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Analytical and numerical study on the solutions of a new (2+1)-dimensional conformable shallow water wave equation

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Abstract. The (2+1)-dimensional conformable nonlinear shallow water wave equation is examined in this work. Initially, definitions and properties of suitable derivatives are presented. Subsequently, exact solutions to this equation are derived using the $\exp(-\phi(\xi))$-expansion and the modified extended tanh-function methods. Then, a numerical method, namely the residual power series method, is utilized to obtain approximate solutions. The interplay between analytical and numerical approaches is explored to validate the solutions. This study fills a gap in the literature on fractional shallow water models, particularly in (2+1)-dimensions, and offers new insights into wave dynamics governed by fractional derivatives. The physical implications of the findings are illustrated through 3D and 2D contour surfaces of some obtained data, offering insight into the physical interpretation of geometric structures. A table is also presented to compare the obtained results. These solutions highlight the practical uses of the investigated model and other nonlinear models in applied sciences. These techniques can potentially yield significant results in solving various fractional differential equations.

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1. Introduction

Fractional differential equations are essential in various fields spanning social and fundamental sciences and engineering disciplines. Recently, their importance has increased due to their indispensable contribution to understanding complex physical processes in areas such as control theory, electrical circuits, and wave propagation. In particular, fractional differential equations arise in various applications, including electrical circuits, chemical engineering, biostatistics and epidemiology, mechanical systems, computer science, optimization, drug development, social sciences, medicine and biology, weather and climate models, robotics and artificial intelligence, and signal processing.

These equations are valuable tools for modeling, analyzing, and designing solutions for numerous engineering problems. Their ability to vividly illustrate nonlinear physical features makes them an essential framework for guiding future work. Consequently, finding solutions to these equations is a remarkable achievement in related fields. Several authors utilized various techniques to compute these solutions and gain a deeper understanding of the essential features of material structures in various settings.

A variety of analytical methods have been employed to pursue solutions and nuanced comprehension of these equations. It has become evident that no single technique can universally address all types of nonlinear problems with precision. This realization has given rise to numerous methods, including the modified simplest equation method [31,32], the auxiliary equation method [18,19], the modified extended tanh-function method [16], the Bernoulli sub-equation function method [35,36], the $\exp(-\phi(\xi))$-expansion method [11], the sine-Gordon expansion method [37], the modified exponential function method [38], the rational sine-Gordon expansion method [39], the $(1/G')$-expansion method [13,40], the (G'/G^2)-expansion

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method [14], the modified (G'/G)-expansion method [5], the φ^6-model expansion method [20], and the homotopy perturbation method [15, 25, 26] etc.

The allure of high-dimensional fractional partial differential equations (FPDEs) has captivated the attention of academics in recent years. Their prevalence extends across biology, chemistry, physics, engineering, mechanics, and economics, among other branches. Various derivative definitions have been proposed for fractional differential equations, including the Riemann-Liouville [17], Caputo [34], and conformable [2] derivatives. The Riemann-Liouville derivative, stemming from the contributions of Riemann and Liouville, stands out for its frequent application in contemporary mathematical discourse. Additionally, the conformable fractional derivative approach has gained popularity among mathematicians due to its simplicity and reliability. The term "conformable" refers to the use of conformable fractional derivatives in the equation, which generalizes the classical shallow water wave equation to account for non-integer-order calculus. This fractional framework allows the model to better describe physical processes that exhibit memory, nonlocal effects, or complex dynamics, making the equation more adaptable to real-world phenomena.

As a well-known FPDE, shallow water wave equations model wave behavior in shallow bodies of water like seas, rivers, or coastal regions. The (2+1)-dimensions account for two spatial variables and time, which allows for more complex interactions like wave breaking, dispersion, and nonlinear effects. Here, we address the following shallow water wave problem in (2+1)-dimensions [3],

$$A\mathcal{D}_t^{\vartheta} u_x + au_{xx} + b(u)_{xx}^2 + cu_{xxxx} + du_{yy} = 0. \tag{1}$$

where $\mathcal{D}_t^{\vartheta}$ denotes the conformable derivative, and A, a, b, c, d are arbitrary constants. This equation serves as a descriptive model for the propagation of gravity waves on a water surface, particularly in scenarios where oblique waves directly interact with the surface [22]. Besides, the conformable shallow water wave equation describes the behavior of shallow water waves, typically focusing on how waves propagate in fluids where the horizontal length scale is much larger than the vertical depth.

Although there is a body of work on integer-order (2+1)-dimensional shallow water wave equations, the fractional (conformable) extension in (2+1)-dimensions is less studied. Research is particularly limited in deriving exact solutions for this model. For instance, in [27], the authors have obtained multiple rogue wave solutions to the model using the Hirota bilinear transformation and the trial function method. Besides, in this research paper, innovative methodologies are employed to present exact traveling wave solutions as well as the numerical solutions to Eq. (1). The objective is to surmount the limitations associated with conventional methods and offer effective solutions to this intricate equation.

The paper is organized as follows. Basic definitions are given in Section 2. The $\exp(-\phi(\xi))$-expansion method is described in detail in Section 3. The modified extended tanh-function approach is detailed in Section 4. A numerical approach, the residual power series method (RPSM), is introduced in Section 5. Section 6 contains analytical and approximate solutions of the studied equation. In Section 7, the paper presents the results.

2. Conformable Derivative

The conformable derivative is a relatively recent approach to fractional calculus that preserves many properties of the standard derivative, making it easier to apply to physical systems. Conformable derivatives have already been applied to classical models, improving the flexibility of solutions to represent more realistic physical phenomena.

Definition 1. The conformable derivative of a function, $h : [0, \infty) \to \mathbb{R},\ t > 0,\ \theta \in (0, 1)$ of order θ is as follows defined:

$$\mathcal{D}_t^{\theta}(h)(t) = \lim_{\gamma \to 0} \frac{h(t + \gamma t^{1-\theta}) - h(t)}{\gamma}. \tag{2}$$

Additionally, in the event that h is-differentiable within a given interval $(0, k)$, where $k > 0$, and the $\lim_{t \to 0^+} \mathcal{D}_t^{\theta}(h)(t)$ exists, then definition is formed

$$\mathcal{D}_t^{\theta}(h)(0) = \lim_{t \to 0^+} \mathcal{D}_t^{\theta}(h)(t). \tag{3}$$

Lemma 1. Let h_1 and h_2 be θ-differentiable at $t>0$ for $0<\theta\leq 1$ [12,24,28]. There after,

- $\mathcal{D}_t^\theta(t^{s_1}) = s_1 t^{s_1-\theta}$, $s_1 \in \mathbb{R}$,
- $\mathcal{D}_t^\theta(s_1h_1+s_2h_2) = s_1\mathcal{D}_t^\theta(h_1)+s_2\mathcal{D}_t^\theta(h_2)$, $s_1, s_2 \in \mathbb{R}$,
- $\mathcal{D}_t^\theta(\frac{h_1}{h_2}) = \frac{h_2.\mathcal{D}_t^\theta(h_1)-h_1\mathcal{T}_t^\theta(h_2)}{h_2^2}$,
- $\mathcal{D}_t^\theta(h_1.h_2) = h_1.\mathcal{D}_t^\theta(h_2)+h_2.\mathcal{D}_t^\theta(h_1)$,
- $\mathcal{D}_t^\theta(h_1)(t) = t^{1-\theta}\frac{dh_1(t)}{dt}$,
- $\mathcal{D}_t^\theta(C) = 0$, when C is a const.

Definition 2. Let the function h with n variables be defined as $(y_1, y_2, \ldots, y_n)$. The partial derivatives of h with respect to y_i of order $\theta \in (0,1]$ are given as [29,33]:

$$\frac{d^\theta}{dy_i^\theta}h(y_1,y_2,\ldots,y_n) = \lim_{\gamma\to 0}\frac{h(y_1,y_2,\ldots,y_{i-1},y_i+\gamma y_i^{1-\theta},y_n)-h(y_1,y_2,\ldots,y_n)}{\gamma}.$$

The next section is reserved to introduce the $\exp(-\phi(\xi))$-expansion, the modified extended tanh-function, and the RPS methods.

3. The Exp$(-\phi(\xi))$-Expansion Method

Examine the nonlinear equation, which is presented as follows:

$$\mathcal{P}(u, \mathcal{D}_t^\theta u, \mathcal{D}_x u, \mathcal{D}_y u, \mathcal{D}_x^2 u, \mathcal{D}_y^2 u, \ldots) = 0. \tag{4}$$

In this case, $\mathcal{D}_t^\theta$ represents the conformable derivative operator of the function. When $\mathcal{P}$ is a polynomial of $u(x,y,\ldots,t)$ and its derivatives, and the subscripts signifying partial derivatives. During utilizing the $\exp(-\phi(\xi))$-expansion method [1,21,23] for obtaining wave solutions of Eq. (4), it is crucial to carry out the next procedures.

- The real variables $x, y, z, \ldots, t$ are combined using ξ as a compound variable.

$$\xi = kx+ly+\cdots+\frac{mt^\theta}{\theta}, \quad u(x,y,z,\ldots,t) = u(\xi). \tag{5}$$

where the $k, l, \ldots, m$ are arbitrary values to be determined later.

- The following ordinary differential equation (ODE) is what is left after reducing Eq. (4),

$$\mathcal{H}(u(\xi), u'(\xi), u''(\xi), \ldots) = 0. \tag{6}$$

- The following finite series can be used to construct the precise solutions:

$$u(\xi) = B_0 + \sum_{r=1}^{N} B_r(\exp(\xi(-\phi)))^r, B_N \neq 0. \tag{7}$$

- The following ODE is satisfied by $\phi = \phi(\xi)$.

$$\phi'(\xi) = exp(-\phi(\xi)) + \eta exp(\phi(\xi)) + \lambda. \tag{8}$$

- Eq. (8) shows the following solutions when $\eta \neq 0$ and $\lambda^2 - 4\eta > 0$, depending on certain parameters.

$$u_1(\xi) = \frac{\ln\left(-\sqrt{(\lambda^2-4\eta)}\tanh\left(\frac{\sqrt{(\lambda^2-4\eta)}}{2}(h+\xi)\right)-\lambda\right)}{2\eta}, \tag{9}$$

in the case of $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$

$$u_2(\xi) = \frac{\ln\left(\sqrt{(4\eta-\lambda^2)}\tanh\left(\frac{\sqrt{(4\eta-\lambda^2)}}{2}(h+\xi)\right)-\lambda\right)}{2\eta}, \tag{10}$$

in the case of $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$u_3(\xi) = -\ln\left(\frac{\lambda}{\sinh(\lambda(h+\xi)) + \cosh(\lambda(h+\xi)) - 1}\right), \tag{11}$$

in the case of $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$ and $\eta \neq 0$,

$$u_4(\xi) = \ln\left(-\frac{2(\lambda(h+\xi)+2)}{\lambda^2(h+\xi)}\right), \tag{12}$$

in the case of $\lambda^2 - 4\eta = 0$, $\lambda = 0$ and $\eta = 0$,

$$u_5(\xi) = \ln(h+\xi), \tag{13}$$

where the constant for integration is h.

- The determination of the N value in Eq. (7) involves considering the balance principle between the largest nonlinear terms and the highest order derivatives of $u(\xi)$ as outlined in Eq. (6). Upon replacing Eq. (7) with Eq. (8) into Eq. (6) and consolidating terms with identical powers of $\exp(-\phi)$, the left-hand side of Eq. (6) undergoes a transformation into a polynomial. This transformation results in a system of algebraic equations involving variables B_r, $(r = 0, 1, 2, 3, \ldots, N), c, \lambda$, and η. The solution to Eq. (6) can be obtained by setting all the coefficients of this polynomial to zero, solving the resulting system of algebraic equations, and then substituting the solutions back into Eq. (7).

4. Modified Extended Tanh-Function Method

Let us explore a specific partial differential equation (PDE) to illustrate the core concept of the modified extended tanh-function method [4, 30, 41].

$$\mathcal{B}(v, \mathcal{D}_t^\theta v, \mathcal{D}_x v, \mathcal{D}_y v, \mathcal{D}_x^2 v, \mathcal{D}_y^2 v, \ldots) = 0, \tag{14}$$

where $\mathcal{B}$ is a polynomial in $v(x, y, z, \ldots, t)$ with nonlinear components in its partial derivatives. The transformation,

$$\xi = kx + ly + \ldots + \frac{mt^\theta}{\theta}, \quad v(x, y, z, \ldots, t) = v(\xi), \tag{15}$$

converts Eq. (14) into an ODE presented in the subsequent form,

$$\mathcal{B}(v(\xi), v'(\xi), v''(\xi), \ldots) = 0. \tag{16}$$

Assume that the solution to Eq. (16) takes on the following form,

$$v(\xi) = A_0 + \sum_{r=1}^{N}\left(A_r\phi^r(\xi) + B_r\phi^{-r}(\xi)\right). \tag{17}$$

Here, $\phi(\xi)$ satisfies the following Riccati equation,

$$\phi'(\xi) = \sigma + \phi(\xi)^2, \tag{18}$$

where σ is a constant that will be found out afterward. As may be seen below, Eq. (18) admits several different solutions as,

- If $\sigma < 0$

$$\phi(\xi) = -\sqrt{-\sigma}tanh(\sqrt{-\sigma}\xi) \quad or \quad \phi(\xi) = -\sqrt{-\sigma}coth(\sqrt{-\sigma}\xi).$$

- If $\sigma > 0$

$$\phi(\xi) = \sqrt{\sigma} tan(\sqrt{\sigma}\xi) \quad or \quad \phi(\xi) = -\sqrt{\sigma} cot(\sqrt{\sigma}\xi).$$

- If $\sigma = 0$

$$\phi(\xi) = -\frac{1}{\xi}. \tag{19}$$

Determining the positive integer N in Eq. (17) involves achieving a balance between the highest order derivatives and the nonlinear variables. Symbolic calculations can be used to find the values of A_r and B_r by replacing Eq. (17), and Eq. (18) in Eq. (16). Following this path, by gathering terms with the same power ϕ^r, where $(r = 0, 1, 2, \cdots, N)$, and setting them to zero, produces the unknown constants. The exact solutions to Eq. (14) can subsequently be derived by replacing the determined values, along with the Eq. (17).

5. Residual Power Series Method (RPSM)

To illustrate the principle of the RPSM [6–10] algorithm, examine the following nonlinear fractional differential equation (FDE).

$$\mathcal{D}_t^{\theta} u(x,y,t) + R[x,y]u(x,y,t) + N[x,y]u(x,y,t) = h(x,y,t). \tag{20}$$

where $R[x,y]$ is a linear and $N[x,y]$ is a nonlinear operator. The initial condition of the equation is expressed as

$$u(x,y,0) = f_0(x,y) = f(x,y). \tag{21}$$

Subject to the constraint of Eq. (21), the approach entails expanding a fractional series at $t = 0$ to find the solution to Eq. (20),

$$f_{n-1}(x,y) = h(x,y) = \mathcal{D}_t^{(n-1)\theta} u(x,y,0). \tag{22}$$

As seen below, the solution can be stated as a series expansion,

$$u(x,y,t) = \sum_{n=0}^{\infty} f_n(x,y) \frac{t^{n\theta}}{\theta^n n!}. \tag{23}$$

Thus, for $\mathbb{R}^{\frac{1}{v}}$ be the radius of convergence, $0 \le t < \mathbb{R}^{\frac{1}{v}}$ and $0 < \theta \le 1$, the $k-th$ truncated series of $u(x,y,t)$, represented as,

$$u_k(x,y,t) = f(x,y) + \sum_{n=1}^{k} f_n(x,y) \frac{t^{n\theta}}{\theta^n n!}, k = 1, 2, 3, \dots \tag{24}$$

Therefore, the $k-th$ residual function's initial expression is

$$Resu_k(x,y,t) = \mathcal{D}_t^{\theta} u_k(x,y,t) + R[x,y]u_k(x,y,t) + N[x,y]u_k(x,y,t) - h(x,y,t). \tag{25}$$

It is evident that for $t \ge 0$, $Resu(x,y,t) = 0$ and
$\lim_{k\to\infty} Resu_k(x,y,z,t) = Resu(x,y,z,t)$.

Calculating out $Resu_1(x,y,z,0) = 0$, yields the first unknown function, $f_1(x,y,z)$. The fractional derivative of a constant is 0 in the conformable sense, hence
$\mathcal{D}_t^{(n-1)\omega} Resu_k(x,y,z,t) = 0$ relative to $n = 1, 2, 3, ..., k$. The desired $f_n(x,y,z)$ coefficients are obtained by solving this equation for $t = 0$. Thus, $u_n(x,y,z,t)$ solutions may be determined, respectively.

The $\exp(-\phi(\xi))$-expansion and the modified extended tanh-function method can generate a variety of exact solutions, including exponential, solitons, periodic, and rational solutions. They are highly adaptable to different types of nonlinear equations. Besides, unlike many other techniques (such as perturbation methods), the RPSM does not require linearization or small parameter assumptions, making it suitable for strongly nonlinear PDEs. These methods are adaptable to a wide variety of nonlinear PDEs. This adaptability makes them suitable for models where other methods might fail or require substantial modification. For example, if your PDE includes fractional derivatives, nonlinear terms, or higher-order terms, these methods can often be extended to handle such complexities.

6. Application of the techniques

For the analytical methods, if we examine Eq. (1) in this context,

$$A\mathcal{D}_t^\theta u_x + au_{xx} + b(u)^2_{xx} + cu_{xxxx} + du_{yy} = 0. \tag{26}$$

Utilizing $u(x, y, t) = u(\xi)$ with $\xi = kx + ly + \frac{mt^\theta}{\theta}$ and performing the integration results in,

$$ak^2u(\xi) + Akmu(\xi) + bk^2u(\xi)^2 + ck^4u''(\xi) + dl^2u(\xi) = 0. \tag{27}$$

Balancing, $u^2 = 2N, u'' = N + 2$ results in $N = 2$. Upon substitution it into Eq. (7) and Eq. (17), the following exact solutions are derived.

6.1. Analytical solutions by exp$(-\phi(\xi))$-expansion method. Given that $N = 2$, upon substituting Eq. (7), the series of sums is as follows:

$$u = B_0 + B_1 \exp(-\phi(\xi)) + B_2 \exp(-\phi(\xi))^2. \tag{28}$$

When combined with Eq. (8), the algebraic system that follows is created.

$$\begin{gathered}
ak^2B_0 + dl^2B_0 + AkmB_0 + bk^2B_0^2 + ck^4\eta\lambda B_1 + 2ck^4\eta^2B_2 = 0,\\
ak^2B_1 + dl^2B_1 + AkmB_1 + 2ck^4\eta B_1 + ck^4\lambda^2B_1 + 2bk^2B_0B_1 + 6ck^4\eta\lambda B_2 = 0,\\
3ck^4\lambda B_1 + bk^2B_1^2 + ak^2B_2 + dl^2B_2 + AkmB_2 + 8ck^4\eta B_2 + 4ck^4\lambda^2B_2\\
+2bk^2B_0B_2 = 0,\\
2ck^4B_1 + 10ck^4\lambda B_2 + 2bk^2B_1B_2 = 0,\\
6ck^4B_2 + bk^2B_2^2 = 0.
\end{gathered}$$

Two cases and two sets of solutions for B_0, B_1, B_2, and m are obtained.

Case 1.

$$\begin{aligned}
B_0 &= -\frac{6c\eta k^2}{b},\ B_1 = -\frac{6ck^2\lambda}{b},\ B_2 = -\frac{6ck^2}{b},\\
m &= -\frac{ak^2 + ck^4\left(\lambda^2 - 4\eta\right) + dl^2}{Ak},\ \xi = kx + ly + \frac{mt^\theta}{\theta}.
\end{aligned}$$

Set 1.
For $\lambda^2 - 4\eta > 0$, $\eta \neq 0$,

$$\begin{aligned}
u_1(x, y, t) = -\frac{6c\eta\, k^2}{b} &- \frac{12c\,\eta k^2\lambda}{b\left(-\sqrt{\lambda^2 - 4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\eta}\Psi\right) - \lambda\right)}\\
&- \frac{24c\eta^2k^2}{b\left(-\sqrt{\lambda^2 - 4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\eta}\Psi\right) - \lambda\right)^2},
\end{aligned} \tag{29}$$

For $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$,

$$\begin{aligned}
u_2(x, y, t) = -\frac{6c\eta\, k^2}{b} &- \frac{12c\,\eta\lambda k^2}{b\left(\sqrt{4\eta - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta - \lambda^2}\Psi\right) - \lambda\right)}\\
&- \frac{24c\eta^2k^2}{b\left(\sqrt{4\eta - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta - \lambda^2}\Psi\right) - \lambda\right)^2},
\end{aligned} \tag{30}$$

where $\Psi = \left(-\frac{t^\theta\left(ak^2 + ck^4\left(\lambda^2 - 4\eta\right) + dl^2\right)}{A\theta k} + h + kx + ly\right)$.

For $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$\begin{aligned} u_3(x,y,t) &= -\frac{6ck^2\lambda^2}{b\left(\sinh\left(\lambda\Omega\right)+\cosh\left(\lambda\Omega\right)-1\right)} \\ &\quad -\frac{6ck^2\lambda^2}{b\left(\sinh\left(\ (\lambda\Omega\right)+\cosh\left(\lambda\Omega\right)-1\right)^2}, \end{aligned} \tag{31}$$

where $\Omega = \left(-\frac{t^\theta\left(ak^2+c\lambda^2k^4+dl^2\right)}{A\theta k} + h + kx + ly\right)$.

For $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$ and $\eta \neq 0$,

$$u_4(x,y,t) = -\frac{6c\eta k^2}{b} + \frac{3ck^2\lambda^3\Lambda}{b\left(\lambda\Lambda+2\right)} - \frac{3ck^2\lambda^4\Lambda^2}{2b\left(\lambda\Lambda+2\right)^2}, \tag{32}$$

where $\Lambda = \left(-\frac{t^\theta\left(ak^2+dl^2\right)}{A\theta k} + h + kx + ly\right)$

For $\lambda^2 - 4\eta = 0$, $\lambda = 0$ and $\eta = 0$,

$$u_5(x,y,t) = -\frac{6ck^2}{b\left(-\frac{t^\theta\left(ak^2+4c\eta k^4+dl^2\right)}{A\theta k} + h + kx + ly\right)^2}. \tag{33}$$

Case 2.

$$\begin{aligned} B_0 &= -\frac{ck^2\left(2\eta+\lambda^2\right)}{b},\ B_1 = -\frac{6ck^2\lambda}{b},\ B_2 = -\frac{6ck^2}{b}, \\ m &= \frac{-ak^2+ck^4\left(\lambda^2-4\eta\right)-dl^2}{Ak},\ \xi = kx+ly+\frac{mt^\theta}{\theta} \end{aligned}$$

Set 2.
For $\lambda^2 - 4\eta > 0$, $\eta \neq 0$,

$$\begin{aligned} u_6(x,y,t) = -\frac{ck^2\left(2\eta+\lambda^2\right)}{b} &- \frac{12c\eta k^2\lambda}{b\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\Upsilon\right)-\lambda\right)} \\ &- \frac{24c\eta^2k^2}{b\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\Upsilon\right)-\lambda\right)^2}, \end{aligned} \tag{34}$$

For $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$,

$$\begin{aligned} u_7(x,y,t) = -\frac{ck^2\left(2\eta+\lambda^2\right)}{b} &- \frac{12c\eta k^2\lambda}{b\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\Upsilon\right)-\lambda\right)} \\ &- \frac{24c\eta^2k^2}{b\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\Upsilon\right)-\lambda\right)^2}, \end{aligned} \tag{35}$$

where $\Upsilon = \left(\frac{t^\theta\left(-ak^2+ck^4\left(\lambda^2-4\eta\right)-dl^2\right)}{A\theta k} + h + kx + ly\right)$

For $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$\begin{aligned} u_8(x,y,t) &= -\frac{6ck^2\lambda^2}{b\left(\sinh\left(\lambda\Phi\right)+\cosh\left(\lambda\Phi\right)-1\right)} \\ &\quad -\frac{6ck^2\lambda^2}{b\left(\sinh\left(\lambda\Phi\right)+\cosh\left(\lambda\Phi\right)-1\right)^2} - \frac{ck^2\lambda^2}{b}, \end{aligned} \tag{36}$$

where $\Phi = \left(\frac{t^\theta\left(-ak^2+c\lambda^2k^4-dl^2\right)}{A\theta k} + h + kx + ly\right)$

For $\lambda^2-4\eta=0$, $\lambda\neq 0$ and $\eta\neq 0$,

$$u_9(x,y,t)=-\frac{6c\eta k^2}{b}+\frac{3ck^2\lambda^3\Xi}{b\left(\lambda\Xi+2\right)}-\frac{3ck^2\lambda^4\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+h+kx+ly\right)^2}{2b\left(\lambda\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+h+kx+ly\right)+2\right)^2}, \tag{37}$$

where $\Xi=\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+h+kx+ly\right)$

For $\lambda^2-4\eta=0$, $\lambda=0$ and $\eta=0$,

$$u_{10}(x,y,t)=-\frac{4c\eta k^2}{b}-\frac{6ck^2}{b\left(\frac{t^\theta\left(-ak^2+4c\eta k^4-dl^2\right)}{A\theta k}+h+kx+ly\right)^2}. \tag{38}$$

6.2. **The modified extended tanh-function method solutions.** By taking $N=2$, Eq. (17) becomes,

$$v=A_0+A_1\phi(\xi)+B_1\phi(\xi)^{-1}+A_2\phi(\xi)^2+B_2\phi(\xi)^{-2}, \tag{39}$$

and when considered together with the Eq. (18) here, the following algebraic system of equations is obtained,

$$\begin{gathered}
ak^2A_0+dl^2A_0+AkmA_0+bk^2A_0^2+2ck^4\sigma^2A_2+2bk^2A_1B_1+2ck^4B_2\\
+2bk^2A_2B_2=0,\\
bk^2A_1^2+ak^2A_2+dl^2A_2+AkmA_2+8ck^4\sigma A_2+2bk^2A_0A_2=0,\\
ak^2A_1+dl^2A_1+AkmA_1+2ck^4\sigma A_1+2bk^2A_0A_1+2bk^2A_2B_1=0,\\
bk^2B_1^2+ak^2B_2+dl^2B_2+AkmB_2+8ck^4\sigma B_2+2bk^2A_0B_2=0,\\
ak^2B_1+dl^2B_1+AkmB_1+2ck^4\sigma B_1+2bk^2A_0B_1+2bk^2A_1B_2=0,\\
2ck^4\sigma^2B_1+2bk^2B_1B_2=0,\\
6ck^4\sigma^2B_2+bk^2B_2^2=0,\\
2ck^4A_1+2bk^2A_1A_2=0,\\
6ck^4A_2+bk^2A_2^2=0.
\end{gathered}$$

Four cases and four sets of solutions for A_0, A_1 A_2, B_1, B_2 and m are obtained here.

Case 3.

$$\begin{aligned}
A_0 &= -\frac{12ck^2\sigma}{b},\ A_1=0,\ B_1=0,\ A_2=-\frac{6ck^2}{b},\ B_2=-\frac{6ck^2\sigma^2}{b},\\
m &= \frac{-ak^2+16ck^4\sigma-dl^2}{Ak},\ \xi=kx+ly+\frac{mt^\theta}{\theta}.
\end{aligned}$$

Set 3.

For $\sigma<0$,

$$\begin{aligned}
v_1(x,y,t)=&-\frac{12ck^2\sigma}{b}+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}\\
&+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b},
\end{aligned} \tag{40}$$

or

$$v_2(x,y,t)=-\frac{12ck^2\sigma}{b}+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}$$

$$+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{41}$$

For $\sigma > 0$,

$$v_3(x,y,t) = -\frac{12ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}$$
$$-\frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{42}$$

or

$$v_4(x,y,t) = -\frac{12ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}$$
$$-\frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{43}$$

For $\sigma = 0$,

$$v_5(x,y,t) = -\frac{6ck^2}{b\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+kx+ly\right)^2}. \tag{44}$$

Case 4.

$$\begin{aligned} A_0 &= -\frac{6ck^2\sigma}{b},\ A_1 = 0,\ B_1 = 0,\ A_2 = -\frac{6ck^2}{b},\ B_2 = 0, \\ m &= \frac{-ak^2+4ck^4\sigma-dl^2}{Ak},\ \xi = kx+ly+\frac{mt^\theta}{\theta}. \end{aligned}$$

Set 4.
For $\sigma < 0$,

$$v_6(x,y,t) = -\frac{6ck^2\sigma}{b} + \frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{45}$$

or

$$v_7(x,y,t) = -\frac{6ck^2\sigma}{b} + \frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{46}$$

For $\sigma > 0$,

$$v_8(x,y,t) = -\frac{6ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{47}$$

or

$$v_9(x,y,t) = -\frac{6ck^2\sigma}{b} - \frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2+4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{48}$$

For $\sigma = 0$,

$$v_{10}(x,y,t) = -\frac{6ck^2}{b\left(\frac{t^\theta\left(-ak^2-dl^2\right)}{A\theta k}+kx+ly\right)^2}. \tag{49}$$

Case 5.

$$\begin{aligned} A_0 &= -\frac{2ck^2\sigma}{b},\ A_1=0,\ B_1=0,\ A_2=-\frac{6ck^2}{b},\ B_2=0, \\ m &= \frac{-ak^2-4ck^4\sigma-dl^2}{Ak},\ \xi=kx+ly+\frac{mt^\theta}{\theta}. \end{aligned}$$

Set 5.
For $\sigma<0$,

$$v_{11}(x,y,t)=-\frac{2ck^2\sigma}{b}+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{50}$$

or

$$v_{12}(x,y,t)=-\frac{2ck^2\sigma}{b}+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{51}$$

For $\sigma>0$,

$$v_{13}(x,y,t)=-\frac{2ck^2\sigma}{b}-\frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{52}$$

or

$$v_{14}(x,y,t)=-\frac{2ck^2\sigma}{b}-\frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-4ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \tag{53}$$

For $\sigma=0$,

$$v_{15}(x,y,t)=-\frac{6ck^2}{b\left(\frac{t^\theta(-ak^2-dl^2)}{A\theta k}+kx+ly\right)^2}. \tag{54}$$

Case 6.

$$\begin{aligned} A_0 &= \frac{4ck^2\sigma}{b},\ A_1=0, B_1=0,\ A_2=-\frac{6ck^2}{b},\ B_2=-\frac{6ck^2\sigma^2}{b}, \\ m &= \frac{-ak^2-16ck^4\sigma-dl^2}{Ak},\ \xi=kx+ly+\frac{mt^\theta}{\theta}. \end{aligned}$$

Set 6.
For $\sigma<0$,

$$\begin{aligned} v_{16}(x,y,t)=\frac{4ck^2\sigma}{b}&+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)^2}{A\theta k}+kx+ly\right)\right)}{b} \\ &+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \end{aligned} \tag{55}$$

or

$$\begin{aligned} v_{17}(x,y,t)=\frac{4ck^2\sigma}{b}&+\frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b} \\ &+\frac{6ck^2\sigma\coth\left(\sqrt{-\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \end{aligned} \tag{56}$$

For $\sigma > 0$,

$$v_{18}(x,y,t) = \frac{4ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b} - \frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \quad (57)$$

or

$$v_{19}(x,y,t) = \frac{4ck^2\sigma}{b} - \frac{6ck^2\sigma\tan\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b} - \frac{6ck^2\sigma\cot\left(\sqrt{\sigma}\left(\frac{t^\theta\left(-ak^2-16ck^4\sigma-dl^2\right)}{A\theta k}+kx+ly\right)\right)^2}{b}, \quad (58)$$

For $\sigma = 0$,

$$v_{20}(x,y,t) = -\frac{6ck^2}{b\left(\frac{t^\theta(-ak^2-dl^2)}{A\theta k}+kx+ly\right)^2}. \quad (59)$$

Next we present 3D, contour, and 2D plots of some of the obtained analytical solutions.

6.3. **Approximate solutions by RPSM.** First, we assume an initial condition for $t = 0$, using an exact solutions found previously. Thus, from Eq. (45), the initial condition is taken as

$$v_6(x,y,0) = -\frac{6ck^2\sigma}{b} + \frac{6ck^2\sigma\tanh\left(\sqrt{-\sigma}\,(kx+ly)\right)^2}{b}. \quad (60)$$

For the approximate solutions to the (2+1)-dimensional shallow water wave equation (26), where $t \geq 0$, $0 < \theta \leq 1$, the RPSM solution is in the form of Eq. (24). Thus, Eq. (25) can be written as,

$$Resu_k(x,y,t) = A\mathcal{D}_t^\theta(u_k)_x + a(u_k)_{xx} + b(u_k)_{xx}^2 + c(u_k)_{xxxx} + d(u_k)_{yy} = 0. \quad (61)$$

Hence, $Resu_1(x,y,t)$ is obtained as,

$$\begin{aligned} Resu_1(x,y,t) &= A\left(f_1\right)_x + a\left(f_{xx} + \frac{t^\theta\left(f_1\right)_{xx}}{\theta}\right) \\ &+b\left(2\left(f_x + \frac{t^\theta\left(f_1\right)_x}{\theta}\right)^2 + 2\left(f + \frac{t^\theta f_1}{\theta}\right)\left(f_{xx} + \frac{t^\theta\left(f_1\right)_{xx}}{\theta}\right)\right) \\ &+c\left(f_{xxxx} + \frac{t^\theta\left(f_1\right)_{xxxx}}{\theta}\right) + d\left(f_{yy} + \frac{t^\theta\left(f_1\right)_{yy}}{\theta}\right), \end{aligned} \quad (62)$$

where $f = f(x,y)$ and $f_1 = f_1(x,y)$. The first unknown parameter is obtained by setting $t = 0$ as,

$$f_1 = \frac{12ck\sigma^2\left(ak^2 - 4ck^4\sigma + dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right) \times\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\sqrt{-\sigma}}, \quad (63)$$

is determined, and consequently, $u_1 = u_1(x,y,t)$ is obtained as

$$\begin{aligned} u_1 &= \frac{12ck\sigma^2 t^\theta\left(ak^2 - 4ck^4\sigma + dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\theta\sqrt{-\sigma}} \\ &+\frac{6ck^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+ly)\right)}{b} - \frac{6ck^2\sigma}{b}. \end{aligned} \quad (64)$$

Similarly, the next residual term is

$$
\begin{aligned}
Resu_2 \;=\;& d\left(f_{yy}+\frac{t^{\theta}\left(f_1\right)_{yy}}{\theta}+\frac{t^{2\theta}\left(f_2\right)_{yy}}{2\theta^2}\right) \\
&+At^{1-\theta}\left(t^{\theta-1}\left(f_1\right)_x+\frac{t^{2\theta-1}\left(f_2\right)_x}{\theta}\right) \\
&+a\left(f_{xx}+\frac{t^{\theta}\left(f_1\right)_{xx}}{\theta}+\frac{t^{2\theta}\left(f_2\right)_{xx}}{2\theta^2}\right) \\
&+2b\left(f_x+\frac{t^{\theta}\left(f_1\right)_x}{\theta}+\frac{t^{2\theta}\left(f_2\right)_x}{2\theta^2}\right)^2 \\
&+2b\left(f+\frac{t^{\theta}\left(f_1\right)}{\theta}+\frac{t^{2\theta}\left(f_2\right)}{2\theta^2}\right)\left(f_{xx}+\frac{t^{\theta}\left(f_1\right)_{xx}}{\theta}+\frac{t^{2\theta}\left(f_2\right)_{xx}}{2\theta^2}\right) \\
&+c\left(f_{xxxx}+\frac{t^{\theta}\left(f_1\right)_{xxxx}}{\theta}+\frac{t^{2\theta}\left(f_2\right)_{xxxx}}{2\theta^2}\right), \qquad (65)
\end{aligned}
$$

where $f_2=f_2(x,y)$ is written. For $t=0$, the second unknown parameter can be obtained as follows by taking the first order derivative,

$$
f_2=\frac{\begin{array}{c}12c\sigma^2\left(ak^2-4ck^4\sigma+dl^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right)-2\right)\\ \times\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)\end{array}}{A^2b}, \qquad (66)
$$

Thus, $u_2=u_2(x,y,t)$ solution becomes

$$
\begin{aligned}
u_2 \;=\;& \frac{\begin{array}{c}6c\sigma^2t^{2\theta}\left(ak^2-4ck^4\sigma+dl^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right)-2\right)\\ \times\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)\end{array}}{A^2b\theta^2} \\
&+\frac{12ck\sigma^2t^{\theta}\left(ak^2-4ck^4\sigma+dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\theta\sqrt{-\sigma}} \\
&+\frac{6ck^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+ly)\right)}{b}-\frac{6ck^2\sigma}{b}. \qquad (67)
\end{aligned}
$$

Similarly, the other solution is calculated as

$$
\begin{aligned}
u_3 \;=\;& -\frac{\begin{array}{c}4c\sigma^3t^{3\theta}\left(ak^2-4ck^4\sigma+dl^2\right)^3\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right)-5\right)\\ \times\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)\end{array}}{A^3b\theta^3k\sqrt{-\sigma}} \\
&+\frac{\begin{array}{c}6c\sigma^2t^{2\theta}\left(ak^2-4ck^4\sigma+dl^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+ly)\right)-2\right)\\ \times\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+ly)\right)\end{array}}{A^2b\theta^2} \\
&+\frac{12ck\sigma^2t^{\theta}\left(ak^2-4ck^4\sigma+dl^2\right)\tanh\left(\sqrt{-\sigma}(kx+ly)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+ly)\right)}{Ab\theta\sqrt{-\sigma}} \\
&+\frac{6ck^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+ly)\right)}{b}-\frac{6ck^2\sigma}{b}. \qquad (68)
\end{aligned}
$$

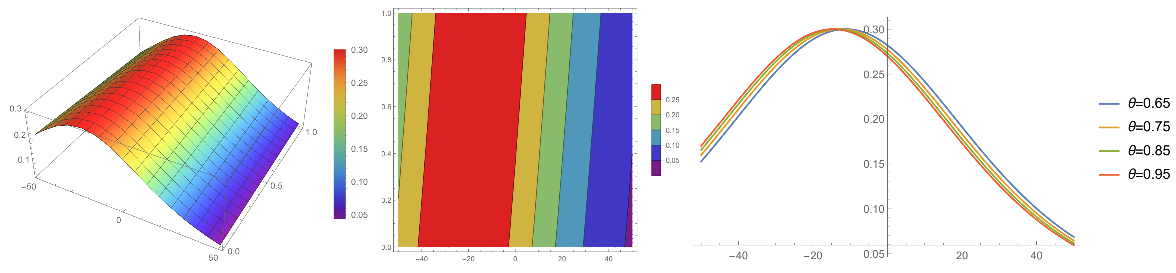
Next we present a comparison table and some 3D comparison plots with RPSM and the exact solutions.

Figure 1 and Figure 2 display the surface graphics of the analytical solutions, whereas Figures 3, 4, and 5 displays the surface graphics of the approximate solutions. Meanwhile, by taking the following values and ranges, approximate and exact solution were compared in Table 1.

Table 1. Comparing exact and RPSM solutions of Eq. (68) with exact solution of Eq. (45).

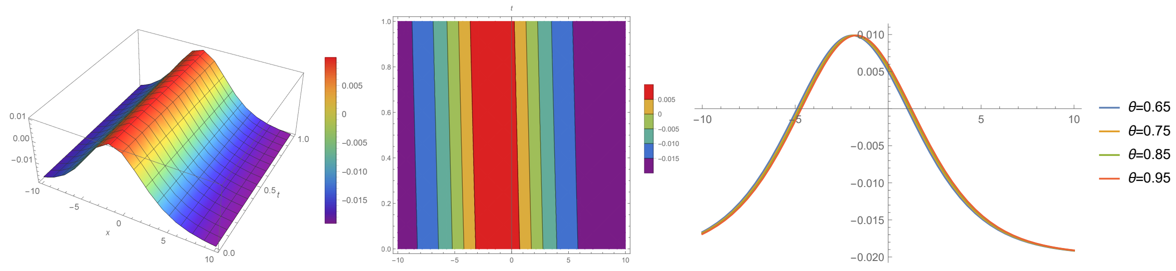
	$\theta = 0.75$			$\theta = 0.85$			$\theta = 0.95$		
t	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*
0.0	0.575491	0.575491	0.00000	0.575491	0.575491	0.00000	0.575491	0.575491	0.00000
0.1	0.583687	0.583687	5.29179×10^{-9}	0.581233	0.581233	1.27760×10^{-9}	0.579571	0.579571	3.2607×10^{-10}
0.2	0.589284	0.589284	4.22828×10^{-8}	0.585848	0.585848	1.34733×10^{-8}	0.583378	0.583378	4.53833×10^{-9}
0.3	0.594195	0.594195	1.42543×10^{-7}	0.590117	0.590116	5.34286×10^{-8}	0.587089	0.587089	2.11688×10^{-8}
0.4	0.598709	0.598709	3.37514×10^{-7}	0.594176	0.594176	1.41958×10^{-7}	0.590740	0.590740	6.31116×10^{-8}
0.5	0.602949	0.602948	6.58507×10^{-7}	0.598087	0.598086	3.02863×10^{-7}	0.594348	0.594348	1.47233×10^{-7}
0.6	0.606981	0.606980	1.13671×10^{-6}	0.601883	0.601882	5.62409×10^{-7}	0.597921	0.597921	2.94118×10^{-7}
0.7	0.610850	0.610848	1.80317×10^{-6}	0.605586	0.605585	9.48980×10^{-7}	0.601467	0.601466	5.27882×10^{-7}
0.8	0.614582	0.614579	2.68881×10^{-6}	0.609212	0.609210	1.49283×10^{-6}	0.604988	0.604987	8.76021×10^{-7}
0.9	0.618200	0.618196	3.82442×10^{-6}	0.612770	0.612768	2.22590×10^{-6}	0.608488	0.608487	1.36928×10^{-6}
1.0	0.621718	0.621712	5.24064×10^{-6}	0.616270	0.616267	3.18163×10^{-6}	0.611970	0.611967	2.04154×10^{-6}

- Figure 1 $k = 0.2, c = 1$, $b = 0.01$, $y = 0.1$, $z = 0.5$, $h = 0.1$, $\eta = 0.05$, $\lambda = 0.5$, $d = 0.1$, $l = 0.5$, $a = 0.1$, $A = 0.1$ and $\theta = 0.95$, $-50 \le x \le 50$ for (A), (B), and $t = 0.99$ for (C).
- Figure 2 $c = 0.0001$, $k = 0.202$, $b = 0.001$, $\sigma = -1.21$, $l = 0.45$, $a = 0.221$, $d = 0.05$, $A = -1.01$, $y = 0.55$ and $\theta = 0.98$, $-10 \le x \le 10$ for (A), (B), and $t = 0.99$ for (C).
- Table 1 $x = 2$, $y = 1$, $c = 0.99$, $k = 0.22$, $b = 0.2$, $\sigma = -0.64$, $l = 0.45$, $a = 0.05$, $d = 0.006$, $A = 0.71$ and $0 \le t \le 1$.
- Figure 3 $x = -1$, $y = 1$, $c = 0.45$, $k = 0.01$, $b = 0.1$, $\sigma = 0.9$, $l = 0.01$, $a = 0.001$, $d = 0.6$, $A = 1$ and $\theta = 0.75$, $-50 \le x \le 50$ for (A) and (B) $0 \le t \le 1$.
- Figure 4 $x = 2$, $y = 1$, $c = 0.57$, $k = 0.12$, $b = 0.2$, $\sigma = -0.12$, $l = 0.01$, $a = 0.03$, $d = 0.01$, $A = 0.7$ and $\theta = 0.85$, $-50 \le x \le 50$ for (A) and (B) $0 \le t \le 1$.
- Figure 5 $x = 2$, $y = 1$, $c = 0.99$, $k = 0.22$, $b = 0.2$, $\sigma = -0.64$, $l = 0.45$, $a = 0.05$, $d = 0.006$, $A = 0.71$ and $\theta = 0.95$, $-50 \le x \le 50$ for (A) and (B) $0 \le t \le 1$.



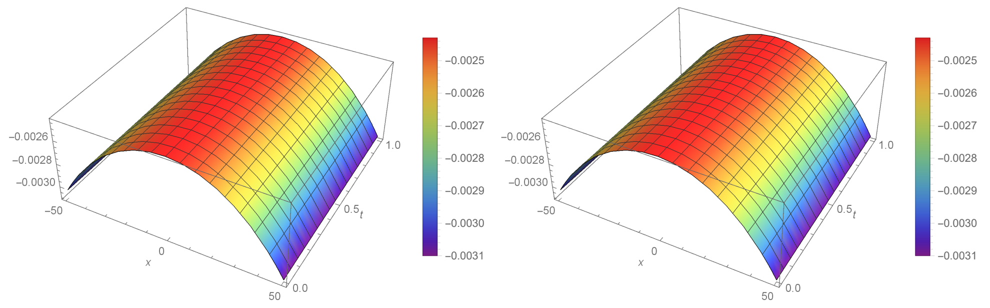
(A) (B) (C)

Figure 1. (A) 3D, (B) contour and (C) 2D plots of the $\exp(-\phi(\xi))$-expansion method solution $u_1(x, y, t)$ of Eq. (29).



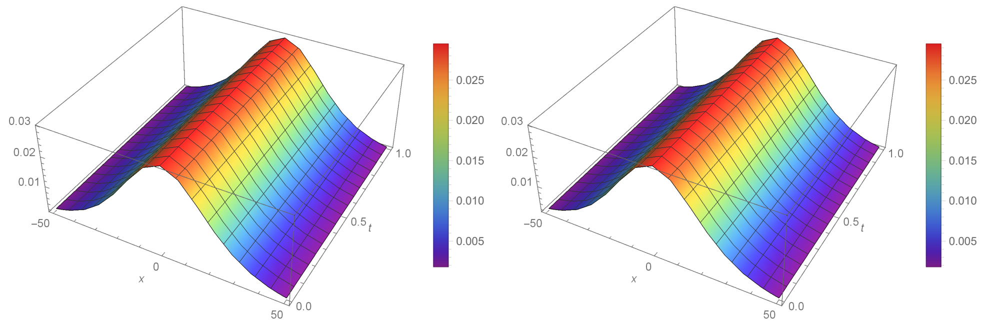
(A) (B) (C)

Figure 2. (A) 3D, (B) contour and (C) 2D plots of the modified extended tanh-function method solution $v_{11}(x, y, t)$ of Eq. (50).



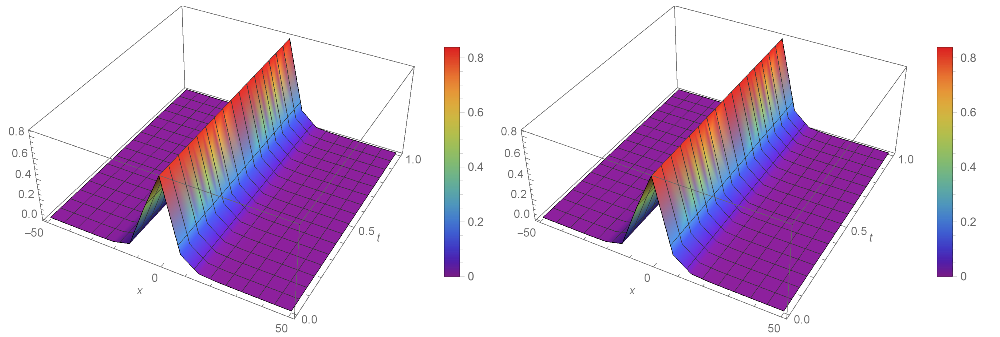
(A) RPSM solution (B) Exact solution

FIGURE 3. Comparison plots of the u_3 solution according to Eq. (68) with the exact solution.



(A) RPSM solution (B) Exact solution

FIGURE 4. Comparison plots of the u_3 solution according to Eq. (68) with the exact solution.



(A) RPSM solution (B) Exact solution

FIGURE 5. Comparison plots of the u_3 solution according to Eq. (68) with the exact solution.

Some new solutions to the present equation are displayed in the surface plots, which can be helpful in solving additional differential equations of arbitrary order.

7. Conclusion

This study investigated solutions to the (2+1)-dimensional shallow water wave equation with conformable derivative by use of the modified extended tanh-function and the $\exp(-\phi(\xi))$-expansion methods. Additionally, the RPSM was used to get approximations of the solutions. Many exact solutions with low computational complexity were obtained using the mentioned analytical approaches. Moreover, the RPSM is a straightforward method, and its independent calculation for each iteration step facilitates computations up to higher-order iterations. We also compared our analytical solutions with the numerical solutions to verify the validity of the results. This provide insights into the applicability of these methods for real-world modeling.

To visually represent the obtained solutions, 3D, contour, and 2D plots were generated. Analytical and approximate results, surface plots, and a comparison table illustrate the accuracy of the techniques. The solutions exhibit distinct features with important physical attributes not previously addressed before. In some interpretations of the figures, the physical behavior of the exact solutions is illustrated for specific numerical values. Understanding these applications is essential for their potential real-world implementations.

The accomplished solutions are crucial for comprehending the physical behavior of the problem. The suggested techniques are reliable and beneficial, providing light on the physical properties of various complicated non-linear models. This study contributes to understanding of higher-dimensional wave phenomena under fractional calculus, paving the way for future research on fractional fluid models.

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