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Some Scattering and Spectral Properties of a Difference Equation with an Interface Condition and Hyperbolic Eigenparameter

Guler Basak Oznur

Department of Mathematics, Faculty of Science, Gazi University, 06500, Ankara, Turkey.

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Abstract. The purpose aim of this study is to investigate the properties of scattering solutions and the scattering function of an difference equation with the interface conditions and hyperbolic parameter. We also investigate resolvent operator, Green function, continuous spectrum and the set of the eigenvalues of this problem. Finally, we present an example to demonstrate the application of our results. This work is important because the boundary condition is depend on quadratic hyperbolic parameter. This difference provides a new perspective of the problem.

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1. Introduction and Preliminaries

Examining the spectral analysis of boundary value problems first started with the continuous case. As technology and science developed, many mathematicians began to investigate difference equations because differential equations were not sufficient to model the problems encountered in the most realistic way. In recent years, with the developments in modeling studies for problems in engineering, economics and other fields, it has become important to examine the spectral and scattering analysis of differential equations and difference operators, which are discrete counterparts of the operators containing these equations. While difference equations solve differential equations approximately, they also appear as mathematical models of many daily events. Applications of the theory of difference equations are widely available in biology, genetics, engineering and many other fields. The first studies in this field were made by Atkinson, Agarwal and Wong, Kelly and Peterson [3–5,19]. Additionally, many scientists have studied the spectral and scattering theory of difference equations [1, 2, 8, 14, 16].
On the other hand, instant and sharp situation changes may be encountered at some scientific process stages. Although these changes are negligible compared to the entire process, these sudden changes are encountered in many events in nature. Since the effect of the change is short-term, these effects are called impulse effects. With the impulse effect, certain subranges are formed and a condition is needed to obtain a general solution for the equation by establishing a relationship between the solutions in each subrange. This condition is called impulsive condition. In the literature, this condition is also called interface condition, transmission condition, jump condition and point interaction [6,9,11,27,29]. Such problems are generally related to discontinuous matter properties such as heat-mass transfer and vibration. In the literature, differential and difference equations with interface condition have a very important place because the

Email address: basakoznur@gazi.edu.tr (G.B. Oznur)

emerge as a natural description of many phenomena observed in the world. In addition, impulsive equations have recently been developed in non-stationary biological systems such as heart rhythms, blood flows, population dynamics, physical phenomena with a variable structure such as theoretical physics, atomic physics, radiophysics, pharmacokinetics, and mathematical economics, chemical technology, electrical technology, metallurgy, ecology, industrial robotics and medicine. There are a lot of interest in boundary value problems with interface conditions due to the multitude of application areas [17, 18, 22, 32]. Additionally, there are many studies examining the scattering and spectral properties of these problems [7, 10, 20, 28, 30, 33–35]. Unlike other studies, the boundary conditions in this study depends on the hyperbolic parameter in quadratic form. Since the boundary condition depends on the hyperbolic parameter, representation of Jost solution has changed. Because of this, analicity region of Jost solution and application region of Naimark conditions have shifted from upper half-plane to left half-plane [31]. This new approach will provide a wide perspective on applications of these problems in physics, economics and engineering. The fact that the boundary condition depends on the spectral parameter represents important concepts such as critical force, potential energy and characteristic frequency in some problems. Boundary value problems involving spectral parameters in both differential equations and boundary conditions were started to be studied by Birkhoff in 1908 [12, 13] and have been subject of intense study until today.

Now, let us consider a difference operator $\mathcal{L}$ in the Hilbert space $l_2(\mathbb{N})$ generated by the difference expression

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}\backslash\{d_0-1, d_0, d_0+1\} \tag{1.1}$$

and the boundary conditions

$$\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) y_1 + \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right) y_0 = 0, \quad |\beta_2| + |\alpha_2| \neq 0 \tag{1.2}$$

with the interface condition

$$\begin{bmatrix} y_{d_0+1} \\ \Delta y_{d_0+1} \end{bmatrix} = P \begin{bmatrix} y_{d_0-1} \\ \nabla y_{d_0-1} \end{bmatrix}, \qquad P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \tag{1.3}$$

where β_j, α_j are real numbers for $j = 0, 1, 2$, $\det P \neq 0$, P_i are real numbers for $i = 1, 2, 3, 4$ and $\lambda = 2\cosh z$. In (1.3), Δ denotes the forward and ∇ denotes the backward difference operators defined by $\Delta y_n = y_{n+1} - y_n$ and $\nabla y_n = y_n - y_{n-1}$, respectively. Throughout this paper, we assume that $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are real sequences in $l_2(\mathbb{N})$ fulfilling $a_n \neq 0$ for all $n \in \mathbb{N}\cup\{0\}$ and the following condition

$$\sum_{n\in\mathbb{N}} n\left(|1-a_n| + |b_n|\right) < \infty. \tag{1.4}$$

We define two-semi strips used in this study as follows:

$$T_0 := \left\{z \in \mathbb{C} : \mathtt{Rez} < 0, -\frac{\pi}{2} \leq \mathtt{Imz} \leq \frac{3\pi}{2}\right\}, \quad T := T_0 \cup T_1,$$

where $T_1 := \left\{z \in \mathbb{C} : \mathtt{Rez} = 0, -\frac{\pi}{2} \leq \mathtt{Imz} \leq \frac{3\pi}{2}\right\}$. Next, we will denote the fundamental solutions of (1.1) for $z \in T$ and $n = 0, 1, 2, ..., d_0 - 1$ by $\{R_n(z)\}$ and $\{Q_n(z)\}$ satisfying the initial conditions

$$R_0(z) = 0, \qquad R_1(z) = 1$$

and

$$Q_0(z) = \frac{1}{a_0}, \qquad Q_1(z) = 0,$$

respectively. For each $n \geq 0$, $R_n(z)$ is a polynomial of degree $(n-1)$ and $Q_n(z)$ is a polynomial of degree $(n-2)$. Since it is well known that the Wronskian of two solutions $y = \{y_n(z)\}$ and $u = \{u_n(z)\}$ of the difference equation (1.1) defined by

$$W[y, u] := a_n\left[y_n(z)u_{n+1}(z) - y_{n+1}(z)u_n(z)\right],$$

we easily find

$$W[R_n(z), Q_n(z)] = -1, \quad z \in \mathbb{C}.$$

It is clear that, $R_n(z)$ and $Q_n(z)$ are linear independent solutions of (1.1) and entire functions of z. By using the fundamental solutions of (1.1) and the boundary condition (1.2), we express other solution of the equation (1.1) as follows:

$$\phi_n(z) = -\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right) R_n(z) + a_0\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) Q_n(z) \tag{1.5}$$

for $z \in T$ and $n = 0, 1, 2, ..., d_0 - 1$.

On the other hand, the equation (1.1) has another solution $e(z) = \{e_n(z)\}$ represented by

$$e_n(z) = r_n e^{nz}\left(1 + \sum_{m=1}^{\infty} A_{nm} e^{mz}\right), \quad n = d_0 + 1, d_0 + 2, ...,$$

where r_n and A_{nm} are given in terms of the sequences $\{a_n\}$ and $\{b_n\}$ as

$$r_n := \prod_{k=n}^{\infty} a_k^{-1},$$

$$A_{n1} := -\sum_{k=n+1}^{\infty} b_k,$$

$$A_{n2} := \sum_{k=n+1}^{\infty}\left\{1 - a_k^2 + b_k \sum_{p=k+1}^{\infty} b_p\right\},$$

$$A_{n,m+2} := A_{n+1,m} + \sum_{k=n+1}^{\infty}\left\{\left(1 - a_k^2\right) A_{k+1,m} - b_k A_{k,m+1}\right\}$$

for $m \geq 1$ [21]. The function $e_n(z)$ is analytic according to z in $\mathbb{C}_{left} := \{z \in \mathbb{C} : \mathtt{Re}z < 0\}$, continuous in $\overline{\mathbb{C}}_{left} := \{z \in \mathbb{C} : \mathtt{Re}z \leq 0\}$ and 2π periodic. We remark that, $e_n(z)$ is called Jost solution of (1.1) satisfying the following asymptotic equation for $z \in T$

$$\lim_{n\to\infty} e^{-nz} e_n(z) = 1.$$

Besides Jost solution, we also shall define the unbounded solution of (1.1) by $\breve{e}_n(z)$ fulfilling $\lim_{n\to\infty} e^{nz}\breve{e}_n(z) = 1$, $z \in \overline{\mathbb{C}}_{left}$. From the definition of Wronskian, it is found that

$$W\left[e_n(z), \breve{e}_n(z)\right] = -2\sinh z, \quad n = d_0 + 1, d_0 + 2, ... \quad z \in T \backslash \{0, \pi i\}.$$

Outline of the paper is as follows:

- The Jost solution of $\mathcal{L}$ is initially determined, and the Jost function is subsequently obtained by utilizing the boundary condition in the Jost solution.
- The determination of the scattering function of (1.1)-(1.3) and analyze its properties.
- The resolvent operator and Green function of $\mathcal{L}$ are derived. The set of eigenvalues of $\mathcal{L}$ is derived from the resolvent operator. The asymptotic equality of the Jost function is determined.
- Finally, the unperturbed difference equation with hyperbolic eigen parameter and interface condition is considered as a special example of (1.1)-(1.3). The Jost solution, scattering function, and set of eigenvalues for this example are obtained.

So far, spectral and scattering theory of various operators, even matrix valued operators have been studied with general boundary conditions in continusous case [24–26]. The results obtained in the continuous and discontinuous cases are parallel to each other. When there are discontinuous points in the interval, new solutions are defined by using linear combinations of solutions in continuous case.

2. Jost Solutions and Scattering Function of $\mathcal{L}$

In this section, we present some new results which different from this continuous case.

Firstly, let us consider the following solution of $\mathcal{L}$ for $z \in T$

$$\widetilde{E}_n(z) = \begin{cases} \nu_1(z)R_n(z) + \nu_2(z)Q_n(z); & n = 0, 1, 2, ..., d_0 - 1 \\ e_n(z); & n = d_0 + 1, d_0 + 2, ... \end{cases}.$$

By the help of the interface condition (1.3), we get the following equalities

$$\nu_1(z)R_{d_0-1}(z) + \nu_2(z)Q_{d_0-1}(z) = \frac{1}{P_1P_4 - P_2P_3}\left(P_4 e_{d_0+1} - P_2 e_{d_0+1}\right)$$

$$\nu_1(z)\nabla R_{d_0-1}(z) + \nu_2(z)\nabla Q_{d_0-1}(z) = \frac{1}{P_1P_4 - P_2P_3}\left(P_1 e_{d_0+1} - P_3 e_{d_0+1}\right).$$

From the last equalities, $\nu_1(z)$ and $\nu_2(z)$ can be written as follows:

$$\begin{aligned}\nu_1(z) = -\frac{a_{d_0-2}}{P_1P_4 - P_2P_3}&[e_{d_0+1}(z)\{P_3Q_{d_0-1}(z) + P_4\nabla Q_{d_0-1}(z)\}\\ &-\Delta e_{d_0+1}(z)\{P_1Q_{d_0-1}(z) + P_2\nabla Q_{d_0-1}(z)\}]\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\nu_2(z) = \frac{a_{d_0-2}}{P_1P_4 - P_2P_3}&[e_{d_0+1}(z)\{P_3R_{d_0-1}(z) + P_4\nabla R_{d_0-1}(z)\}\\ &-\Delta e_{d_0+1}(z)\{P_1R_{d_0-1}(z) + P_2\nabla R_{d_0-1}(z)\}],\end{aligned} \tag{2.2}$$

respectively. The function $\widetilde{E}(z) = \left\{\widetilde{E}_n(z)\right\}$ is called the Jost solution of (1.1)-(1.3). It is obvious that $\nu_1(-z) = \overline{\nu_1(z)}$ and $\nu_2(-z) = \overline{\nu_2(z)}$.

Next, using (1.5) and $e_n(z)$, the equation (1.1) has another solution $\widetilde{F}(z) = \left\{\widetilde{F}_n(z)\right\}$ represented by

$$\widetilde{F}_n(z) = \begin{cases} \phi_n(z); & n = 0, 1, 2, ..., d_0 - 1 \\ \nu_3(z)e_n(z) + \nu_4(z)e_n(-z); & n = d_0 + 1, d_0 + 2, ... \end{cases}$$

for $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \backslash \{0, \pi i\}$. To get the coefficients $\nu_3(z)$ and $\nu_4(z)$, we will use same way as finding $\nu_1(z)$ and $\nu_2(z)$. If we apply the interface condition (1.3) to $\widetilde{F}_n(z)$, we get

$$\begin{aligned}\nu_3(z) = -\frac{a_{d_0+1}}{2\sinh z}&[P_1\phi_{d_0-1}(z)\Delta e_{d_0+1}(-z) + P_2\nabla\phi_{d_0-1}(z)\Delta e_{d_0+1}(-z)\\ &-P_3\phi_{d_0-1}(z)e_{d_0+1}(-z) - P_4\nabla\phi_{d_0-1}(z)e_{d_0+1}(-z)]\end{aligned}$$

and

$$\begin{aligned}\nu_4(z) = \frac{a_{d_0+1}}{2\sinh z}&[P_1\phi_{d_0-1}(z)\Delta e_{d_0+1}(z) + P_2\nabla\phi_{d_0-1}(z)\Delta e_{d_0+1}(z)\\ &-P_3\phi_{d_0-1}(z)e_{d_0+1}(z) - P_4\nabla\phi_{d_0-1}(z)e_{d_0+1}(z)].\end{aligned}$$

From (2.1) and (2.2), the coefficient $\nu_4(z)$ can be written as

$$\nu_4(z) = \frac{a_{d_0+1}}{a_{d_0-2}}\frac{P_1P_4 - P_2P_3}{2\sinh z}\left[\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)\nu_2(z) + a_0\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)\nu_1(z)\right]. \tag{2.3}$$

By using $R_n(z) = R_n(-z)$ and $Q_n(z) = Q_n(-z)$, we obtain that $\nu_4(z) = \nu_3(-z) = \overline{\nu_3(z)}$.

Lemma 2.1. *For all $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \backslash \{0, \pi i\}$, the Wronskian of the solutions $\widetilde{E}_n(z)$ and $\widetilde{F}_n(z)$ is given by*

$$W[\widetilde{E}_n(z), \widetilde{F}_n(z)] = \begin{cases} -\frac{a_{d_0-2}}{a_{d_0+1}}\frac{2\sinh z}{P_1P_4 - P_2P_3}\nu_4(z); & n = 0, 1, 2, ..., d_0 - 1 \\ -2\sinh z\,\nu_4(z); & n = d_0 + 1, d_0 + 2, ... \end{cases}.$$

Proof. By the help of the definition of Wronskian for $n = 0, 1, 2, ..., d_0 - 1$, we find

$$W[\widetilde{E}_n(z), \widetilde{F}_n(z)] = a_0\left[\widetilde{E}_0(z)\widetilde{F}_1(z) - \widetilde{F}_0(z)\widetilde{E}_1(z)\right].$$

It is known that, $R_0(z) = 0$, $R_1(z) = 1$, $Q_0(z) = \frac{1}{a_0}$ and $Q_1 = 0$, then we can write

$$W[\widetilde{E}_n(z), \widetilde{F}_n(z)] = -\left[\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)\nu_2(z) + a_0\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)\nu_1(z)\right]$$

for $n = 0, 1, 2, ..., d_0 - 1$. From (2.3), the last equation gives

$$W[\widetilde{E}_n(z), \widetilde{F}_n(z)] = -\frac{a_{d_0-2}}{a_{d_0+1}}\frac{2\sinh z}{P_1P_4 - P_2P_3}\nu_4(z), \quad n = 0, 1, 2, ..., d_0 - 1.$$

In a similar way, we get

$$W[\widetilde{E}_n(z), \widetilde{F}_n(z)] = a_{d_0+1}\left[\widetilde{E}_{d_0+1}(z)\widetilde{F}_{d_0+2}(z) - \widetilde{F}_{d_0+1}(z)\widetilde{E}_{d_0+2}(z)\right]$$

for $n = d_0 + 1, d_0 + 2, \ldots$. By using the solutions $\widetilde{E}_n(z)$ and $\widetilde{F}_n(z)$, we easily obtain

$$W[\widetilde{E}_n(z), \widetilde{F}_n(z)] = -2\sinh z \nu_4(z), \quad n = d_0 + 1, d_0 + 2, \ldots$$

The proof is completed. □

Now, we define the Jost function of $\mathcal{L}$ by applying the boundary condition (1.2) to the Jost solution $\widetilde{E}(z)$ of (1.1)-(1.3)

$$M(z) = \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)\nu_1(z) + \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)\frac{\nu_2(z)}{a_0}.$$

By using (2.3), the Jost function of $\mathcal{L}$ can be rewritten as

$$M(z) = \frac{a_{d_0-2}}{a_0 a_{d_0+1}} \frac{2\sinh z}{P_1P_4 - P_2P_3}\nu_4(z). \tag{2.4}$$

It is easy to see that $M(z)$ is analytic in T_0 and continuous in T. In addition, the function $M(z)$ has the following property

$$M(-z) = \overline{M(z)}.$$

Theorem 2.2. *For all* $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$, $M(z) \neq 0$.

Proof. To prove the Theorem 2.2, we assume that there exists a point z_0 in $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$ such that $M(z_0) = 0$. If we use $M(z_0) = 0$ in (2.4), we find $\nu_4(z_0) = 0$. It is known that $\nu_4(z) = \nu_3(-z) = \overline{\nu_3(z)}$, then we obtain $\nu_4(z_0) = \nu_3(-z_0) = \overline{\nu_3(z_0)} = 0$. It gives us $\widetilde{F}_n(z_0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, but it gives a contradiction. It completes the proof of Theorem 2.2. □

Definition 2.3. The function

$$S(z) := \frac{\overline{M(z)}}{M(z)}, \qquad z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$$

is called the scattering function of $\mathcal{L}$.

Scattering function can be also given with the help of coefficient $\nu_4(z)$ as

$$S(z) = -\frac{\nu_4(-z)}{\nu_4(z)}, \qquad z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}.$$

Theorem 2.4. *The scattering function satisfies the following equalities*

$$S(-z) = S^{-1}(z) = \overline{S(z)}, \quad |S(z)| = 1$$

for $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$.

Proof. By the help of the Definition 2.3, we get

$$S(-z) = \frac{\overline{M(-z)}}{M(-z)} \quad \text{and} \quad \overline{S(z)} = \frac{\overline{\left[\overline{M(z)}\right]}}{\overline{M(z)}}.$$

Since $\overline{\left[\overline{M(z)}\right]} = M(z)$, $\overline{M(-z)} = M(z)$ and $\overline{M(z)} = M(-z)$, we can say

$$S(-z) = S^{-1}(z) = \overline{S(z)}.$$

In addition, since $|S(z)| = \overline{S(z)}S(z)$, we write $|S(z)| = 1$. □

3. Resolvent Operator and Continuous Spectrum of $\mathcal{L}$

In this section, we give unbounded solution of (1.1)-(1.3). We investigate resolvent operator, discrete spectrum and continuous spectrum of $\mathcal{L}$. Furthermore, we obtain an important asymptotic equation for $M(z)$.
For $z \in T\backslash\{0, \pi i\}$, we define the following solution of $\mathcal{L}$

$$\widetilde{G}_n(z) = \begin{cases} \phi_n(z); & n = 0, 1, 2, ..., d_0 - 1 \\ \nu_5(z)e_n(z) + \nu_6(z)\breve{e}_n(z); & n = d_0 + 1, d_0 + 2, ... \end{cases}.$$

The solution $\widetilde{G}(z) = \left\{\widetilde{G}_n(z)\right\}$ is called the unbounded solution of (1.1)-(1.3). If the interface condition (1.3) implies in the solution $\widetilde{G}(z)$, the coefficients $\nu_5(z)$ and $\nu_6(z)$ can be written as

$$\nu_5(z) = \frac{a_{d_0+1}}{2\sinh z}[P_1\Delta\breve{e}_{d_0+1}(z)\phi_{d_0-1}(z) + P_2\Delta\breve{e}_{d_0+1}(z)\nabla\phi_{d_0-1}(z) \\ -P_3\breve{e}_{d_0+1}(z)\phi_{d_0-1}(z) - P_4\breve{e}_{d_0+1}(z)\nabla\phi_{d_0-1}(z)]$$

and

$$\nu_6(z) = \frac{a_{d_0+1}}{2\sinh z}[P_1\Delta e_{d_0+1}(z)\phi_{d_0-1}(z) + P_2\Delta e_{d_0+1}(z)\nabla\phi_{d_0-1}(z) \\ -P_3 e_{d_0+1}(z)\phi_{d_0-1}(z) - P_4 e_{d_0+1}(z)\nabla\phi_{d_0-1}(z)]$$

for $z \in T\backslash\{0, \pi i\}$. It is seen that $\nu_6(z)$ can be written in terms of $\nu_4(z)$ such that

$$\nu_4(z) = \nu_6(z).$$

Similar to Lemma 2.1, the Wronskian of the solutions $\widetilde{E}(z)$ and $\widetilde{G}(z)$ is found that

$$W[\widetilde{E}_n(z), \widetilde{G}_n(z)] = \begin{cases} -\frac{a_{d_0-2}}{a_{d_0+1}}\frac{2\sinh z}{P_1P_4 - P_2P_3}\nu_4(z); & n = 0, 1, 2, ..., d_0 - 1 \\ -2\sinh z\nu_4(z); & n = d_0 + 1, d_0 + 2, ... \end{cases}$$

for $z \in T\backslash\{0, \pi i\}$.

Theorem 3.1. *The resolvent operator of $\mathcal{L}$ is defined by*

$$(R_\lambda(\mathcal{L})g)_n := \sum_{k\in\mathbb{Z}} I_{nk}(z)g_k, \quad g := g_k \in l_2(\mathbb{N}),$$

where

$$I_{n,k}(z) = \begin{cases} -\frac{\widetilde{E}_n(z)\widetilde{G}_k(z)}{W[\widetilde{E}_k, \widetilde{G}_k]}; & k \le n \\ -\frac{\widetilde{G}_n(z)\widetilde{E}_k(z)}{W[\widetilde{E}_k, \widetilde{G}_k]}; & k > n \end{cases}$$

is the Green function for $z \in T\backslash\{0, \pi i\}$ and $k, n \neq d_0$.

Proof. To introduce the resolvent operator of $\mathcal{L}$, we need to solve the following equation

$$\nabla(a_n\Delta y_n) + h_n y_n - \lambda y_n = g_n, \tag{3.1}$$

where $h_n = a_{n-1} + a_n + b_n$. Since $E_n(z)$ and $G_n(z)$ are linearly independent fundamental solutions of (1.1)-(1.3), we write the general solution of (3.1)

$$y_n(z) = u_n\widetilde{E}_n(z) + v_n\widetilde{G}_n(z), \tag{3.2}$$

where u_n, v_n are coefficients and they are different from zero. From the method of variation of parameters, the following coefficients are obtained. Using the method of variation of parameters for $k \neq d_0$, the coefficients u_n, v_n must be as follows:

$$u_n = -\sum_{k=1}^{n}\frac{\widetilde{G}_k g_k}{W\left[\widetilde{E}_k, \widetilde{G}_k\right]}, \tag{3.3}$$

$$v_n = -\sum_{k=n+1}^{\infty}\frac{\widetilde{E}_k g_k}{W\left[\widetilde{E}_k, \widetilde{G}_k\right]}. \tag{3.4}$$

When the coefficients u_n, v_n obtained in (3.3) and (3.4) are substituted in (3.2), the Green function and resolvent operator of $\mathcal{L}$ are found. □

Theorem 3.1 is the basic tool for finding the set of eigenvalues of $\mathcal{L}$. By using the definition of eigenvalues [23] and this theorem, we get the set of eigenvalues of (1.1)-(1.3) as

$$\sigma_d := \{\lambda \in \mathbb{C} : \lambda = 2\cosh z, z \in T_0, \nu_4(z) = 0\}.$$

Theorem 3.2. *If the condition* (1.4) *satisfies, then the continuous spectrum of* $\mathcal{L}$ *is* $[-2, 2]$*, i.e.,* $\sigma_c(\mathcal{L}) = [-2, 2]$.

Proof. Firstly, we denote the following difference operators in $l_2(\mathbb{N})$

$$(\mathcal{L}_0 y)_n = y_{n-1} + y_{n+1}, \quad n \in \mathbb{N}\backslash\{d_0 - 1, d_0 + 1\},$$

$$(\mathcal{L}_1 y)_n = (a_{n-1} - 1)\, y_{n-1} + b_n y_n + (a_n - 1)\, y_{n+1}, \quad n \in \mathbb{N}\backslash\{d_0 - 1, d_0, d_0 + 1\}.$$

It is easy to see that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. Under the condition (1.4), from the compactness criteria in $l_2(\mathbb{N})$, we obtain that $\mathcal{L}_1$ is a compact operator in $l_2(\mathbb{N})$ [23]. Also, we write $\mathcal{L}_0 = \mathcal{L}_2 + \mathcal{L}_3$, where $\mathcal{L}_2$ is a selfadjoint operator with $\sigma_c(\mathcal{L}_2) = [-2, 2]$ and $\mathcal{L}_3$ is a finite dimensional operator in $l_2(\mathbb{N})$. Hence, by the help of Weyl theorem [15] of a compact perturbation, we find the continuous spectrum of (1.1)-(1.3). □

Theorem 3.3. *Assume* (1.4)*. For all* $z \in T$*, the function* $\nu_4(z)$ *satisfies the following asymptotic equation*

$$\nu_4(z) = e^{5z}\left[\widetilde{D} + o(1)\right], \quad |z| \to \infty, \quad \alpha_1 \neq 0,$$

where

$$\widetilde{D} = \left(\prod_{k=1}^{d_0-3}\right)^{-1} a_{d_0+1} P_2 \alpha_2 r_{d_0+1}.$$

Proof. It is known that the polynomial function $R_n(z)$ is of $(n-1)$ degree and polynomial function $Q_n(z)$ is of $(n-2)$ degree with respect to λ, we can write that

$$\lim_{|z|\to\infty} R_n(z) e^{nz} = \left(\prod_{k=1}^{n-1} a_k\right)^{-1}, \quad n = 0, 1, 2, ..., d_0 - 1 \tag{3.5}$$

and it is known that

$$\lim_{|z|\to\infty} e_n(z) e^{-nz} = r_n(z), \quad n = d_0 + 1, d_0 + 2, ..., \tag{3.6}$$

where $r_n := \left(\prod_{k=n}^{\infty} a_k\right)^{-1}$. From (2.3), (3.5) and (3.6), we obtain

$$\lim_{|z|\to\infty} \frac{\nu_4(z)}{a_{d_0+1}} = (-P_1 + P_2 - P_3 + P_4)\left(\prod_{k=1}^{d_0-2} a_k\right)^{-1} e^{3z}\alpha_2 r_{d_0+1} - (P_2 + P_4)\left(\prod_{k=1}^{d_0-3} a_k\right)^{-1} e^{4z}\alpha_2 r_{d_0+1}$$
$$+ (P_1 - P_2)\left(\prod_{k=1}^{d_0-2} a_k\right)^{-1} e^{4z}\alpha_2 r_{d_0+2} + P_2\left(\prod_{k=1}^{d_0-3} a_k\right)^{-1} e^{5z}\alpha_2 r_{d_0+1}.$$

From the last asymptotic equation, we get

$$\lim_{|z|\to\infty} \nu_4(z) e^{-5z} = \left(\prod_{k=1}^{d_0-3}\right)^{-1} a_{d_0+1} P_2 \alpha_2 r_{d_0+1}.$$

It completes the proof. □

4. An Application

In the remainder of this study, we will consider an example where we will apply the results we obtained. In that way, it gives an opportunity to readers applying the main results on a simple example. Let us define the following difference equation

$$y_{n-1} + y_{n+1} = 2\cosh z y_n, \quad n \in \mathbb{N}\backslash\{2,3,4\} \tag{4.1}$$

and the boundary conditions

$$\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) y_1 + \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right) y_0 = 0, \quad |\beta_2| + |\alpha_2| \neq 0 \tag{4.2}$$

with the interface condition

$$\begin{bmatrix} y_4 \\ \Delta y_4 \end{bmatrix} = P \begin{bmatrix} y_2 \\ \nabla y_2 \end{bmatrix}, \qquad P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \tag{4.3}$$

where β_j, α_j are real numbers for $j = 0,1,2$, $\det P \neq 0$ and P_i are real numbers for $i = 1,2,3,4$. In the problem (1.1)-(1.3), suppose that $a_n \equiv 1$, $b_n \equiv 0$ for all $n \in \mathbb{N}$ and $d_0 = 3$. It is evident that $e_n(z) = e^{nz}$ and the fundamental solutions $R_n(z)$ and $Q_n(z)$ of (1.1) satisfy the following conditions for $n = 0,1,2$

$$R_0(z) = 0, \qquad R_1(z) = 1, \qquad R_2(z) = \lambda$$
$$Q_0(z) = 1, \qquad Q_1(z) = 0, \qquad Q_2(z) = -1.$$

Firstly, we obtain the Jost solution of (4.1)-(4.3) as follows:

$$\widetilde{E}_n(z) = \begin{cases} \nu_1(z)R_n(z) + \nu_2(z)Q_n(z); & n = 0,1,2 \\ e_n(z); & n = 4,5,6,... \end{cases}$$

for $z \in T$. If we apply the interface condition (4.3) to $\widetilde{E}_n(z)$, the coefficients $\nu_1(z)$ and $\nu_2(z)$ are found as

$$\nu_1(z) = \frac{1}{P_1P_4 - P_2P_3}\left[e^{4z}\left(P_1 + P_2 + P_3 + P_4\right) - e^{5z}\left(P_1 + P_2\right)\right]$$

and

$$\nu_2(z) = \frac{1}{P_1P_4 - P_2P_3}\left[e^{4z}\left\{\lambda\left(P_1 + P_2 + P_3 + P_4\right) - \left(P_2 + P_4\right)\right\} - e^{5z}\left\{\lambda\left(P_1 + P_2\right) - P_2\right\}\right],$$

recpectively.

For the simplicity on calculations, if we take $\beta_0 = \beta_1 = \alpha_0 = \alpha_2 = 0, \beta_2 = 1$ and $\alpha_1 = -1$ in the boundary condition (4.2), we obtain the Jost function of this problem

$$M(z) = \frac{e^{4z}\lambda}{P_1P_4 - P_2P_3}\left(P_2 + P_4 - P_2e^{z}\right), \qquad z \in T. \tag{4.4}$$

By the help of the definition of scattering function and (4.4), the scattering function of (4.1)-(4.3) is found as

$$S(z) = e^{-8z}\left[\frac{P_2 + P_4 - P_2e^{-z}}{P_2 + P_4 - P_2e^{z}}\right], \qquad z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]\backslash\{0, \pi i\}.$$

In addition, continuous spectrum of this example is $[-2,2]$ from Theorem 3.2. From the definition of eigenvalues, to examine the eigenvalues of (4.1)-(4.3), it is sufficient to find the zeros of $M(z)$ for $z \in T_1$. Since $M(z) = 0$ and $\lambda = 2\cosh z$, it is easy to say

$$e^{3z}\left[P_2\left(e^{2z} - e^{3z} - e^{z} + 1\right) + P_4\left(e^{2z} + 1\right)\right] = 0.$$

From the last equation, we get

$$e^{z} = \frac{P_4}{P_2} + 1. \tag{4.5}$$

Let $P_4 = \widetilde{A}P_2$, $\widetilde{A} \in \mathbb{R}$. Using (4.5), we write $e^{z} = \widetilde{A} + 1$. Then, we have

$$z_k = -\frac{i}{2}\ln|1 + \widetilde{A}| + \frac{1}{2}Arg\left(1 + \widetilde{A}\right) + k\pi, \quad k \in \mathbb{Z}. \tag{4.6}$$

By the help of (4.6), we easily see that the problem (4.1)-(4.3) has eigenvalues if and only if $\ln|1 + \widetilde{A}| < 0$. This gives that $-2 < \widetilde{A} < 0$. These eigenvalues are real and lie on $(-\infty,-2) \cup (2,\infty)$. $\widetilde{A}$ can not be zero, because the interface condition will not work when $\widetilde{A} = 0$. Also, $\widetilde{A} \neq 2$. Since $\widetilde{A} = 2$, then $z_k = i(2k+1)\pi$, $k \in \mathbb{Z}$. But only for $k = 0$, $z_0 \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$. For $z_0 = i\pi$, we get $\lambda_0 = 2\cosh z_0 = -2$. It is known that $\lambda = -2$ is in continuous spectrum, it is not an eigenvalue of this example.

5. Conclusion

In this study, we have handled a difference equation with the boundary conditions that depend on quadratic hyperbolic parameter and interface condition. We have examined the scattering and spectral properties of $\mathcal{L}$ such as scattering solutions, Jost function, scattering function, resolvent operator and eigenvalues. We have investigated the results obtained in the study by giving an example. Since the boundary conditions depend on the hyperbolic parameter, the representations of the scattering solutions and the regions where the solutions are defined change. This gives a new perspective to the problem. This paper prepares a groundwork for many researchers working on scattering analysis and spectral theory. The results obtained in this study can be examined by generalizing the boundary condition.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Author has read and agreed to the published version of the manuscript.

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