http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 4, Pages 1050–1071 (2024)
DOI:10.31801/cfsuasmas.1439744
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: February 19, 2024; Accepted: September 17, 2024

BIAS CORRECTED MAXIMUM LIKELIHOOD ESTIMATORS FOR THE PARAMETERS OF THE GENERALIZED NORMAL DISTRIBUTION

Hasan Huseyin GUL[1], Fatma Zehra DOĞRU[2]

[1,2]Department of Statistics, Giresun University, Giresun, TÜRKİYE

Abstract. The generalized normal (GN) distribution was defined as a generalization of the normal, Laplace, and uniform distributions, with extensive application areas modeling different data settings. At the same time, its maximum likelihood estimators (MLEs) are biased in finite samples. Since such biases may affect the accuracy of estimates, we consider constructing unbiased estimators for unknown parameters of GN distribution. This article adopts the bias-corrected approach, following the analytical methodology suggested by Cox and Snell [1]. Additionally, we explore both regular biases and parametric Bootstrap bias correction techniques. A comprehensive Monte Carlo simulation is conducted to compare the performances of these estimators in estimating GN parameters. Finally, a real data example is presented to illustrate the application of methods.

1. Introduction

It is well-known that the most popular distribution is the normal distribution, widely used due to its tractability and extensive application areas. However, it is quite common to encounter non-normality in real-world examples. Distributions like the Laplace distribution can handle non-normality, for instance, in modeling speech signals. Moreover, a more flexible generalized normal (GN) distribution can contain both the normal and Laplace distributions. The GN distribution has defined a generalization of the normal, Laplace, and uniform distributions, providing

2020 *Mathematics Subject Classification.* 62F10, 62F30, 62F40.

Keywords. Bootstrap bias-correction, Cox-Snell bias-correction, generalized normal distribution, maximum likelihood estimators, Monte-Carlo simulation.

1 hasan.huseyin@giresun.edu.tr-Corresponding author; 0000-0001-9905-8605;

2 fatma.dogru@giresun.edu.tr; 0000-0001-8220-2375.

extensive application fields and enabling the modeling of various datasets. Some researchers used the GN distribution in their studies. Some researchers have utilized the GN distribution in their studies. For instance, Briassouli et al. [2] used GN distribution to add watermarks to images; Kokkinakis and Nandi [3] modeled speech signals with the GN distribution; atmospheric noise, Sharifi and Leon-Garcia [4] considered GN distribution for subband coding of audio and video signals; Choi et al. [5] applied in impulsive noise, the direction of arrival, modeling of the independent component analysis; and Wu and Principe [6] proposed to use GN distribution for blind signal separation.

There are different types of GN distribution in literature. This study considers Version 1, a parametric family of symmetrical distributions known as exponential power distribution or generalized error distribution. Subbotin [7] first proposed the exponential power distribution, and later Nadarajah [8] renamed it the GN distribution. Several studies have focused on the parameter estimation of the GN distribution. Notable contributions include those by Varanasi and Aazhang [9], Nadarajah [8], and Roenko et al. [10]. More recently, Eskin [11] and Eskin and Doğru [12] proposed methods for parameter estimation in joint location and scale models of the GN distribution.

When estimating parameters from any probability distribution, the choice of estimation methodology is very important. Among all the classical estimation methods, the most frequently used method is the maximum likelihood estimation (MLE) due to its several attractive properties. For instance, ML estimators are asymptotically unbiased, consistent, and asymptotically normally distributed. However, most of these properties depend on the large sample size condition. Therefore, properties such as unbiasedness may not hold for small or moderate sample sizes.

The primary objective of this article is to develop modified MLEs that are nearly unbiased, with a particular focus on obtaining second-order unbiasedness. To achieve this, we focus on two different approaches.

First, we propose bias-corrected MLEs (BCEs) for the parameters of GN distribution, following the methodology introduced by [1]. This method corrects the bias by subtracting the estimated bias from the original MLEs.

Next, we consider the parametric bootstrap-based bias-correction approach introduced by Efron [13], with further details provided by Efron and Tibshirani [14]. This estimator, referred to as the bootstrap bias-corrected estimator (PBE), applies bias correction numerically, without needing an analytical bias expression.

The bias-correction technique has been extensively applied to various distributions in the literature. For instance, Cordeiro et al. [15] applied it to the Beta distribution, while Saha and Paul [16] utilized it for the negative binomial distribution. It was also used by Lemonte et al. [17] for the Birnbaum-Saunders distribution and by Giles and Feng [18] for the Gamma distribution. Other applications include the Kumaraswamy distribution by Lemonte [19], the Topp-Leone distribution by Giles [20], the Lomax distribution by Giles et al. [21], and the Nakagami distribution by Schwartz et al. [22]. Zhang and Liu [23] applied the technique to the skew-normal distribution, while Schwartz and Giles [24] focused on the zero-inflated Poisson distribution. Wang and Wang [25] extended it to the weighted Lindley distribution, and Reath et al. [26] used it for the log-logistic distribution. Further examples include the generalized half-normal distribution by Mazucheli and Dey [27], the unit-Gamma distribution by Mazucheli et al. [28], the inverse Weibull distribution by Mazucheli et al. [29], the Johnson S_B distribution by Menezes and Mazucheli [30], and the unit-Weibull distribution by Menezes et al. [31].

The article is designed in this manner. Section 2 defines the GN distribution and outlines its key distributional properties. Section 3 introduces the bias-corrected approach for deriving MLEs that are bias-free to the second error, along with the MLE and PBE methods. Section 4 conducts a Monte Carlo simulation to compare these methods, supplemented by real data for practical illustration. Finally, Section 5 concludes the article.

2. Generalized Normal Distribution

Let X be a GN-distributed random variable with location parameter μ, scale parameter σ, and the shape parameter s as considered in [8]. The probability density function (pdf) of GN distribution is defined as:

$$f(x) = \frac{s}{2\sigma\Gamma(1/s)} exp\left\{-\left|\frac{x-\mu}{\sigma}\right|^s\right\}, x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+, \ s \in \mathbb{R}^+. \tag{1}$$

It is known that the pdf given in 1 has two special cases in terms of s values. When s equals 1, the distribution reduces to the Laplace distribution. Further, when s equals 2, the distribution is normal. Figure 1 shows the different pdf graphs of the GN distribution. We can see from Figure 1 that the distribution is leptokurtic and heavy-tailed for small values of shape parameter s. The pdf is the bell-shaped curve for certain values of s, and the pdfs have the peaky shape of maximum. It can also be observed from the figure that the GN distribution is symmetric around the location parameter, and varying the shape and scale parameters allows for different types of pdfs ([9]). This flexibility gives the GN distribution a wide range of tail behavior, from thinner to thicker tails compared to the normal distribution.

The cumulative distribution function (cdf) of the GN distribution, as given in [8], can be written as:

$$F(x) = \frac{\mathit{\Gamma}\left(1/s, ((\mu-x)/\sigma)^s\right)}{2\mathit{\Gamma}(1/s)}, \ for \ x \le \mu, \tag{2}$$

$$F(x) = 1 - \frac{\mathit{\Gamma}\left(1/s, ((\mu-x)/\sigma)^s\right)}{2\mathit{\Gamma}(1/s)}, \ for \ x > \mu, \tag{3}$$

where $\mathit{\Gamma}(\alpha, x) = \int_x^\infty t^{\alpha-1} \exp(-t)\, dt$ shows the incomplete gamma function. The n^{th} moment of the GN distribution about the origin for each positive n integer was

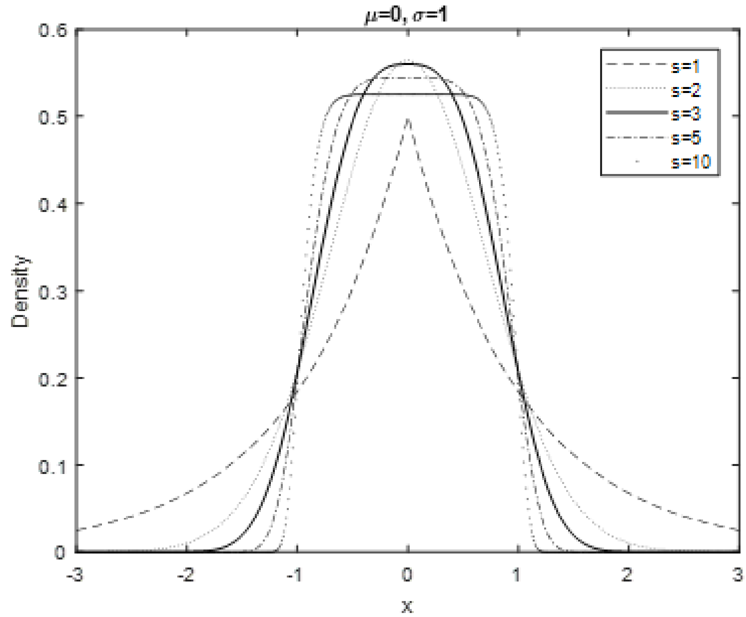

FIGURE 1. Some pdf examples of the GN distribution for $\mu = 0, \sigma = 1$ and different parameter values of s

introduced by [8] to obtain some distributional measures of the GN distribution. This moment is:

$$E\left(X^n\right) = \frac{\mu^n \sum_{k=0}^{n} \binom{n}{k} \left(\sigma/\mu\right)^k \left\{1 + (-1)^k\right\} \Gamma\left((k+1)/s\right)}{2\Gamma\left(1/s\right)}. \tag{4}$$

Further, the expectation, variance, kurtosis, and skewness of the GN distribution can be obtained with the help of the n^{th} moment of the GN distribution given in 5:

$$E\left(X\right) = \mu, \quad Var\left(X\right) = \frac{\sigma^2 \Gamma\left(3/s\right)}{\Gamma\left(1/s\right)},$$

$$Skewness = 0, Kurtosis = \frac{\Gamma\left(1/s\right)\Gamma\left(5/s\right)}{\Gamma^2\left(\frac{3}{s}\right)}. \tag{5}$$

It can be observed that the center of the distribution is μ and the skewness is zero. The variance and kurtosis are related to the parameter s, their values change concerning s.

3. Parameter Estimation

This section presents the ML estimation, along with Cox-Snell bias-corrected, and bootstrap-based bias-corrected inferences for the location and scale parameters of the GN distribution. To simplify computations, the shape parameter s will be estimated using the ML method across all estimation techniques.

3.1. **ML Estimation.** Let $x = x_1, x_2, \ldots, x_n$ be a random sample of size n from a GN distribution with parameter vector $\theta = (\mu, \sigma, s)$. The log-likelihood function for this sample can be written as:

$$l\left(\theta|x\right) = nlog\left(\frac{s}{2\sigma\Gamma\left(1/s\right)}\right) - \sum_{i=1}^{n}\left|\frac{x_i - \mu}{\sigma}\right|^{s}. \tag{6}$$

The maximum likelihood estimates for the parameters μ, σ, and s, $\widehat{\mu}$, $\widehat{\sigma}$, and $\hat{s}$ respectively, can be obtained by the maximization of 6, or equivalently solving the following nonlinear equations:

$$\frac{\partial l}{\partial \mu} = \frac{s}{\sigma^s}\left\{\sum_{x_i \geq \mu}\left(x_i - \mu\right)^{s-1} - \sum_{x_i < \mu}\left(\mu - x_i\right)^{s-1}\right\}, \tag{7}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{s}{\sigma^{s+1}}\sum_{i=1}^{n}\left|x_i - \mu\right|^{s}, \tag{8}$$

$$\frac{\partial l}{\partial s} = \frac{n}{s}\left\{\frac{1}{s}\Psi\left(\frac{1}{s}\right) + 1\right\} - \sum_{i=1}^{n}\left|\frac{x_i - \mu}{\sigma}\right|\log\left|\frac{x_i - \mu}{\sigma}\right|. \tag{9}$$

We note that solving these nonlinear equations requires the use of numerical methods due to their complexity. Some numerical optimization algorithms can be employed to find the related ML estimates.

3.2. **Cox-Snell Bias-Corrected ML Estimation.** Let $l\left(\theta\right)$ be the log-likelihood function based on a sample of n observations and p-dimensional parameter vector θ. $l\left(\theta\right)$ is assumed to be regular, meaning that all derivatives up to and including the third order exist and are continuous. Here, we first estimate the parameter s using the ML estimation method. Then, for a given estimate $\hat{s}$, the joint cumulants of derivates of $l\left(\theta^*\right)$ are defined as follows, where $\theta^* = (\mu, \sigma)$:

$$\kappa_{ij} = E\left(\frac{\partial^2 l}{\partial\theta_i^*\partial\theta_j^*}\right), \qquad i, j = 1, 2, \ldots, p, \tag{10}$$

$$\kappa_{ijl} = E\left(\frac{\partial^3 l}{\partial\theta_i^*\partial\theta_j^*\partial\theta_l^*}\right), \qquad i, j, l = 1, 2, \ldots, p, \tag{11}$$

$$\kappa_{ij,l} = E\left(\frac{\partial^3 l}{\partial\theta_i^*\partial\theta_j^*\partial\theta_l^*}\right)E\left[\left(\frac{\partial^2 l}{\partial\theta_i^*\partial\theta_j^*}\right)\left(\frac{\partial l}{\partial\theta_l^*}\right)\right], \qquad i, j, l = 1, 2, \ldots, p. \tag{12}$$

The derivates of the second-order cumulants are denoted as follows:

$$\kappa_{ij}^{(l)} = \frac{\partial \kappa_{ij}}{\partial \theta_l^*}, \qquad i, j, l = 1, 2, \ldots, p. \tag{13}$$

All of the expressions in 10 to 13 are assumed to be of the order $O_{(n)}$. [1] showed that when the sample data are independent (but not necessarily identically distributed), the bias of the r^{th} element of the ML of θ^*, denoted as $\widehat{\theta}^*$, can be expressed as:

$$Bias\left(\widehat{\theta}_r^*\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} \kappa^{ri}\kappa^{jl}\left[0.5k_{ijl} + k_{ij,l}\right] + \mathcal{O}\left(n^{-2}\right), \tag{14}$$

where $r = 1, \ldots, p$, κ_{ijl} is the $(i,j)^{th}$ element of the inverse of the information matrix denoted as $K = \{-\kappa_{ij}\}$. Corderio and Klein [32] noted that even when all equations in 10 to 13 are of the order $O_{(n)}$, Eq. 14 still holds if the data are non-independent. Eq. 14 can be rewritten in the following form:

$$Bias\left(\widehat{\theta}_r^*\right) = \sum_{r=1}^{p} \kappa^{ri} \sum_{j=1}^{p}\sum_{l=1}^{p} \left[\kappa_{ij}^{(l)} - \frac{1}{2}\kappa_{ijl}\right] + \mathcal{O}\left(n^{-2}\right), \qquad r = 1, 2, \ldots, p. \tag{15}$$

Now, let $a_{ij}^{(l)} = \kappa_{ij}^{(l)} - (\kappa_{ijl}/2)$, for $i, j, l = 1, 2, \ldots, p$ and define the following matrices:

$$A^{(l)} = \left\{a_{ij}^{(l)}\right\}, \qquad i, j, l = 1, 2, \ldots, p, \tag{16}$$

$$A = \left[A^{(1)}|A^{(2)}|\cdots|A^{(p)}\right]. \tag{17}$$

They also showed that the $\mathcal{O}\left(n^{-1}\right)$ bias of the MLE of θ^* in Eq.15 can be re-expressed as:

$$Bias\left(\widehat{\theta}_r^*\right) = K^{-1}Avec\left(K^{-1}\right) + \mathcal{O}\left(n^{-2}\right). \tag{18}$$

Then, the BCE for θ^* can be obtained as:

$$\widehat{\theta}_{r(BCE)}^* = \widehat{\theta}_r^* - \hat{K}^{-1}\hat{A}vec\left(\hat{K}^{-1}\right), \tag{19}$$

where $\hat{K} = K|_{\widehat{\theta}^*}$ and $\hat{A} = A|_{\widehat{\theta}^*}$, and it can be shown that the bias of $\widetilde{\theta}^*$ will be $\mathcal{O}\left(n^{-2}\right)$.

3.3. **Some Inferential Aspects.** To proceed, we require the derivates of the log-likelihood function up to the third order. The derivates can be obtained as:

$$\begin{aligned}
\frac{\partial^2 l}{\partial \mu^2} =& -\frac{s(s-1)}{\sigma^2}\left|\frac{x-\mu}{\sigma}\right|^{s-2} \\
\frac{\partial^3 l}{\partial \mu^3} =& \frac{s(s-1)(s-2)}{\sigma^3}\left|\frac{x-\mu}{\sigma}\right|^{s-3} \\
\frac{\partial^2 l}{\partial \sigma^2} =& \frac{1}{\sigma^2}\left\{1-s(s+1)\left|\frac{x-\mu}{\sigma}\right|^{s}\right\} \\
\frac{\partial^3 l}{\partial \sigma^3} =& -\frac{1}{\sigma^3}\left\{2-s(s+1)(s+2)\left|\frac{x-\mu}{\sigma}\right|^{s}\right\} \\
\frac{\partial^2 l}{\partial \mu \partial \sigma} =& \frac{s}{\sigma^2} sign(\mu-x)\left|\frac{x-\mu}{\sigma}\right|^{s-1} \\
\frac{\partial^3 l}{\partial \mu \partial \sigma^2} =& -\frac{(s+1)s^2}{\sigma^3} sign(\mu-x)\left|\frac{x-\mu}{\sigma}\right|^{s-1} \\
\frac{\partial^3 l}{\partial \mu^2 \partial \sigma} =& -\frac{s^2(s-1)}{\sigma^3} sign(\mu-x)\left|\frac{x-\mu}{\sigma}\right|^{s-2}.
\end{aligned}$$

The joint cumulants of the derivates of the log-likelihood function are found as follows:

$$\begin{aligned}
\kappa_{11} =& E\left[\partial^2 l/\partial \mu^2\right] = -\frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^2\Gamma\left(\frac{1}{s}\right)} \\
\kappa_{12} =& \kappa_{21} = E\left[\partial^2 l/\partial \mu \partial \sigma\right] = 0 \\
\kappa_{22} =& E\left[\partial^2 l/\partial \sigma^2\right] = -\frac{s}{\sigma^2} \\
\kappa_{111} =& E\left[\partial^3 l/\partial \mu^3\right] = \frac{s(s-1)(s-2)}{\sigma^3\Gamma\left(\frac{1}{s}\right)} \\
\kappa_{112} =& \kappa_{121} = \kappa_{211} = E\left[\partial^3 l/\partial \mu^2 \partial \sigma\right] = \frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)} \\
\kappa_{222} =& E\left[\partial^3 l/\partial \sigma^3\right] = \frac{s(s+3)}{\sigma^3} \\
\kappa_{122} =& \kappa_{212} = \kappa_{221} = E\left[\partial^3 l/\partial \sigma^2 \partial \mu\right] = \frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\kappa_{11}^{(1)} =& \partial \kappa_{11}/\partial \mu = 0 \\
\kappa_{12}^{(1)} =& \partial \kappa_{12}/\partial \mu = 0 \\
\kappa_{22}^{(1)} =& \partial \kappa_{22}/\partial \mu = 0
\end{aligned}$$

$$\kappa_{11}^{(2)} = \partial\kappa_{11}/\partial\sigma = \frac{2s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}$$
$$\kappa_{12}^{(2)} = \partial\kappa_{12}/\partial\sigma = 0$$
$$\kappa_{22}^{(2)} = \partial\kappa_{22}/\partial\sigma = \frac{2s}{\sigma^3}.$$

So, we obtain the elements of $A^{(1)}$:

$$a_{11}^{(1)} = \kappa_{11}^{(1)} - 0.5\kappa_{111} = -0.5\left[\frac{s(s-1)(s-2)\Gamma\left(\frac{s-2}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right]$$
$$a_{12}^{(1)} = \kappa_{12}^{(1)} - 0.5\kappa_{121} = -0.5\left[\frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right]$$
$$a_{22}^{(1)} = \kappa_{22}^{(1)} - 0.5\kappa_{122} = -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right].$$

The elements of $A^{(2)}$ are:

$$a_{11}^{(2)} = \kappa_{11}^{(2)} - 0.5\kappa_{112} = \frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\left[2-\frac{s}{2}\right]$$
$$a_{12}^{(2)} = \kappa_{12}^{(2)} - 0.5\kappa_{122} = 0 - 0.5\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}$$
$$a_{22}^{(2)} = \kappa_{22}^{(2)} - 0.5\kappa_{222} = \frac{s^2}{\sigma^3}\left[2-\frac{(s+3)}{2}\right].$$

Finally, the information matrix yields as:

$$K = \{-\kappa_{ij}\} = n\begin{bmatrix} -\frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^2\Gamma\left(\frac{1}{s}\right)} & 0 \\ 0 & -\frac{s}{\sigma^2} \end{bmatrix}.$$

Let define $A = \left[A^{(1)}|A^{(2)}\right]$. Then, we have:

$$A = n\begin{bmatrix} -0.5\left[\frac{s(s-1)(s-2)\Gamma\left(\frac{s-2}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & -0.5\left[\frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & \frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\left[2-\frac{s}{2}\right] & -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] \\ -0.5\left[\frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & \frac{s^2}{\sigma^3}\left[2-\frac{(s+3)}{2}\right] \end{bmatrix}$$

Using Corderio and Klein's [32] modification of Coxx and Snell's [1]) method, we can write the bias of $\widehat{\theta}_r^*$ in the following way:

$$Bias\left(\widehat{\theta}_r^*\right) = Bias\begin{pmatrix} \widehat{\mu} \\ \widehat{\sigma} \end{pmatrix} = K^{-1}Avec\left(K^{-1}\right),$$

where $\hat{K} = K|_{\mu=\widehat{\mu};\sigma=\widehat{\sigma}}$ and $\hat{A} = A|_{\mu=\widehat{\mu};\sigma=\widehat{\sigma}}$.

3.4. **Bootstrap-Based Bias Corrected ML Estimation.** An alternative approach that we consider to derive bias-corrected MLEs for the unknown parameters of GN distribution is a bootstrap resampling method by [13]. The bootstrap method uses the MLEs of the data to generate random samples from GN distribution to estimate the bias, and then subtract the bias from the MLE. For a parameter vector θ^*, the estimated bias of $\widehat{\theta}$ is given by:

$$Bias\left(\widehat{\theta}^*_r\right) = \frac{1}{B}\sum_{j=1}^{B}\widehat{\theta}^*_{r(j)} - \widehat{\theta}^*_r, \tag{20}$$

where $\widehat{\theta}^*_{r(j)}$ is the MLE of θ^* obtained from the *j-th* Bootstrap sample. Hence, the bootstrap bias-corrected estimator is defined as:

$$\widehat{\theta}^*_{r(PBE)} = 2\widehat{\theta}^*_r - \frac{1}{B}\sum_{j=1}^{B}\widehat{\theta}^*_{r(j)}. \tag{21}$$

4. Applications

This section includes a simulation study and a real data example to demonstrate the proposed estimators' performance for the GN distribution's location and scale parameters. Some computational aspects are provided below:

Computational details:

1) We used MATLAB R2017b software for all numerical calculations.

2) The *Nelder-Mead* algorithm for the *fminsearch* function in MATLAB was employed to obtain the ML estimates.

3) For generating the random sample from the GN distribution in the simulation study, we followed the random number-generating algorithm outlined below:

Random number generating algorithm from GN distribution:

Step 1. Sample $X \sim Gamma\left(1/s, 1\right)$.

Step 2. Generate a random sample from the independent random variable Z:

$$Z \sim \frac{1}{2}\left[Z = -1\right] + \frac{1}{2}\left[Z = 1\right].$$

Step 3. Generate a random sample from the $GN\left(\mu, \sigma, s\right)$ with the help of the following equation:

$$Y = \mu + \sigma Z |X|^{1/s} \sim GN\left(\mu, \sigma, s\right).$$

4.1. **Simulation study.** This section presents the results of a Monte Carlo simulation study that compares the performances of the MLE, BCE, and PBE. This evaluation is based on biases and mean squared error (MSE) values, calculated using the following formulas:

$$\widehat{bias}\left(\widehat{\theta}\right) = \overline{\theta} - \theta, \quad \widehat{MSE}\left(\widehat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)^2,$$

where θ is the true parameter value, $\widehat{\theta}_j$ is the estimate of θ for the ith simulated dataset and $\overline{\theta} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\theta}_j$. The simulation studies are carried out $N = 10000$ times with sample sizes n=10, 20, 30, 40, and 50. The true parameters are μ= 1 and 2, s=2, 3 and 5, and σ=0.5, 1, 1.5, 2, 2.5, and 3. Note that the ML estimation method is employed for estimating the shape parameter s in all scenarios. The simulation focuses on comparing the location and scale parameters of the GN distribution.

To assess and compare the performances of the estimators, we employed the integrated bias squared (IBSQ) and average root mean square error (ARMSE) criteria, as introduced by Cribari-Neto and Vasconcellos [33]. These criteria are defined as follows:

$$IBSQ_{(n)} = \sqrt{\frac{1}{36}\sum_{h=1}^{36}\left(r_{h,n}\right)^2}, \quad ARMSE_{(n)} = \frac{1}{36}\sum_{h=1}^{36}RMSE_{h,n},$$

where $r_{h,n}$ and $RMSE_{h,n}$ present the estimated bias and estimated root mean squared error for the h-th scenario, $h = 1, \ldots, 36$. These criteria provide comprehensive measures for the overall performance of the estimators across a range of scenarios.

Simulation results:

The accuracy of the parameter estimates is evaluated by reporting bias and MSE values in Tables 1 - 6. The Monte Carlo experiments involve 1000 bootstrap replications. The results, show that, for the location parameter μ, PBE yields the smallest absolute biases, indicating that the bootstrap correction effectively reduces bias. BCE achieves the smallest MSE values, suggesting that the Cox-Snell correction improves the estimator's precision. MLE and PBE demonstrate similar performances according to MSE criteria for μ, and all estimates show consistency as MSE values decrease with increasing sample sizes. Regarding the scale parameter σ, BCE consistently provides the best results in terms of absolute bias in most simulation cases, with PBE following closely. Moreover, BCE outperforms other estimation methods in terms of MSE values for σ. Similar to μ, all estimates exhibit consistency for σ as indicated by decreasing MSE values with increasing sample sizes.

The results are presented in Tables 7 and 8, which include IBSQ and ARMSE values for different sample sizes. Upon examination of these tables, it is noted that for the parameter μ, IBSQ values display similarity among the estimators. However, ARMSE values highlight the superior performance of BCE. Additionally, concerning the parameter σ, both IBSQ and ARMSE values indicate that BCE and

PBE outperform MLE. Consequently, while BCE and PBE methods demonstrate similar performance, BCE is computationally more straightforward than PBE.

4.2. **Real Data Application.** This section investigates the analysis of a real dataset to illustrate the efficacy of the proposed BCE in comparison to MLE for the parameters of the GN distribution. The dataset relates to hurricanes in the Atlantic, USA, sourced from the Atlantic track files maintained by the US National Hurricane Center. Covering major storm events from 1851 to 2000, this dataset characterizes the weeks of the hurricane season. The hurricane season, as reported by the National Hurricane Center, commences on June 1, and the hurricane dates are transformed into "weeks of the season". These weeks are aggregated in 2-week intervals, starting from June 1-14 (weeks 1 and 2) and concluding in early January (weeks 31 and 32). The dataset covers a total of 755 weeks of the season, and it can be accessed through the link: `https://seattlecentral.edu/qelp/sets/070/070.html#About`.

In this part, we compare the performances of the estimation methods discussed in this paper by applying them to the hurricane dataset. To evaluate the goodness of fit of each estimation method for the given dataset, we utilize the Kolmogorov-Smirnov (KS) test statistics. The steps for calculating the KS statistics are summarized as follows:

- Sort the dataset in descending order.
- Calculate the maximum absolute difference:

$$D = \max_{i=1,2\ldots,n} \{|F_n(x) - F(x)|\} \tag{22}$$

  where, $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \leq x}$ is the empirical cdf and $F(x)$ the theoretical cdf of the distribution being tested.
- The smallest value of D indicates the best fit between the theoretical distribution and the observed data.

The estimation results for MLE, BCE, and PBE are summarized in Table 9. Notably, the shape parameter (s) is estimated as 1.1084 using the ML estimation method. Alternatively, we can estimate the shape parameter s using the profile likelihood estimation method. To illustrate this, we present the profile log-likelihood graph for different values of s in Figure 2. From this figure, we observed that the estimated s value is about 1.0, which aligns closely with the ML estimate of s.

Table 9 provides the KS values, estimates, and bootstrap standard errors in brackets. The results indicate that MLE and BCE yield similar results for the parameter μ. Conversely, PBE demonstrates superior performance for estimating the parameter σ. This suggests that, in the context of this real dataset, PBE outperforms both MLE and BCE in estimating the scale parameter of the GN distribution. Notably, the KS values for all estimators are closely aligned. In terms of the overall fit of the estimators to the dataset, the ML estimator exhibits the smallest KS test statistic. Furthermore, Figure 3 displays the histogram of the

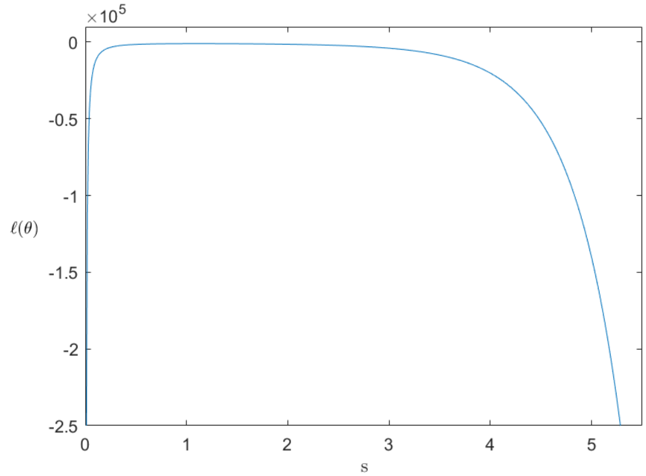

FIGURE 2. The profile log-likelihood graph for different values of s

hurricane dataset alongside the estimated pdfs obtained from MLE, BCE, and PBE. It is evident from the figure that all estimators provide similar results, effectively capturing the characteristics of the entire dataset.

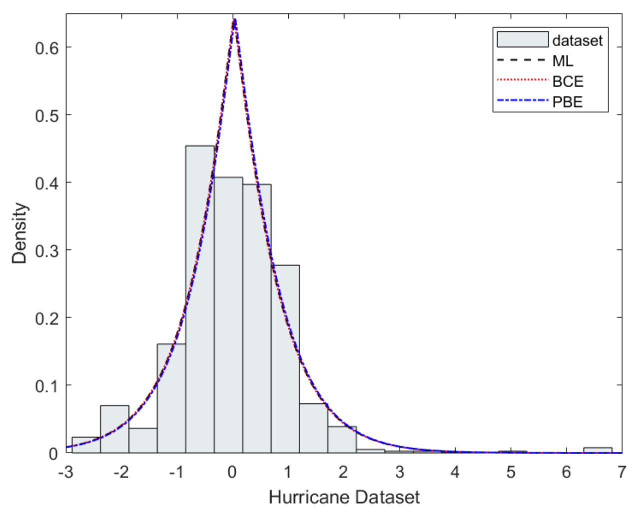

FIGURE 3. The histogram of the hurricane dataset along with the estimated pdfs obtained from MLE, BCE and PBE

TABLE 1. Estimated bias (MSE) for μ and σ, $\mu = 2, s = 2$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0737 (0.0057)	-0.0045 (0.0252)	-0.0009 (0.0247)	-0.0491 (0.0025)	-0.1529 (0.0475)	0.0484 (0.0298)
	20	-0.0384 (0.0015)	-0.0014 (0.0125)	-0.0004 (0.0131)	-0.0256 (0.0007)	-0.1812 (0.0451)	0.0270 (0.0151)
	30	-0.0259 (0.0007)	-0.0006 (0.0083)	-0.0001 (0.0083)	-0.0173 (0.0003)	-0.1895 (0.0441)	0.0172 (0.0092)
	40	-0.0195 (0.0004)	-0.0004 (0.0062)	-0.0001 (0.0063)	-0.0130 (0.0002)	-0.1939 (0.0440)	0.0130 (0.0068)
	50	-0.0157 (0.0003)	-0.0002 (0.0050)	-0.0000 (0.0051)	-0.0105 (0.0001)	-0.1965 (0.0437)	0.0104 (0.0052)
1.0	10	-0.1044 (0.0115)	-0.0017 (0.0494)	-0.0002 (0.0496)	-0.0696 (0.0051)	-0.0741 (0.0533)	0.0683 (0.0599)
	20	-0.0541 (0.0030)	-0.0019 (0.0253)	-0.0001 (0.0248)	-0.0361 (0.0013)	-0.0380 (0.0261)	0.0365 (0.0290)
	30	-0.0367 (0.0014)	-0.0004 (0.0169)	-0.0002 (0.0168)	-0.0244 (0.0006)	-0.0250 (0.0174)	0.0249 (0.0188)
	40	-0.0277 (0.0008)	-0.0003 (0.0126)	-0.0001 (0.0126)	-0.0184 (0.0003)	-0.0194 (0.0129)	0.0186 (0.0137)
	50	-0.0222 (0.0005)	-0.0002 (0.0100)	-0.0000 (0.0100)	-0.0148 (0.0002)	-0.0138 (0.0102)	0.0148 (0.0108)
1.5	10	-0.1271 (0.0171)	-0.0013 (0.0737)	-0.0001 (0.0737)	-0.0847 (0.0076)	-0.3692 (0.2080)	0.0830 (0.0901)
	20	-0.0665 (0.0045)	-0.0002 (0.0376)	-0.0002 (0.0375)	-0.0443 (0.0020)	-0.3219 (0.1395)	0.0449 (0.0410)
	30	-0.0449 (0.0020)	-0.0003 (0.0250)	-0.0003 (0.0252)	-0.0299 (0.0009)	-0.3058 (0.1183)	0.0302 (0.0275)
	40	-0.0339 (0.0012)	-0.0003 (0.0186)	-0.0001 (0.0187)	-0.0226 (0.0005)	-0.2989 (0.1082)	0.0227 (0.0202)
	50	-0.0272 (0.0007)	-0.0005 (0.0153)	-0.0001 (0.0153)	-0.0181 (0.0003)	-0.2935 (0.1011)	0.0183 (0.0162)
2.0	10	-0.1472 (0.0229)	-0.0007 (0.0974)	-0.0007 (0.0975)	-0.0981 (0.0102)	-0.6927 (0.5760)	0.1061 (0.1361)
	20	-0.0768 (0.0061)	-0.0009 (0.0505)	-0.0004 (0.0506)	-0.0512 (0.0027)	-0.6424 (0.4613)	0.0529 (0.0593)
	30	-0.0519 (0.0027)	-0.0004 (0.0329)	-0.0002 (0.0329)	-0.0346 (0.0012)	-0.6234 (0.4220)	0.0353 (0.0376)
	40	-0.0391 (0.0015)	-0.0009 (0.0252)	-0.0002 (0.0252)	-0.0260 (0.0007)	-0.6120 (0.3996)	0.0268 (0.0270)
	50	-0.0314 (0.0010)	-0.0005 (0.0197)	-0.0001 (0.0197)	-0.0209 (0.0004)	-0.6084 (0.3897)	0.0210 (0.0217)
2.5	10	-0.1650 (0.0288)	-0.0024 (0.1233)	-0.0012 (0.1235)	-0.1099 (0.0128)	-1.0373 (1.1964)	0.1185 (0.1659)
	20	-0.0857 (0.0075)	-0.0008 (0.0617)	-0.0006 (0.0618)	-0.0571 (0.0033)	-0.9810 (1.0244)	0.0588 (0.0721)
	30	-0.0578 (0.0034)	-0.0004 (0.0415)	-0.0001 (0.0416)	-0.0385 (0.0015)	-0.9596 (0.9616)	0.0389 (0.0464)
	40	-0.0438 (0.0019)	-0.0002 (0.0316)	-0.0001 (0.0316)	-0.0292 (0.0009)	-0.9443 (0.9229)	0.0288 (0.0334)
	50	-0.0351 (0.0012)	-0.0010 (0.0249)	-0.0000 (0.0249)	-0.0234 (0.0006)	-0.9425 (0.9130)	0.0239 (0.0261)
3.0	10	-0.1807 (0.0345)	-0.0023 (0.1480)	-0.0003 (0.1482)	-0.1204 (0.0153)	-1.3970 (2.0950)	0.1310 (0.1983)
	20	-0.0938 (0.0090)	-0.0005 (0.0741)	-0.0002 (0.0741)	-0.0625 (0.0040)	-1.3361 (1.8589)	0.0655 (0.0883)
	30	-0.0634 (0.0041)	-0.0003 (0.0498)	-0.0001 (0.0499)	-0.0422 (0.0018)	-1.3118 (1.7704)	0.0433 (0.0553)
	40	-0.0480 (0.0023)	-0.0002 (0.0379)	-0.0001 (0.0379)	-0.0320 (0.0010)	-1.3015 (1.7307)	0.0321 (0.0411)
	50	-0.0384 (0.0015)	-0.0011 (0.0298)	-0.0000 (0.0299)	-0.0256 (0.0007)	-1.2941 (1.7045)	0.0258 (0.0323)

TABLE 2. Estimated bias (MSE) for μ and σ, $\mu = 2, s = 3$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0686 (0.0049)	-0.0014 (0.0172)	-0.0004 (0.0174)	-0.0537 (0.0030)	-0.1464 (0.0391)	0.0565 (0.0258)
	20	-0.0360 (0.0013)	-0.0006 (0.0086)	-0.0001 (0.0086)	-0.0282 (0.0008)	-0.1759 (0.0393)	0.0284 (0.0106)
	30	-0.0244 (0.0006)	-0.0013 (0.0055)	-0.0001 (0.0056)	-0.0191 (0.0004)	-0.1870 (0.0405)	0.0188 (0.0066)
	40	-0.0184 (0.0003)	-0.0002 (0.0041)	-0.0000 (0.0042)	-0.0144 (0.0002)	-0.1920 (0.0410)	0.0143 (0.0049)
	50	-0.0148 (0.0001)	-0.0001 (0.0033)	-0.0000 (0.0033)	-0.0116 (0.0001)	-0.1963 (0.0418)	0.0115 (0.0036)
1.0	10	-0.0970 (0.0098)	-0.0018 (0.0343)	-0.0007 (0.0347)	-0.0760 (0.0060)	-0.0855 (0.0413)	0.0802 (0.0523)
	20	-0.0510 (0.0026)	-0.0013 (0.0172)	-0.0002 (0.0172)	-0.0399 (0.0016)	-0.0434 (0.0190)	0.0403 (0.0217)
	30	-0.0344 (0.0012)	-0.0007 (0.0111)	-0.0001 (0.0111)	-0.0270 (0.0007)	-0.0284 (0.0119)	0.0272 (0.0133)
	40	-0.0260 (0.0007)	-0.0015 (0.0082)	-0.0000 (0.0082)	-0.0204 (0.0005)	-0.0196 (0.0088)	0.0204 (0.0095)
	50	-0.0210 (0.0004)	-0.0005 (0.0066)	-0.0000 (0.0066)	-0.0164 (0.0003)	-0.0172 (0.0071)	0.0161 (0.0072)
1.5	10	-0.1188 (0.0147)	-0.0011 (0.0515)	-0.0007 (0.0521)	-0.0930 (0.0090)	-0.3821 (0.1982)	0.0975 (0.0769)
	20	-0.0624 (0.0040)	-0.0003 (0.0258)	-0.0003 (0.0259)	-0.0489 (0.0024)	-0.3289 (0.1336)	0.0489 (0.0308)
	30	-0.0422 (0.0018)	-0.0008 (0.0166)	-0.0001 (0.0167)	-0.0330 (0.0011)	-0.3105 (0.1128)	0.0330 (0.0197)
	40	-0.0319 (0.0010)	-0.0003 (0.0123)	-0.0000 (0.0124)	-0.0250 (0.0006)	-0.3004 (0.1028)	0.0248 (0.0141)
	50	-0.0257 (0.0007)	-0.0013 (0.0100)	-0.0000 (0.0100)	-0.0201 (0.0004)	-0.2950 (0.0970)	0.0199 (0.0109)
2.0	10	-0.1372 (0.0196)	-0.0011 (0.0687)	-0.0006 (0.0965)	-0.1074 (0.0120)	-0.7133 (0.5781)	0.1124 (0.1030)
	20	-0.0721 (0.0053)	-0.0021 (0.0344)	-0.0001 (0.0345)	-0.0564 (0.0032)	-0.6453 (0.4501)	0.0568 (0.0421)
	30	-0.0487 (0.0024)	-0.0014 (0.0222)	-0.0003 (0.0222)	-0.0381 (0.0015)	-0.6243 (0.4122)	0.0379 (0.0258)
	40	-0.0368 (0.0014)	-0.0027 (0.0164)	-0.0001 (0.0165)	-0.0288 (0.0008)	-0.6140 (0.3940)	0.0286 (0.0191)
	50	-0.0296 (0.0009)	-0.0004 (0.0133)	-0.0000 (0.0133)	-0.0232 (0.0005)	-0.6112 (0.3870)	0.0231 (0.0145)
2.5	10	-0.1534 (0.0245)	-0.0017 (0.0858)	-0.0005 (0.0869)	-0.1201 (0.0150)	-1.0563 (1.2034)	0.1281 (0.1312)
	20	-0.0806 (0.0066)	-0.0009 (0.0430)	-0.0002 (0.0431)	-0.0631 (0.0041)	-0.9868 (1.0161)	0.0624 (0.0528)
	30	-0.0545 (0.0030)	-0.0016 (0.0277)	-0.0001 (0.0278)	-0.0426 (0.0018)	-0.9615 (0.9527)	0.0428 (0.0326)
	40	-0.0412 (0.0017)	-0.0012 (0.0206)	-0.0002 (0.0206)	-0.0322 (0.0010)	-0.9532 (0.9298)	0.0320 (0.0237)
	50	-0.0331 (0.0011)	-0.0008 (0.0166)	-0.0001 (0.0166)	-0.0259 (0.0007)	-0.9435 (0.9068)	0.0257 (0.0182)
3.0	10	-0.1681 (0.0294)	-0.0018 (0.1030)	-0.0007 (0.1042)	-0.1316 (0.0180)	-1.4215 (2.1238)	0.1394 (0.1573)
	20	-0.0883 (0.0079)	-0.0022 (0.0516)	-0.0004 (0.0517)	-0.0691 (0.0049)	-1.3427 (1.8539)	0.0701 (0.0644)
	30	-0.0597 (0.0036)	-0.0010 (0.0333)	-0.0001 (0.0334)	-0.0467 (0.0022)	-1.3154 (1.7641)	0.0465 (0.0394)
	40	-0.0451 (0.0021)	-0.0004 (0.0247)	-0.0001 (0.0247)	-0.0353 (0.0013)	-1.3040 (1.7254)	0.0349 (0.0283)
	50	-0.0363 (0.0013)	-0.0010 (0.0199)	-0.0001 (0.0199)	-0.0284 (0.0008)	-1.2964 (1.7014)	0.0281 (0.0221)

TABLE 3. Estimated bias (MSE) for μ and σ, $\mu = 2, s = 5$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0648 (0.0043)	-0.0005 (0.0121)	-0.0008 (0.0127)	-0.0570 (0.0034)	-0.1352 (0.0303)	0.0590 (0.0204)
	20	-0.0344 (0.0012)	-0.0004 (0.0055)	-0.0003 (0.0056)	-0.0303 (0.0009)	-0.1737 (0.0357)	0.0296 (0.0075)
	30	-0.0233 (0.0006)	-0.0006 (0.0035)	-0.0001 (0.0036)	-0.0206 (0.0004)	-0.1864 (0.0383)	0.0197 (0.0044)
	40	-0.0176 (0.0003)	-0.0007 (0.0026)	-0.0001 (0.0026)	-0.0155 (0.0002)	-0.1906 (0.0390)	0.0148 (0.0031)
	50	-0.0142 (0.0002)	-0.0008 (0.0021)	-0.0000 (0.0021)	-0.0125 (0.0001)	-0.1940 (0.0397)	0.0120 (0.0024)
1.0	10	-0.0916 (0.0087)	-0.0012 (0.0242)	-0.0001 (0.0254)	-0.0807 (0.0067)	-0.1020 (0.0348)	0.0851 (0.0414)
	20	-0.0487 (0.0024)	-0.0005 (0.0110)	-0.0001 (0.0112)	-0.0428 (0.0019)	-0.0473 (0.0133)	0.0420 (0.0146)
	30	-0.0330 (0.0011)	-0.0012 (0.0071)	-0.0000 (0.0072)	-0.0291 (0.0009)	-0.0307 (0.0080)	0.0281 (0.0087)
	40	-0.0250 (0.0006)	-0.0007 (0.0052)	-0.0000 (0.0052)	-0.0220 (0.0005)	-0.0242 (0.0058)	0.0213 (0.0064)
	50	-0.0201 (0.0004)	-0.0008 (0.0042)	-0.0000 (0.0042)	-0.0177 (0.0003)	-0.0184 (0.0045)	0.0171 (0.0047)
1.5	10	-0.1122 (0.0130)	-0.0015 (0.0363)	-0.0003 (0.0380)	-0.0988 (0.0101)	-0.4002 (0.1970)	0.1028 (0.0599)
	20	-0.0596 (0.0036)	-0.0017 (0.0165)	-0.0001 (0.0168)	-0.0525 (0.0028)	-0.3329 (0.1273)	0.0500 (0.0220)
	30	-0.0404 (0.0016)	-0.0013 (0.0106)	-0.0001 (0.0107)	-0.0356 (0.0013)	-0.3132 (0.1089)	0.0341 (0.0132)
	40	-0.0306 (0.0009)	-0.0014 (0.0078)	-0.0000 (0.0078)	-0.0269 (0.0007)	-0.3025 (0.0994)	0.0258 (0.0092)
	50	-0.0246 (0.0006)	-0.0012 (0.0062)	-0.0000 (0.0063)	-0.0216 (0.0005)	-0.2968 (0.0943)	0.0209 (0.0071)
2.0	10	-0.1296 (0.0173)	-0.0002 (0.0484)	-0.0004 (0.0507)	-0.1141 (0.0134)	-0.7257 (0.5749)	0.1194 (0.0807)
	20	-0.0688 (0.0048)	-0.0002 (0.0220)	-0.0009 (0.0224)	-0.0606 (0.0037)	-0.6536 (0.4497)	0.0590 (0.0404)
	30	-0.0467 (0.0022)	-0.0008 (0.0142)	-0.0002 (0.0143)	-0.0411 (0.0017)	-0.6315 (0.4129)	0.0392 (0.0254)
	40	-0.0353 (0.0013)	-0.0001 (0.0103)	-0.0001 (0.0104)	-0.0311 (0.0010)	-0.6183 (0.3928)	0.0301 (0.0188)
	50	-0.0284 (0.0008)	-0.0005 (0.0083)	-0.0000 (0.0084)	-0.0250 (0.0006)	-0.6121 (0.3831)	0.0240 (0.0094)
2.5	10	-0.1449 (0.0216)	-0.0014 (0.0605)	-0.0015 (0.0634)	-0.1275 (0.0168)	-1.0798 (1.2271)	0.1329 (0.1014)
	20	-0.0770 (0.0060)	-0.0012 (0.0275)	-0.0007 (0.0280)	-0.0678 (0.0046)	-0.9969 (1.0215)	0.0661 (0.0375)
	30	-0.0522 (0.0027)	-0.0013 (0.0177)	-0.0004 (0.0179)	-0.0460 (0.0021)	-0.9704 (0.9592)	0.0443 (0.0213)
	40	-0.0395 (0.0016)	-0.0006 (0.0129)	-0.0002 (0.0130)	-0.0347 (0.0012)	-0.9567 (0.9282)	0.0336 (0.0153)
	50	-0.0318 (0.0010)	-0.0007 (0.0104)	-0.0001 (0.0105)	-0.0279 (0.0008)	-0.9479 (0.9089)	0.0271 (0.0119)
3.0	10	-0.1587 (0.0260)	-0.0021 (0.0726)	-0.0018 (0.0761)	-0.1397 (0.0201)	-1.4400 (2.1469)	0.1445 (0.1207)
	20	-0.0843 (0.0072)	-0.0012 (0.0330)	-0.0004 (0.0336)	-0.0742 (0.0056)	-1.3491 (1.8544)	0.0727 (0.0443)
	30	-0.0572 (0.0033)	-0.0003 (0.0213)	-0.0001 (0.0215)	-0.0503 (0.0026)	-1.3234 (1.7728)	0.0483 (0.0258)
	40	-0.0432 (0.0019)	-0.0007 (0.0155)	-0.0000 (0.0156)	-0.0380 (0.0015)	-1.3087 (1.7289)	0.0364 (0.0186)
	50	-0.0348 (0.0012)	-0.0003 (0.0125)	-0.0000 (0.0126)	-0.0306 (0.0009)	-1.3010 (1.7055)	0.0295 (0.0142)

TABLE 4. Estimated bias (MSE) for μ and σ, $\mu = 1, s = 2$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0735 (0.0057)	-0.0006 (0.0248)	-0.0004 (0.0248)	-0.0490 (0.0025)	-0.1545 (0.0486)	0.0532 (0.0342)
	20	-0.0383 (0.0015)	-0.0003 (0.0124)	-0.0001 (0.0124)	-0.0256 (0.0007)	-0.1799 (0.0446)	0.0264 (0.0146)
	30	-0.0259 (0.0007)	-0.0003 (0.0083)	-0.0000 (0.0083)	-0.0173 (0.0003)	-0.1885 (0.0437)	0.0177 (0.0092)
	40	-0.0196 (0.0004)	-0.0007 (0.0063)	-0.0000 (0.0063)	-0.0131 (0.0002)	-0.1949 (0.0442)	0.0132 (0.0068)
	50	-0.0157 (0.0002)	-0.0004 (0.0051)	-0.0000 (0.0051)	-0.0105 (0.0001)	-0.1956 (0.0433)	0.0104 (0.0053)
1.0	10	-0.1039 (0.0114)	-0.0008 (0.0499)	-0.0001 (0.0500)	-0.0692 (0.0051)	-0.0761 (0.0550)	0.0761 (0.0686)
	20	-0.0543 (0.0030)	-0.0005 (0.0251)	-0.0004 (0.0251)	-0.0362 (0.0013)	-0.0382 (0.0266)	0.0377 (0.0300)
	30	-0.0366 (0.0014)	-0.0022 (0.0165)	-0.0002 (0.0166)	-0.0244 (0.0006)	-0.0253 (0.0169)	0.0248 (0.0183)
	40	-0.0276 (0.0008)	-0.0003 (0.0126)	-0.0001 (0.0126)	-0.0184 (0.0003)	-0.0190 (0.0127)	0.0185 (0.0134)
	50	-0.0222 (0.0005)	-0.0012 (0.0101)	-0.0000 (0.0101)	-0.0148 (0.0002)	-0.0156 (0.0101)	0.0151 (0.0107)
1.5	10	-0.1278 (0.0173)	-0.0008 (0.0740)	-0.0006 (0.0741)	-0.0851 (0.0077)	-0.3693 (0.2119)	0.0932 (0.1054)
	20	-0.0664 (0.0045)	-0.0007 (0.0370)	-0.0001 (0.0371)	-0.0442 (0.0020)	-0.3242 (0.1420)	0.0459 (0.0439)
	30	-0.0448 (0.0020)	-0.0010 (0.0249)	-0.0004 (0.0249)	-0.0299 (0.0009)	-0.3042 (0.1147)	0.0305 (0.0278)
	40	-0.0339 (0.0012)	-0.0026 (0.0189)	-0.0002 (0.0189)	-0.0226 (0.0005)	-0.2968 (0.1068)	0.0230 (0.0205)
	50	-0.0272 (0.0007)	-0.0008 (0.0149)	-0.0001 (0.0149)	-0.0181 (0.0003)	-0.2924 (0.1002)	0.0182 (0.0157)
2.0	10	-0.1475 (0.0230)	-0.0053 (0.1987)	-0.0016 (0.0988)	-0.0983 (0.0102)	-0.6940 (0.5794)	0.1063 (0.1365)
	20	-0.0766 (0.0060)	-0.0006 (0.0494)	-0.0001 (0.0494)	-0.0511 (0.0027)	-0.6407 (0.4599)	0.0541 (0.0590)
	30	-0.0517 (0.0027)	-0.0004 (0.0332)	-0.0000 (0.0332)	-0.0345 (0.0012)	-0.6192 (0.4155)	0.0354 (0.0362)
	40	-0.0392 (0.0016)	-0.0031 (0.0253)	-0.0001 (0.0253)	-0.0261 (0.0007)	-0.6089 (0.3959)	0.0263 (0.0276)
	50	-0.0314 (0.0010)	-0.0010 (0.0199)	-0.0000 (0.0199)	-0.0209 (0.0004)	-0.6073 (0.3888)	0.0210 (0.0215)
2.5	10	-0.1650 (0.0288)	-0.0047 (0.1233)	-0.0018 (0.1235)	-0.1099 (0.0128)	-1.0382 (1.2006)	0.1227 (0.1737)
	20	-0.0857 (0.0075)	-0.0012 (0.0617)	-0.0003 (0.0618)	-0.0571 (0.0033)	-0.9781 (1.0186)	0.0585 (0.0727)
	30	-0.0578 (0.0034)	-0.0006 (0.0415)	-0.0001 (0.0416)	-0.0385 (0.0015)	-0.9587 (0.9615)	0.0394 (0.0474)
	40	-0.0438 (0.0019)	-0.0015 (0.0316)	-0.0000 (0.0316)	-0.0292 (0.0009)	-0.9508 (0.9343)	0.0295 (0.0332)
	50	-0.0351 (0.0012)	-0.0019 (0.0249)	-0.0000 (0.0249)	-0.0234 (0.0006)	-0.9436 (0.9157)	0.0235 (0.0273)
3.0	10	-0.1807 (0.0345)	-0.0006 (0.1480)	-0.0021 (0.1481)	-0.1204 (0.0153)	-1.3986 (2.0994)	0.1306 (0.1997)
	20	-0.0938 (0.0090)	-0.0016 (0.0741)	-0.0010 (0.0741)	-0.0625 (0.0040)	-1.3327 (1.8492)	0.0652 (0.0871)
	30	-0.0634 (0.0041)	-0.0002 (0.0498)	-0.0003 (0.0499)	-0.0422 (0.0018)	-1.3151 (1.7787)	0.0428 (0.0554)
	40	-0.0480 (0.0023)	-0.0007 (0.0379)	-0.0004 (0.0379)	-0.0320 (0.0010)	-1.2994 (1.7252)	0.0324 (0.0402)
	50	-0.0384 (0.0015)	-0.0021 (0.0298)	-0.0001 (0.0299)	-0.0256 (0.0007)	-1.2946 (1.7057)	0.0259 (0.0318)

TABLE 5. Estimated bias (MSE) for μ and σ, $\mu = 1, s = 3$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0686 (0.0049)	-0.0015 (0.0172)	-0.0002 (0.0174)	-0.0537 (0.0030)	-0.1461 (0.0388)	0.0566 (0.0262)
	20	-0.0360 (0.0013)	-0.0005 (0.0086)	-0.0001 (0.0086)	-0.0282 (0.0008)	-0.1769 (0.0394)	0.0281 (0.0102)
	30	-0.0244 (0.0006)	-0.0007 (0.0057)	-0.0000 (0.0057)	-0.0191 (0.0004)	-0.1868 (0.0404)	0.0190 (0.0064)
	40	-0.0184 (0.0003)	-0.0005 (0.0042)	-0.0000 (0.0042)	-0.0144 (0.0002)	-0.1919 (0.0410)	0.0144 (0.0046)
	50	-0.0148 (0.0002)	-0.0002 (0.0033)	-0.0000 (0.0033)	-0.0116 (0.0001)	-0.1952 (0.0414)	0.0115 (0.0036)
1.0	10	-0.0972 (0.0099)	-0.0006 (0.0341)	-0.0006 (0.0345)	-0.0761 (0.0060)	-0.0889 (0.0426)	0.0801 (0.0515)
	20	-0.0509 (0.0026)	-0.0012 (0.0163)	-0.0002 (0.0163)	-0.0398 (0.0016)	-0.0436 (0.0193)	0.0399 (0.0216)
	30	-0.0345 (0.0012)	-0.0003 (0.0115)	-0.0001 (0.0115)	-0.0270 (0.0007)	-0.0281 (0.0125)	0.0269 (0.0136)
	40	-0.0261 (0.0007)	-0.0011 (0.0083)	-0.0000 (0.0083)	-0.0204 (0.0004)	-0.0214 (0.0088)	0.0200 (0.0093)
	50	-0.0210 (0.0004)	-0.0004 (0.0067)	-0.0000 (0.0067)	-0.0164 (0.0003)	-0.0175 (0.0071)	0.0163 (0.0074)
1.5	10	-0.1188 (0.0147)	-0.0003 (0.0532)	-0.0006 (0.0540)	-0.0930 (0.0090)	-0.3824 (0.1981)	0.0981 (0.0779)
	20	-0.0623 (0.0040)	-0.0006 (0.0258)	-0.0006 (0.0259)	-0.0488 (0.0024)	-0.3256 (0.1314)	0.0489 (0.0315)
	30	-0.0421 (0.0018)	-0.0008 (0.0169)	-0.0002 (0.0169)	-0.0330 (0.0011)	-0.3114 (0.1136)	0.0328 (0.0193)
	40	-0.0319 (0.0010)	-0.0012 (0.0124)	-0.0001 (0.0124)	-0.0249 (0.0006)	-0.3001 (0.1025)	0.0248 (0.0139)
	50	-0.0256 (0.0007)	-0.0025 (0.0104)	-0.0001 (0.0104)	-0.0201 (0.0004)	-0.2961 (0.0978)	0.0201 (0.0111)
2.0	10	-0.1372 (0.0196)	-0.0019 (0.0687)	-0.0022 (0.0695)	-0.1074 (0.0120)	-0.7104 (0.5738)	0.1132 (0.1035)
	20	-0.0721 (0.0053)	-0.0003 (0.0344)	-0.0001 (0.0345)	-0.0564 (0.0032)	-0.6450 (0.4503)	0.0561 (0.0427)
	30	-0.0487 (0.0024)	-0.0005 (0.0222)	-0.0003 (0.0222)	-0.0381 (0.0015)	-0.6264 (0.4155)	0.0380 (0.0264)
	40	-0.0368 (0.0014)	-0.0005 (0.0164)	-0.0000 (0.0165)	-0.0288 (0.0008)	-0.6146 (0.3948)	0.0289 (0.0186)
	50	-0.0296 (0.0009)	-0.0022 (0.0133)	-0.0000 (0.0133)	-0.0232 (0.0005)	-0.6121 (0.3882)	0.0230 (0.0145)
2.5	10	-0.1539 (0.0247)	-0.0019 (0.0872)	-0.0001 (0.0886)	-0.1205 (0.0151)	-1.0565 (1.2028)	0.1271 (0.1291)
	20	-0.0808 (0.0067)	-0.0005 (0.0420)	-0.0008 (0.0422)	-0.0633 (0.0041)	-0.9880 (1.0188)	0.0628 (0.0528)
	30	-0.0546 (0.0030)	-0.0010 (0.0280)	-0.0005 (0.0281)	-0.0427 (0.0018)	-0.9616 (0.9534)	0.0424 (0.0331)
	40	-0.0412 (0.0017)	-0.0009 (0.0208)	-0.0002 (0.0208)	-0.0323 (0.0011)	-0.9509 (0.9252)	0.0318 (0.0239)
	50	-0.0331 (0.0011)	-0.0009 (0.0167)	-0.0001 (0.0167)	-0.0259 (0.0007)	-0.9457 (0.9114)	0.0259 (0.0184)
3.0	10	-0.1680 (0.0294)	-0.0036 (0.1059)	-0.0012 (0.1074)	-0.1315 (0.0180)	-1.4258 (2.1359)	0.1389 (0.1572)
	20	-0.0881 (0.0079)	-0.0008 (0.0508)	-0.0013 (0.0509)	-0.0690 (0.0048)	-1.3434 (1.8572)	0.0698 (0.0640)
	30	-0.0597 (0.0036)	-0.0007 (0.0332)	-0.0003 (0.0333)	-0.0467 (0.0022)	-1.3168 (1.7671)	0.0472 (0.0398)
	40	-0.0451 (0.0021)	-0.0017 (0.0244)	-0.0002 (0.0245)	-0.0353 (0.0013)	-1.3047 (1.7281)	0.0350 (0.0278)
	50	-0.0363 (0.0013)	-0.0012 (0.0199)	-0.0001 (0.0200)	-0.0284 (0.0008)	-1.2955 (1.6980)	0.0281 (0.0220)

TABLE 7. Integrated bias squared norm for BCE, MLE, and PBE

	Estimator for μ			Estimator for σ		
n	BCE	MLE	PBE	BCE	MLE	PBE
10	0.1304	0.0020	0.0009	0.0999	0.7983	0.1057
20	0.0679	0.0012	0.0004	0.0525	0.7456	0.0526
30	0.0459	0.0009	0.0002	0.0356	0.7287	0.0352
40	0.0347	0.0011	0.0001	0.0269	0.7204	0.0265
50	0.0279	0.0010	0.0001	0.0216	0.7158	0.0213

TABLE 8. Average root-mean-squared error for BCE, MLE, and PBE

	Estimator for μ			Estimator for σ		
n	BCE	MLE	PBE	BCE	MLE	PBE
10	0.1227	0.2339	0.2336	0.0951	0.6491	0.2809
20	0.0636	0.1604	0.1609	0.0494	0.5831	0.1797
30	0.0431	0.1299	0.1302	0.0334	0.5602	0.1412
40	0.0324	0.1120	0.1122	0.0251	0.5485	0.1198
50	0.0257	0.1002	0.1004	0.0200	0.5416	0.1052

TABLE 9. Estimation results for MLE, BCE, and PBE (Bootstrap standard errors) for hurricane dataset

Estimators	μ	σ	KS-value
MLE	0.0168 (0.0604)	0.8078 (0.1186)	0.0855
BCE	0.0182 (0.0604)	0.8084 (0.1188)	0.0892
PBE	0.0403 (0.1288)	0.8076 (0.1181)	0.1010

5. CONCLUSION

In this paper, we introduced two bias correction methods, BCE and PBE, for estimating the parameters of the GN distribution. Our simulation study results showed that the BCE for the scale parameter consistently outperforms both MLE and PBE across all sample sizes. Notably, the BCE demonstrates significantly improved accuracy and reliability.

On the other hand, for the location parameter, the PBE exhibits minimal biases, while the BCE shows small MSE values in all cases. This numerical analysis highlights the effectiveness of the proposed bias correction methodology, particularly in improving the precision of parameter estimates for both the location and scale parameters.

To further illustrate the practical applicability of the MLE, BCE, and PBE, we provided a real-data example involving the analysis of hurricane data from the Atlantic, USA. The findings from the real-data example indicated that the PBE outperforms bootstrap standard errors for the scale parameter, highlighting its superiority in estimating the scale parameter's variability. Additionally, the bootstrap standard errors for both the MLE and BCE are comparable for the location parameter.

In conclusion, our study suggests that the BCE offers a practical and useful alternative to the MLE, particularly in cases with small to moderate sample sizes. By incorporating Efron's bootstrap procedure, our proposed BCE method contributes to more accurate and reliable parameter estimation within the GN distribution framework.

Author Contribution Statements All authors contributed equally to conceptualization, writing, methodology, supervision, software, visualization, review, and editing.

Declaration of Competing Interests The authors declare that they have no competing interests.

Acknowledgements We are grateful for the support provided by Giresun University (Grant number: FEN-BAP-A-090323-15) through its Type A project of Scientific Research and Development Projects. Additionally, we thank two anonymous referees and the associate editor for their valuable comments and suggestions, which have greatly improved the paper.

References

[1] Cox, D. R., Snell, E. J., A general definition of residuals, *Journal of the Royal Statistical Society: Series B (Methodological)*, 30(2) (1968), 248-265. https://doi.org/10.1111/j.2517-6161.1968.tb00724.x

[2] Briasouli, A., Tsakalides, P., Stouraitis, A., Hidden messages in heavy tails: DCT-Domain watermark detection using AlphaStable models, *IEEE Trans*, 7(4) (2005), 700-715. https://doi.org/10.1109/TMM.2005.850970

[3] Kokkinakis, K., Nandi, A., Exponent parameter estimation for generalized Gaussian probability density functions with application to speech modeling, *Signal Processing*, 85 (2005) 1852-1858. https://doi.org/10.1016/j.sigpro.2005.02.017

[4] Sharifi, K., Leon-Garcia, A., Estimation of shape parameter for generalized Gaussian distributions in subband decompositions of video, *IEEE Transactions on Circuits and Systems for Video Technology*, 5(1) (1995), 52-56. https://doi.org/10.1109/76.350779

[5] Choi, S., Cichocki, A., Amari, S., Flexible independent component analysis. In Neural Networks for Signal Processing 8, *Proceedings of the 1998 IEEE Signal Processing Society Workshop*, (1998), 83-92. https://doi.org/10.1023/A:1008135131269

[6] Wu, H. C., Principe, J., Minimum entropy algorithm for source separation, *In 1998 Midwest Symposium on Circuits and Systems, Notre Dame, USA*, (1998), 242-245. https://doi.org/10.1109/MWSCAS.1998.759478

[7] Subbotin, M. T., On the Law of Frequency of Error, *Maths Books*, 31(2) (1923), 206-301. http://mi.mathnet.ru/sm6854

[8] Nadarajah, S., A generalized normal distribution, *Journal of Applied Statistics*, 32(7) (2005), 685-694. https://doi.org/10.1080/02664760500079464

[9] Varanasi, M. K., Aazhang, B., Parametric generalized Gaussian density estimation. *Journal of the Acoustical Society of America*, 86(4) (1989), 1404-1415. https://doi.org/10.1121/1.398700

[10] Roenko, A. A., Lukin, V. V., Djurović, I., Simeunović, M., Estimation of parameters for generalized Gaussian distribution, *In 2014 6th International Symposium on Communications, Control and Signal Processing (ISCCSP), IEEE*, (2014), 376-379. https://doi.org/10.1109/ISCCSP.2014.6877892

[11] Eskin, E. N., Joint Modelling of the Location and Scale Parameters of the Generalized Normal Distribution, Master's Thesis, Giresun University, Giresun, Turkey, (2022).

[12] Eskin, E. N., Doğru, F. Z., A heteroscedastic regression model with the generalized normal distribution, *Sigma Journal of Engineering and Natural Sciences, in press*, 42(5) (2024), 1480-1489. https://sigma.yildiz.edu.tr/article/1673 doi:10.14744/sigma.2024.00114

[13] Efron, B., The jackknife, the bootstrap and other resampling plans, *Society for Industrial and Applied Mathematics*, (1982). https://doi.org/10.1137/1.9781611970319

[14] Efron, B., Tibshirani, R. J., An Introduction to the Bootstrap, Volume 57 of Monographs on Statistics and Applied Probability, Chapman and Hall, New York, 1994. https://doi.org/10.1201/9780429246593

[15] Cordeiro, G. M., Da Rocha, E. C., Da Rocha, J. G. C., Cribari-Neto, F., Bias-corrected maximum likelihood estimation for the beta distribution, *Journal of Statistical Computation and Simulation*, 58(1) (1997), 21-35. https://doi.org/10.1080/00949659708811820

[16] Saha, K., Paul, S., Bias-corrected maximum likelihood estimator of the negative binomial dispersion parameter, *Biometrics*, 61(1) (2005), 179-185. https://doi.org/10.1111/j.0006-341X.2005.030833.x

[17] Lemonte, A. J., Cribari-Neto, F., Vasconcellos, K. L., Improved statistical inference for the two-parameter Birnbaum–Saunders distribution, *Computational Statistics and Data Analysis*, 51(9) (2007), 4656-4681. https://doi.org/10.1016/j.csda.2006.08.016

[18] Giles, D. E., Feng, H., Bias of the maximum likelihood estimators of the two-parameter gamma distribution revisited, Econometrics Working Paper EWP0906, Department of Economics, University of Victoria, 2009. https://ideas.repec.org/p/vic/vicewp/0908.html

[19] Lemonte, A. J., Improved point estimation for the Kumaraswamy distribution, *Journal of Statistical Computation and Simulation*, 81(12) (2011), 1971-1982. https://doi.org/10.1080/00949655.2010.511621

[20] Giles, D. E., A note on improved estimation for the Topp-Leone distribution, Econometrics Working Paper EWP1203, Department of Economics, University of Victoria, 2012. https://ideas.repec.org/p/vic/vicewp/1703.html

[21] Giles, D. E., Feng, H., Godwin, R. T., On the bias of the maximum likelihood estimator for the two-parameter Lomax distribution, *Communications in Statistics-Theory and Methods*, 42(11) (2013), 1934-1950. https://doi.org/10.1080/03610926.2011.600506

[22] Schwartz, J., Godwin, R. T., Giles, D. E., Improved maximum-likelihood estimation of the shape parameter in the Nakagami distribution, *Journal of Statistical Computation and Simulation*, 83(3) (2013), 434-445. https://doi.org/10.1080/00949655.2011.615316

[23] Zhang, G., Liu, R., Bias-corrected estimators of scalar skew normal, *In New Developments in Statistical Modeling, Inference and Application, Springer, Cham*, (2016), 203-214. https://doi.org/10.1007/978-3-319-42571-9_11

[24] Schwartz, J., Giles, D. E., Bias-reduced maximum likelihood estimation of the zero-inflated Poisson distribution, *Communications in Statistics-Theory and Methods*, 45(2) (2016), 465-478. https://doi.org/10.1080/03610926.2013.824590

[25] Wang, M., Wang, W., Bias-corrected maximum likelihood estimation of the parameters of the weighted Lindley distribution, *Communications in Statistics-Simulation and Computation*, 46(1) (2017), 530-545. https://doi.org/10.1080/03610918.2014.970696
[26] Reath, J., Dong, J., Wang, M., Improved parameter estimation of the log-logistic distribution with applications, *Computational Statistics*, 33(1) (2018), 339-356. https://doi.org/10.1007/s00180-017-0738-y
[27] Mazucheli, J., Dey, S., Bias-corrected maximum likelihood estimation of the parameters of the generalized half-normal distribution, *Journal of Statistical Computation and Simulation*, 88(6) (2018), 1027-1038. https://doi.org/10.1080/00949655.2017.1413649
[28] Mazucheli, J., Menezes, A. F. B., Dey, S., Improved maximum-likelihood estimators for the parameters of the unit-gamma distribution, *Communications in Statistics-Theory and Methods*, 47(15) (2018), 3767-3778. https://doi.org/10.1080/03610926.2017.1361993
[29] Mazucheli, J., Menezes, A. F. B., Dey, S., Bias-corrected maximum likelihood estimators of the parameters of the inverse Weibull distribution, *Communications in Statistics-Simulation and Computation*, 48(7) (2019), 2046-2055. https://doi.org/10.1080/03610918.2018.1433838
[30] Menezes, A. F. B., Mazucheli, J., Improved maximum likelihood estimators for the parameters of the Johnson SB distribution, *Communications in Statistics-Simulation and Computation*, 49(6) (2020), 1511-1526. https://doi.org/10.1080/03610918.2018.1498892
[31] Menezes, A., Mazucheli, J., Alqallaf, F., Ghitany, M. E., Bias-corrected maximum likelihood estimators of the parameters of the Unit-Weibull distribution, *Austrian Journal of Statistics*, 50(3) (2021), 41-53. https://doi.org/10.17713/ajs.v50i3.1023
[32] Cordeiro, G. M., Klein, R., Bias correction in ARMA models, *Statistics and Probability Letters*, 19(3) (1994), 169-176. https://doi.org/10.1016/0167-7152(94)90100-7
[33] Cribari-Neto, F., Vasconcellos, K. L., Nearly unbiased maximum likelihood estimation for the beta distribution, *Journal of Statistical Computation and Simulation*, 72(2) (2002), 107-118. https://doi.org/10.1080/00949650212144