1+\sqrt{2}+\sqrt{3}\right)B}{3\sqrt{2}} \tag{50}$$

is gotten.

Theorem 45. The geodesic curvature $K_g^{\chi_4}$ of the χ_4-Smarandache curve is as follows:

$$K_g^{\chi_4} = \frac{\left(1+\sqrt{2}\right)}{3}.$$

Proof: From (1) and (50), the tangent vector T_{χ_4} of χ_4-Smarandache curve is as follows:

$$T_{\chi_4} = \frac{\left(\sqrt{2}-1\right)T-\sqrt{6}N+\left(1-\sqrt{2}\right)B}{\sqrt{12-4\sqrt{2}}}.$$

So, by the cross product of the vectors χ_4 and T_{χ_4} in (49) and (50),

$$\chi_4 \wedge T_{\chi_4} = \frac{\left(-4+\sqrt{6}+2\sqrt{3}+6\sqrt{2}\right)T+2N+\left(-4-\sqrt{6}-2\sqrt{3}+6\sqrt{2}\right)B}{3\sqrt{24-8\sqrt{2}}}$$

is gotten. If the derivative of the vector T_{χ_4} is taken,

$$T'_{\chi_4} = \frac{\sqrt{6}T+2\left(\sqrt{2}-1\right)N-\sqrt{6}B}{\sqrt{24-8\sqrt{2}}}$$

is obtained. From (3) and (4), the proof is completed.

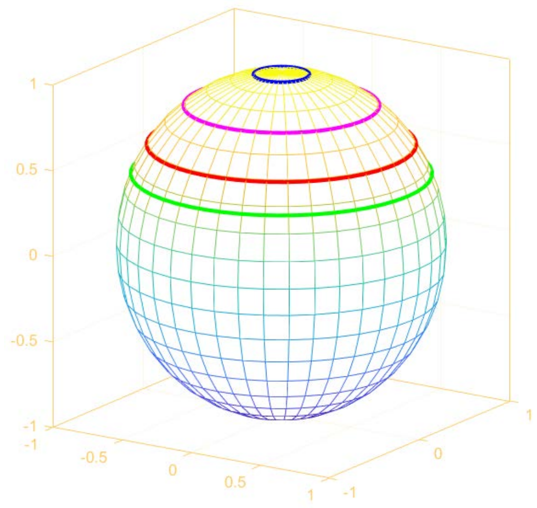

Figure 5. χ_1 (red), χ_2 (blue), χ_3 (green) and χ_4 (magenta) $-$ Smarandache Curves on the Unit Sphere

Conclusion and Suggestions

In this study, the Frenet elements of the curves that are drawn on the unit sphere by the unit vectors obtained from the linear combination of the Frenet vectors of the helix curve were calculated and Smarandache curves were defined by creating Sabban frames of these curves. Similar studies can be done on different curves in various spaces by considering other well-known frames.

Acknowledgement

The authors would like to thank the referees who criticize the paper and the editors who contributed to the article publication process.

Author Contribution

The authors co-wrote, read and approved the manuscript.

Ethics

There are no ethical issues regarding the publication of this article.

Conflict of Interest

The authors declare that they have no conflict of interest.

ORCID

Süleyman Şenyurt https://orcid.org/0000-0003-1097-5541

Sümeyye Gür Mazlum https://orcid.org/0000-0003-2471-1627

References

Abbena, E., Salamon, S., & Gray, A. (2017). *Modern differential geometry of curves and surfaces with Mathematica.* CRC Press (3rd Edition). https://doi.org/10.1201/9781315276038

Alıç, Ş., & Yılmaz, B. (2021). Smarandache Curves according to alternative frame in E^3. *Journal of Universal Mathematics*, *4*(2), 140-156. https://doi.org/10.33773/jum.956862

Ali, A. T. (2010). Special smarandache curves in the euclidean space. *International Journal of Mathematical Combinatorics*, *2*, 30-36. https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=710cb20458012e6eb683d3e29998cd0e57325b1e

Bektaş Ö. and Yüce, S. (2013). Special smarandache curves according to darboux frame in E^3. *Romanian Journal of Mathematics and Computer Science,* *3*(1), 48-59. https://arxiv.org/pdf/1203.4830

Çetin, M., Tunçer, Y., & Karacan, M. K. (2014). Smarandache curves according to Bishop frame in Euclidean 3-space. Gen. *Math. Notes, 20*(2), 50-66. https://fs.unm.edu/SN/MA-SmarandacheCurvesAccordingBishop.pdf

Çetin, M. and Kocayiğit, H. (2013). On the quaternionic smarandache curves in euclidean 3-space. *Int. J. Contemp. Math. Sciences, 8*(3), 139-150. https://www.m-hikari.com/ijcms/ijcms-2013/1-4-2013/cetinIJCMS1-4-2013.pdf

Do Carmo, M. P. (1976). *Differential geometry of curves and surfaces*. Prentice Hall, Englewood Cliffs, NJ.

Gür Mazlum, S. (2023). *On the evolute curves of any non unit speed curve in euclidean 3-space* [Oral presentation]. 8th Asia Pacific International Modern Sciences Congress, Iksad Institute, Delhi, India.

Koenderink, J. (1990). *Solid Shape*. MIT Press, Cambridge, MA.

Şenyurt, S. (2018). D-smarandache curves according to the sabban frame of the spherical indicatrix curve. *Turk. J. Math. Comput. Sci. 9,* 39-49. https://dergipark.org.tr/en/download/article-file/610102

Şenyurt, S., & Canlı, D. (2023). *On the tangent indicatrix of special viviani's curve and its corresponding smarandache curves according to sabban frame* [Oral presentation]. 10th International Başkent Congress on Physical, Ankara, Türkiye.

Şenyurt, S., & Çalışkan, A. (2015). An application according to spatial quaternionic Smarandache curve. *Appl. Math. Sci., 9*(5), 219-228. http://dx.doi.org/10.12988/ams.2015.411961

Şenyurt, S., & Öztürk, B. (2018). Smarandache Curves of Salkowski Curve According Frenet Frame. *Turkish Journal of Mathematics and Computer Science, 10*, 190-201. https://dergipark.org.tr/en/download/article-file/615426

Şenyurt, S., & Sivas, S. (2013). An application of smarandache curve. *Ordu University Journal of Science and Tecnology*, *3*(1), 46-60. https://dergipark.org.tr/tr/pub/ordubtd/issue/11065/132165

Şenyurt, S., Cevahir, C., & Altun, Y. (2019). Smarandache curves according to sabban frame belonging to mannheim curves pair. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math., 68*(1), 500-523. https://doi.org/10.31801/cfsuasmas.431877

Şenyurt, S., & Gür Mazlum, S. (2023). *Some applications on the tangent indicatrix curve of the helix* [Oral presentation]. 2nd International İzmir Congress on Life. İzmir, Türkiye.

Şenyurt, S., Cevahir, C., & Altun, Y. (2020). On the Smarandache curves of spatial quaternionic involute curve. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 90*, 827-837. https://doi.org/10.1007/s40010-019-00640-5

Şenyurt, S., Canlı, D., Can, E., & Gür Mazlum, S. (2022). Some special smarandache ruled surfaces by frenet frame in E^3-II. *Honam Mathematical Journal, 44*(4), 594-617. https://doi.org/10.5831/HMJ.2022.44.4.594

Şenyurt, S., Ayvacı, K. H., & Canlı, D. (2023a). Smarandache curves according to flc-frame in euclidean 3-space. *Fundamentals of Contemporary Mathematical Sciences, 4*(1), 16-30. https://doi.org/10.54974/fcmathsci.1142404

Şenyurt, S., Canlı, D., Can, E., & Gür Mazlum, S. (2023b). Another application of smarandache based ruled surfaces with the darboux vector according to frenet frame in E^3. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 72*(4), 880-906. https://doi.org/10.31801/cfsuasmas.1151064

Taşköprü, K., & Tosun, M. (2014). Smarandache curves according to Sabban frame on S^2. *Boletim da Sociedade Paranaense de Matematica, 32*(1), 51-59. https://arxiv.org/pdf/1206.6229

Turgut, M., & Yılmaz, S. (2008). Smarandache curves in minkowski space-time. *International J. Math. Combin., 3*, 51-55. https://www.mathcombin.com/upload/file/20150126/1422261423330082520.pdf#page=56