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ON THE FINITENESS OF SOME p-DIVISIBLE SETS

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Abstract. For any positive integer n, let H_n denote the n^{th} harmonic number. Given a prime number p, it is not known whether the set of integers $J(p) = \{n \in \mathbb{N} : p \mid H_n\}$ is finite. In this paper, we first investigate a variant of this set, namely, we work on the divisibility properties of the differences of harmonic numbers. For any prime p and a positive integer w, we define the set $D(p, w)$ as $\{n \in \mathbb{N} : p \mid H_n - H_w\}$ and work on the structure of this set. We present some finiteness results on $D(p, w)$ and obtain upper bounds for the number of elements in the set. Next, we consider the differences of generalized harmonic numbers and present an upper bound for the corresponding counting function. Moreover, under some plausible conditions, we prove that the difference set of generalized harmonic numbers is finite. Finally, we point out some directions to pursue.

1. Introduction

The n^{th} harmonic number H_n is defined as the sum

$$\sum_{k=1}^{n} \frac{1}{k}$$

for any positive integer n. These numbers have been investigated in different aspects, where one of the paths is to work on their integerness and related properties, such as divisibilities. It is known that these numbers are non-integers except for the case $n = 1$. Moreover, the difference of two harmonic numbers

$$H_n - H_m$$

is also not an integer whenever $n > m \geq 1$ by [23]. However, we focus on the divisibility properties of these differences as they come with intriguing features.

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Let p be a prime number. We use the notation $p \mid \frac{a}{b} \in \mathbb{Q}$ to mean that p divides the numerator of $\frac{a}{b}$ in its lowest terms. In 1991, the set $J_p = J(p) = \{n \in \mathbb{N} : p \mid H_n\}$ was presented in [16]. Some conjectures were also given in the paper and one of the conjectures was that the set is finite for any prime number p. They showed that J_p is finite for the prime numbers $\{2, 3, 5, 7\}$. Later on, the finiteness of the set was obtained for primes p up to 547, except for $\{83, 127, 397\}$, in [10], but the problem is still open.

However, there are some asymptotic results on the set. Let $J_p(x)$ count the number of elements in $J(p)$ that are less than x, for any positive real number x. Then, it is known by [27] that $J_p(x) < 129p^{\frac{2}{3}}x^{0.765}$, hence one has that

$$J_p(x) = o(x).$$

The upper bound was improved later to $3x^{\frac{2}{3}+\frac{1}{25\log p}}$ in [30].

Moreover, it is known that for any prime p, the elements $\{p-1, p(p-1), p^2-1\}$ are always in the set J_p and if the set consists of only those elements, the prime number p is called harmonic (see [16]).

We, in this paper, will work on a variant of this set, namely we will pick a prime number p, a positive integer w and look for positive integers n so that the prime p divides the difference $H_n - H_w$. We will use the following notation for the set.

Definition 1. *For any prime p and a positive integer w, we define*

$$D(p, w) := \{n \in \mathbb{N} : p \mid H_n - H_w\}.$$

Remark 1. *For any prime number p, if J_p is finite, then $D(p, w)$ is also finite. (See [19], Remark.4.12).*

As we mentioned, it is known [23] that the difference $H_n - H_m$ is never an integer whenever $n > m \geq 1$. In addition to this fact, it was shown in [15] that the equality $H_k - H_m = H_\ell - H_n$ is valid only if $k = \ell$ and $m = n$ holds. However, we work around the divisibility properties of $D(p, w)$ as the differences are interesting enough for this purpose. Consequently, we will need the p-adic order ν_p defined on the rational numbers. Let n be any integer and p be a prime number. We have

$$\nu_p(n) = \begin{cases} k & \text{if } p^k \parallel n \\ \infty & \text{if } n = 0 \end{cases}$$

where $p^k \parallel n$ means that $p^k \mid n$ but $p^{k+1} \nmid n$ with $k \in \mathbb{Z}$. If $n = \frac{a}{b}$ is a rational number, we set

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b).$$

We will start by investigating the congruence relations on $D(p,w)$, and then we will give an upper bound for the counting function

$$D_{p,w}(x) = |\{n \in D(p,w) \ : \ n \leq x\}|.$$

To obtain the upper bound, we first need to bound the number of elements in the intervals of length at most p, lying inside the set $D(p,w)$. The idea is based on the argument given in [27]. Eventually, we will obtain our first main result.

Theorem A. *Let p be a prime number, w be a positive integer and $x \geq 1$ be a real number. Then, we have*

$$D_{p,w}(x) < 3x^{\frac{2}{3}+\frac{1}{25\log p}}.$$

Next, we consider an extension of the harmonic numbers, the generalized harmonic numbers. They are defined as

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}$$

for any positive integers n and s. We extend the difference set to these numbers:

Definition 2. *Let p be a prime number and s, w be any positive integers. Then, we define*

$$G(p,s,w) = G_{p,s,w} = \{n \in \mathbb{N} : p \mid H_n^{(s)} - H_w^{(s)}\}.$$

Next, we define the corresponding counting function

$$G_{p,s,w}(x) = |\{n \in G(p,s,w) \colon n \leq x\}|$$

and obtain our second main result.

Theorem B. *Assume that p is a prime number, s, w are any positive integers and $x \geq 1$ is any real number. Then,*

$$G_{p,s,w}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}$$

holds. Furthermore, whenever $p > se^{\frac{3}{25}}$ holds, we have

$$G_{p,s,w}(x) = o(x).$$

Moreover, we show that $G(p,s,w)$ is finite in some cases and this will be our third main result.

Theorem C. *Let p be a prime number, s, w be positive integers with $s \geq 2$ and $p-1 \nmid s$. If the inequality*

$$\nu_p\left(H_k^{(s)}\right) \leq s-1$$

holds for any $k \in \{1, 2, \ldots p-1\}$, then $G(p,s,w)$ is finite.
Moreover, if $p^m \leq w < p^{m+1}$ for some integer $m \geq 0$, then we have $G(p,s,w) \subseteq \{1, \ldots, p^{m+1}-1\}$.

In Section 5, we obtain some difference sets using [26] together with some of our results, including a counter example for the case when the condition in Theorem C fails, and also discuss the computational process.

Then, in the last section, we present some generalizations of the harmonic numbers and point out some directions to work on the divisibility properties of the differences.

A generalization of the harmonic numbers is the Dedekind harmonic numbers [4]. For any number field K, a finite field extension of the rationals, the n^{th} Dedekind harmonic number is defined as

$$h_K(n) = \sum_{\substack{0 \neq I \subseteq \mathcal{O}_K \\ N(I) \leq n}} \frac{1}{N(I)},$$

such that the sum ranges over all non-zero ideals of $\mathcal{O}_K$ with norm less than or equal to n. These numbers also come with plenty of properties and it was shown in the same paper [4] that the difference of these numbers are non-integer after a while.

Moreover, another generalization of the harmonic numbers is the hyperharmonic numbers. These numbers were defined in [13] recursively as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

for $r \geq 2$, such that $h_n^{(1)} = H_n$.

The integerness of these numbers was an open question proposed in [25]. This property was studied by various authors (see [3,7,8,18]) and recently, it was shown that there are in fact hyperharmonic integers [28]. The set J_p was also extended to the hyperharmonic numbers in [19] and for divisibility properties of the generalized hyperharmonic numbers, which is an simultaneous extension of both generalized harmonic and hyperharmonic numbers, we refer interested readers to [20] and [21].

In [13], it was stated that the n^{th} hyperharmonic number of order r can be written as

$$h_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

Hence, one may work on this identity to continue the investigation on the differences. In fact, the binomial coefficients leads to a conjecture on the harmonic differences, which arises from central binomial coefficients and the Catalan numbers [24].

Lastly, we direct interested readers to [6] for intriguing results on the differences of hyperharmonic numbers.

2. Properties of $D(p, w)$

In this section, we will investigate the structure of the set. First, let us consider the case where $w < p$ and start with an observation.

We have by [9] that

$$H_{p-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{p-2} + \frac{1}{p-1} \equiv 0 \pmod{p} \tag{1}$$

for any prime number $p > 2$. Therefore, we may split the sum

$$H_{p-1} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right) + \left(\frac{1}{r+1} + \cdots + \frac{1}{p-2} + \frac{1}{p-1}\right) \equiv 0 \pmod{p}$$

and write

$$-H_r \equiv \left(\frac{1}{r+1} + \cdots + \frac{1}{p-2} + \frac{1}{p-1}\right) \pmod{p}$$

for any integer $1 \leq r \leq p-1$.

In particular, we have

$$\frac{1}{k} + \frac{1}{p-k} \equiv 0 \pmod{p} \tag{2}$$

for any $1 \leq k \leq p-1$, which implies the following result.

Proposition 1. *For any prime p and $1 \leq r \leq p-1$, we have*

$$H_r \equiv H_{p-1-r} \pmod{p}.$$

Proof. Notice for any prime p and $1 \leq r \leq p-1$ that

$$\begin{aligned} H_{p-1-r} - H_r &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{p-1-r}\right) - \left(\frac{1}{r} + \cdots + \frac{1}{2} + 1\right) \\ &\equiv \left(1 + \frac{1}{2} + \cdots + \frac{1}{p-r-1}\right) + \left(\frac{1}{p-r} + \cdots + \frac{1}{p-2} + \frac{1}{p-1}\right) \\ &= H_{p-1} \equiv 0 \pmod{p} \end{aligned}$$

and we are done. □

Corollary 1. *Let p be a prime number, w be a positive integer and a, b be positive integers with $1 \leq a < b \leq p-1$ such that $a + b = p - 1$. Then, if $a \in D(p, w)$ then we also have $b \in D(p, w)$.*

This corollary indicates that we have a symmetry about $\frac{p-1}{2}$ for any odd prime p.

We can generalize Corollary 1 for integers greater than $p-1$, but first let us introduce some notations. Given a positive integer n and a prime p, we may write $\hat{n}$ to mean that $\lfloor \frac{n}{p} \rfloor$. The interval

$$[pk, p(k+1)-1]$$

will be denoted by I_k for any $k \in \mathbb{Z}^{\geq 0}$. Moreover, if $a = pk + r \in I_k$ for some k, we will use $\bar{a}$ for the integer $pk + (p-1-r)$. Therefore, a quick observation is as follows:

$$a \in D(p,w) \implies \bar{a} \in D(p,w). \tag{3}$$

Before we give the proof, let us first show an argument that will be quite useful when dealing with modular equivalences. Suppose that p is a prime number and $n = pk + r$ is a positive integer with non-negative integers k and $0 \leq r \leq p-1$. Also, note that for any integers a and b, we have

$$\frac{1}{a} \equiv \frac{1}{pb+a} \pmod{p}.$$

Then, as we have

$$\left(1 + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) \equiv 0,$$

we also have

$$\frac{1}{(pm+1)} + \frac{1}{(pm+2)} + \cdots + \frac{1}{(pm+p-1)}$$

for any integer m (see also [11]). Therefore, for a given n as above, we will write

$$H_n \equiv \frac{1}{p} H_k + H_r \pmod{p}$$

throughout the paper (see [16]). Now, we can proceed with the proof.

Proposition 2. *Let $a = pk + r$ be a positive integer with $r, k \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq p-1$. Then, we have*

$$a \in D(p,w) \implies \bar{a} = pk + (p-1-r) \in D(p,w)$$

for any $w \in \mathbb{Z}^{>0}$.

Proof. Let $a = pk + R$ and $w = pm + r$ for some integers k, m with $0 \leq r, R \leq p-1$. Suppose that $a \in D(p,w)$ so that we write

$$H_a - H_w \equiv \frac{1}{p}\left(H_k - H_m\right) + \left(H_R - H_r\right) \equiv 0 \pmod{p}.$$

Then, setting $\bar{a} = pk + (p-1-R)$ yields that

$$H_{\bar{a}} - H_w \equiv \frac{1}{p}\left(H_k - H_m\right) + \left(H_{p-1-R} - H_r\right) \equiv \frac{1}{p}\left(H_k - H_m\right) + \left(H_R - H_r\right) \equiv 0 \pmod{p}$$

by Proposition 1. □

Remark 2. *For any prime number p and positive integer w, we have*

$$\{w, \bar{w}\} \subseteq D(p, w).$$

Moreover, the equality $D(p, w) = D(p, \bar{w})$ holds.

Furthermore, we actually have

$$H_{p-1} \equiv 0 \pmod{p^2} \tag{4}$$

for primes $p > 3$ by [29]. This congruence points out some more elements in $D(p, w)$ whenever $w < p$.

Proposition 3. *Suppose that $p > 3$ is a prime and $0 < w < p$ is an integer. Then if we let $n = p(p-1) + w$, we have $\{n, \bar{n}\} \in D(p, w)$.*

Proof. Equation (4) states that $\nu_p(H_{p-1}) \geq 2$. As a consequence, we have

$$H_n - H_w \equiv \frac{1}{p} H_{p-1} + H_w - H_w \equiv \frac{1}{p} H_{p-1} \equiv 0 \pmod{p}.$$

□

Proposition 4. *Let p be a prime number and $w < p$ be a positive integer. If $n = p\hat{n} + r \in D(p, w)$ then $\hat{n} \in J_p$.*

Proof. Suppose that we have $n = p\hat{n} + r \in D(p, w)$ for some prime p and an integer $0 < w < p$. Then, $H_n - H_w \equiv \frac{1}{p} H_{\hat{n}} + \left(H_r - H_w\right) \equiv 0 \pmod{p}$ implies $\nu_p\left(H_{\hat{n}}\right) \geq 1$ so that we have $\hat{n} \in J_p$. □

The symmetry

$$n \in D(p, w) \iff \bar{n} \in D(p, w)$$

actually points out that there is a symmetry for the set J_p too, which can be seen by taking w as some element in J_p. Namely, the elements of J_p come in pairs. We omit the case when $n = \bar{n}$, so that $n \equiv \frac{p-1}{2} \pmod{p}$.

We note that we did not consider the case when $0 \in J_p$ throughout our investigation. If we set $H_0 = \frac{0}{1}$ as in [16], then we can see that $\{0, p-1, p(p-1), p^2-1\} \subseteq J_p$ where the pairs are $\{0, p-1\}$ and $\{p(p-1), p^2-1\}$ since

$$\bar{0} = p - 1, \;\; \overline{p(p-1)} = p(p-1) + p - 1 = p^2 - 1.$$

However, we may omit this case. Now, if we remove the restriction $w < p$, we obtain the following result.

Lemma 1. *Let $w \in I_k$ for some non-negative integer k. Then, we have*

$$I_{k+1} \cap D(p, w) = \emptyset.$$

Moreover, if n belongs to $D(p, w)$ then $\hat{n}$ belongs to $D(p, \hat{w})$ for any n and w.

Proof. If $w \in I_k = [pk, p(k+1)-1]$ then we can write $w = pk + r$ for some integer $0 \le r \le p-1$. Now, let us take any $n = p(k+1) + R \in I_{k+1} \cap D(p,w)$ with $0 \le R \le p-1$ and write the difference as

$$H_n - H_w \equiv \frac{1}{p}\big(H_{k+1} - H_k\big) + \big(H_R - H_r\big) = \frac{1}{p}\frac{1}{k+1} + \big(H_R - H_r\big) \pmod p. \quad (5)$$

The p-adic valuation of $H_R - H_r$ is always non-negative as both $R, r < p$. However, we have $\nu_p\left(\frac{1}{p}\frac{1}{k+1}\right) \le -1$ so that we end up with $\nu_p\big(H_n - H_w\big) \le -1$.

For the last part, suppose that $w \in I_k$ and there is $n = p\hat{n} + R \in D(p,w)$. Writing $H_n - H_w$ as in (5), we deduce that

$$H_n - H_w \equiv \frac{1}{p}\big(H_{\hat{n}} - H_{\hat{w}}\big) + \big(H_R - H_r\big) \equiv 0 \pmod p$$

so that $\nu_p\big(H_{\hat{n}} - H_{\hat{w}}\big) \ge 0$ yields that $\hat{n} \in D(p, \hat{w})$. □

Lemma 2. *Let p be an odd prime and w be a positive integer. If $D(p,w)$ is finite, then $D(p, pw+r)$ is also finite for any integer $0 \le r \le p-1$.*

Proof. Suppose that $D(p, pw+r)$ is infinite for some $0 \le r \le p-1$ and write $D(p,w) = \{n_1 < n_2 < \cdots < n_k\}$ for some $k \in \mathbb{Z}^{>0}$. Then, choose some

$$n = pk + R \in D(p, pw+r)$$

with $k > \lfloor n_k/p \rfloor$. As $n \in D(p, pw+r)$ we have

$$H_n - H_{pw+r} \equiv \frac{1}{p}\left(H_k - H_w\right) + \Bigg(H_R - H_r\Bigg) \equiv 0 \pmod p$$

so that

$$\nu_p\left(H_k - H_w\right) \ge 1.$$

Thus, $k \in D(p,w)$ must hold but the fact $k > \lfloor n_k/p \rfloor$ yields a contradiction. □

3. Proof of Theorem A

In this section, we prove our first main result, which is to bound the function

$$D_{p,w}(x) = |\{n \in D(p,w) \ : \ n \le x\}|.$$

We begin by dividing the set into intervals of length at most p, next we bound them and then provide the upper bound for the whole set $D(p,w) \cap [1,x]$.

Before we prove Theorem A, we first prove a weaker version of it, with the use of arguments of [27]. Then, using the tools from [30] we will obtain Theorem A.

For any positive integer d, we let

$$f_d(x) = (x+1)(x+2)\dots(x+d). \quad (6)$$

Consequently we get

$$\frac{f_d'(x)}{f_d(x)} = \frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+d}.$$

Then, if $n \in D(p,w)$ for some $n > w$ we can say that

$$H_n - H_w = \frac{1}{w+1} + \frac{1}{w+2} + \cdots + \frac{1}{n} = \frac{f_{n-w}'(w)}{f_{n-w}(w)} \equiv 0 \pmod{p}.$$

Now, we will bound the number of elements in the intersection of $D(p,w)$ with intervals of length at most p. To do so, we use the polynomial $f_d'(x)$. However, we will not consider the particular case $d = n - w, x = w$, as the polynomial $f_{n-w}'(w)$ leads to some other direction that we do not investigate in this paper (see Section 6). Moreover, the condition $n > w$ will not be a concern, as we see in the proof of the next lemma.

Lemma 3. *Assume that p is a prime number, w is a positive integer and x, y are real numbers with $1 \le y < p$. Then, we have*

$$|D(p,w) \cap [x, x+y]| < \frac{3}{2} y^{\frac{2}{3}} + 1.$$

Proof. Let us write

$$D(p,w) \cap [x, x+y] = \{n_1 < n_2 < \cdots < n_k\}$$

for some $k \ge 2$ because otherwise there is nothing to show. Therefore, suppose that $k = |D(p,w) \cap [x, x+y]| > 1$. For any $1 \le i < j \le k$ we have

$$H_{n_i} - H_{n_j} = (H_{n_i} - H_w) - (H_{n_j} - H_w) \equiv 0 \pmod{p}. \tag{7}$$

Then, let us set $d_i = n_{i+1} - n_i$ for $i = 1, 2, \ldots, k-1$ and observe for any i that

$$\frac{f_{d_i}'(n_i)}{f_{d_i}(n_i)} = \frac{1}{n_i+1} + \frac{1}{n_i+2} + \cdots + \frac{1}{n_{i+1}} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p}. \tag{8}$$

by (7) above. Then, the result follows from [27, Lemma 2.2]. □

A partition of J_p was given in [16] as follows. Inductively, we define the sets $J_p^{(1)} = [1, p-1] \cap J_p$ and $J_p^{(k+1)} = \{pn + r \in J_p \ : \ n \in J_p^{(k)},\ 0 \le r \le p-1,\ p \mid H_n\}$ for any positive integer k. It was shown that $J_p^{(k)} = [p^{k-1}, p^k - 1]$. Hence, we can write

$$J_p = \bigcup_{k=1}^{\infty} J_p^{(k)}$$

Fact 5. *By the definition*

$$J_p^{(k+1)} = \{pn + r \in J_p \ : \ n \in J_p^{(k)},\ 0 \le r \le p-1,\ p \mid H_n\},$$

notice that if $J_p^{(k)} = \emptyset$ *for some positive integer* k*, then* $J_p^{(t)} = \emptyset$ *for any* $t \geq k$ *and we get*

$$J_p = \bigcup_{t=1}^{k-1} J_p^{(t)}.$$

Now, we give a partition of $D(p, w)$ for any $w < p$ using the notation above.

Definition 3. *Let* p *be a prime and* $w < p$ *be a positive integer. We define* $D_{p,w}^{(1)} = D(p, w) \cap [1, p-1]$ *and* $D_{p,w}^{(k+1)} = \{pn + r \in D(p, w) : n \in J_p^{(k)},\ \ 0 \leq r \leq p-1\}$.

Next, a similar result can be obtained.

Proposition 6. *The equality*

$$D_{p,w}^{(k)} = D(p, w) \cap [p^{k-1}, p^k - 1]$$

holds for any prime number p *and positive integer* k*.*

Proof. Let us prove by induction on k. For $k = 1$, the result follows. Now, suppose that the equality $D_{p,w}^{(k)} = [p^{k-1}, p^k - 1]$ holds and let

$$pn + r \in D_{p,w}^{(k+1)} = \{pn + r \in D(p, w) \mid n \in J_p^{(k)} \ \ : \ \ 0 \leq r \leq p-1\}.$$

Then, as $n \in J_p^{(k)}$ we know that $p^{k-1} \leq n \leq p^k - 1$ holds, which implies that

$$pn + r \in [p^k, p^{k+1} - 1]$$

and we are done. Conversely, if $m \in D(p, w) \cap [p^k, p^{k+1} - 1]$ then we can write $m = pn + r$ for some $n \in [p^{k-1}, p^k - 1]$ and $0 \leq r \leq p-1$. Furthermore, as the integer $m = pn + r \in D(p, w)$, we have

$$H_m - H_w = H_{pn+r} - H_w \equiv \frac{1}{p} H_n + \big(H_r - H_w\big) \equiv 0.$$

That is, as $\nu_p\big(H_r - H_w\big) \geq 0$ holds, we obtain that $n \in J_p$. The proof is now complete. □

Now, we can prove a weaker version of Theorem A.

Lemma 4. *Let* p *be a prime,* $w < p$ *be a positive integer and* $x \geq 1$ *be a real number. Then, we have*

$$D_{p,w}(x) < 129 p^{\frac{2}{3}} x^{0.765}.$$

Proof. First, let us set $N = \frac{3}{2}(p-1)^{2/3} + 1$. With the help of Lemma 3 and [27, Lemma 2.2] we obtain that

$$|D_{p,w}^{(1)}| = |J_p^{(1)}| < N.$$

Next, we have

$$|D_{p,w}^{(k+1)}| = \sum_{n \in J_p^{(k)}} |D(p, w) \cap [pn, pn + p - 1]| < |J_p^{(k)}| N.$$

Moreover, $|J_p^{(k)}| < N^k$ holds by the proof of [27, Theorem 1.1]. Consequently, we get

$$|D_{p,w}^{(k)}| < N^k$$

and the rest is similar to the cited proof. □

Now, we can prove Theorem A.

Proof of Theorem A. Our aim is to improve the upper bound presented in Lemma 4. To improve the upper bound for $D_{p,w}(x)$, we need to modify Definition 3, investigate the different cases, and then follow the procedure presented in [30] for J_p. In the proof of Lemma 3, we had

$$D(p,w) \cap [x, x+y] = \{n_1 < \cdots < n_k\}$$

with some positive integer $w < p$, real numbers x, y with $1 \leq y < p$ and set $d_i = n_{i+1} - n_i$ for $i = 1, 2, \ldots, k-1$. Then, we observed in (8) that

$$\frac{f'_{d_i}(n_i)}{f_{d_i}(n_i)} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p}.$$

As $f_d(x)$ is a polynomial of degree d and the intersection interval has length at most p, we deduce that there are at most $d-1$ many solutions of

$$f'_{d_i}(n_i) \equiv 0 \pmod{p}.$$

This fact leads that

$$|\{i : n_{i+1} - n_i = d\}| \leq d - 1$$

for any positive integer $d \geq 1$ with $i = 1, 2, \ldots, k$.

At this point, we need to consider the cases where $w \in [p^t, p^{t+1} - 1]$ for some $t \in \mathbb{Z}^{\geq 0}$.

Case 1. $w \in [1, p-1]$.
In this case, we can continue with Definition 3 and set $D_{p,w}^{(1)} = D(p,w) \cap [1, p-1]$ and $D_{p,w}^{(k+1)} = \{pn + r \in D(p,w) : n \in J_p^{(k)},\ 0 \leq r \leq p-1\}$ for any k. Then, together with our argument presented in the proof of Lemma 4, our setup becomes identical with the set up given in [30, Theorem 1.1].

Namely, [30, Lemma 2.4] applies to the difference set so that we have

$$|D(p,w) \cap [x, x+y]| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} \tag{9}$$

for any prime number p, any positive integer w and any real numbers x, y with $\frac{8}{3} \leq y < p$. Here, we do not have to bound w with p by our observation in the proof of Lemma 3. So, we continue with the improved upper bound.

Let us set

$$N = \left(\frac{9}{8}\right)^{\frac{1}{3}} (p-1)^{\frac{2}{3}}.$$

Then, given a real number x, we can find the positive integer m satisfying

$$p^{m-1} \le x < p^m.$$

Then, we can write

$$D_{p,w}(x) = D_{p,w}(p^{m-1}-1) + |D(p,w) \cap [p^{m-1}, x]|. \tag{10}$$

For the first summand, we can write by Definition 3 and Proposition 6, together with (9) that

$$\begin{aligned} D_{p,w}(p^{m-1}-1) &= \sum_{i=1}^{m-1} |D(p,w) \cap [p^{i-1}, p^i - 1]| \\ &= \sum_{i=1}^{m-1} |D_{p,w}^{(i)}| \le \sum_{i=1}^{m-1} N^i = \frac{N}{N-1} N^{m-1}. \end{aligned} \tag{11}$$

Here, we also use the fact that $|D_{p,w}^{(i)}| \le N^i$ for $i \ge 1$ via Lemma 4. For the second summand, we have

$$|D(p,w) \cap [p^{m-1}, x]| \le \sum_{\substack{n \in J_p^{(m-1)} \\ pn \le x}} |D(p,w) \cap [pn, pn+p-1]|$$

so that

$$\begin{aligned} |D(p,w) \cap [p^{m-1}, x]| \le N \sum_{\substack{n \in J_p^{(m-1)} \\ pn \le x}} 1 &= N \left| D(p,w) \cap \left[p^{m-2}, \frac{x}{p}\right] \right| \\ &\le N^2 \left| D(p,w) \cap \left[p^{m-3}, \frac{x}{p^2}\right] \right| \\ &\le \dots \\ &= N^{m-1} \left| D(p,w) \cap \left[1, \frac{x}{p^{m-1}}\right] \right|. \end{aligned}$$

Here, if $x < 3p^{m-1}$ then

$$\left| D(p,w) \cap \left[1, \frac{x}{p^{m-1}}\right] \right| \le 1 \le \left(\frac{9}{8}\right)^{\frac{1}{3}} \left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

Otherwise, if $x \ge 3p^{m-1}$ then by (9) we get

$$\left| D(p,w) \cap \left[1, \frac{x}{p^{m-1}}\right] \right| \le \left(\frac{9}{8}\right)^{\frac{1}{3}} \left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

Thus, we obtain that

$$|D(p,w)\cap[p^{m-1},x]| \le N^{m-1}\left|D(p,w)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \le N^{m-1}\left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

Then, combining this result with (11), we write for (10) that

$$D_{p,w}(x) \le \frac{N}{N-1}N^{m-1} + N^{m-1}\left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

The rest is similar to the proof of [30, Theorem 1.1] and we are done.

Case 2. $w \in [p, p^2-1]$.

In the first case, when we have $w \in I_0 = [1, p-1]$, we had the sets

$$D^{(1)}_{p,w} = D(p,w)\cap[1,p-1]$$

and $D^{(k+1)}_{p,w} = \{pn + r \in D(p,w) : n \in J^{(k)}_p,\ \ 0 \le r \le p-1\}$ for any integer $k \ge 1$. Now, if w belongs to the interval $[p, p^2-1]$, we will need to modify the sets $D^{(k+1)}_{p,w}$ for $k \ge 1$.

We know by Lemma 1 that if $pn + r \in D(p,w)$ then $n \in D(p,\hat{w})$ holds where $0 \le r \le p-1$ and $\hat{w} = \lfloor \frac{w}{p} \rfloor$. However, the positive integer w in the lemma was strictly less than p, and we get $\hat{w} = 0$ so that $D(p,w)$ becomes J_p. That is why we had $J^{(k)}_p$ in Definition 3. However, we need the following definition to have a partition of $D(p,w)$ when $w \in [p,p^2-1]$:

Definition 4. *For any prime number p and a positive integer w, we define*

$$D^{(1)}_{p,w} = D(p,w)\cap[1,p-1] \text{ and } D^{(k+1)}_{p,w} = \{pn + r \in D(p,w) : n \in D^{(k)}_{p,\hat{w}}, r \in [0,p-1]\}$$

where $\hat{w} = \lfloor \frac{w}{p} \rfloor$, $k \in \mathbb{Z}^{>0}$.

Consequently, using Lemma 3, we can write that

$$|D^{(1)}_{p,w}| < \frac{3}{2}(p-1)^{2/3} + 1 = N.$$

In fact, we have

$$|D^{(k+1)}_{p,w}| = \sum_{n\in D^{(k)}_{p,\hat{w}}} |D(p,w)\cap[pn,pn+p-1]| < |D^{(k)}_{p,\hat{w}}|N < N^k N = N^{k+1}$$

by the first case, as $\hat{w} \in [1,p-1]$.

As a consequence, we again obtain the same setup in [30] to bound our set. Moreover, as Definition 4 applies to any $w \in [p^t, p^{t+1}-1]$ with $t \in \mathbb{Z}^{\ge 0}$, we can cover all the cases. The proof of Theorem A is now complete. □

Remark 3. *The authors of [30] examined general harmonic numbers in [12], defined as follows. Let $a, b \geq 1$ be two integers. They introduced*

$$H_{a,b}(n) = \sum_{k=0}^{n-1} \frac{1}{ak+b},$$

such that by setting $a = b = 1$, we recover $H_{1,1}(n) = H_n$. Furthermore, for positive integers $w \leq n$, we can express

$$H_n - H_w = \frac{1}{w+1} + \cdots + \frac{1}{n} = H_{1,w+1}(n-w)$$

and encourage readers to consult [12] for further interesting results.

4. Proof of Theorem B

In this section, we work with the differences of generalized harmonic numbers and then prove Theorem B. Let us introduce these numbers. For any positive integers n and s, the n^{th} generalized harmonic number of order s is defined as

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}.$$

First of all, as

$$1 < \sum_{k=1}^{\infty} \frac{1}{k^s} < 2$$

holds, they are non-integer except for the case, when $n = 1$. Also, one can easily show that the difference $H_n^{(s)} - H_m^{(r)}$ is never an integer, except for the trivial case: $n = m$ and $s = r$.

These numbers also satisfy a Wolstenholme [29] type congruence, the generalized version of (4) by [17]:

Fact 7. *For any prime number p and a positive integer s, the congruence*

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}$$

holds whenever $p - 1 \nmid s$.

This fact is quite useful when we deal with the divisibility properties. In particular, we know that most of the time,

$$\nu\left(H_{p-1}^{(s)}\right) = 1$$

holds (see [22]).

Similar to the harmonic numbers, given a positive integer $n = p\hat{n} + r$ with p a prime number and an integer $0 \leq r \leq p-1$, we have that

$$H_n^{(s)} = H_{p\hat{n}+r}^{(s)} \equiv \frac{1}{p^s} H_{\hat{n}}^{(s)} + H_r^{(s)} \equiv 0 \pmod{p}, \tag{12}$$

whenever $p - 1 \nmid s$ by Fact 7.

Moreover, an extension of $J(p)$ can also be defined as

$$J(p,s) = J_{p,s} := \{n \in \mathbb{N} : \ p \mid H_n^{(s)}\}.$$

Fact 8. *If $p\hat{n} + r \in J(p,s)$, then we have $\hat{n} \in J(p,s)$ whenever $p - 1 \nmid s$.*

The fact comes from (12) as if $\nu_p\left(\frac{1}{p^s} H_{\hat{n}}^{(s)} + H_r^{(s)}\right) \geq 0$, then

$$\nu_p\left(\frac{1}{p^s} H_{\hat{n}}^{(s)}\right) \geq 0$$

must hold by the Archimedean property of ν_p. In other words, $\nu_p\left(H_{\hat{n}}^{(s)}\right) \geq s$ must hold so that we get $\hat{n} \in J(p,s)$ (see also the proof of Proposition 4).

Also, similar to Fact 5, setting

$J_{p,s}^{(1)} = J(p,s) \cap [1, p-1]$ and $J_{p,s}^{(k+1)} = \{pn + r \in J_{p,s} : n \in J_{p,s}^{(k)},\ \ 0 \leq r \leq p-1,\ \ p \mid H_n^{(s)}\}$

for any $k \geq 1$, we have that $J_{p,s}^{(k)} = [p^{k-1}, p^k - 1]$ (see [5, Lemma 3.1]). Hence, we have the following fact.

Fact 9. *If $J_{p,s}^{(k)} = J(p,s) \cap [p^{k-1}, p^k - 1] = \emptyset$ for some positive integer k, then we have $J(p,s) = \bigcup_{t=1}^{k-1} J_{p,s}^{(t)}$.*

Now, let us define the corresponding difference set for generalized harmonic numbers.

Definition 5. *Let p be a prime number and s, w be any positive integers. Then, we define*

$$G(p,s,w) = G_{p,s,w} = \{n \in \mathbb{N} : p \mid H_n^{(s)} - H_w^{(s)}\}.$$

Note by definition that if $w \in J(p,s)$, then the difference set $G(p,s,w)$ becomes identical with $J(p,s)$.

Let us extend some of our results to the generalized harmonic numbers. For the rest of this section, suppose that $p - 1 \nmid s$ holds. Under this condition, we can extend our results from Section 2. For instance, we generalize Lemma 1 and we obtain the following result.

Lemma 5. *Let n, w be positive integers and p be a prime number. Also let $n = p\hat{n}+R$, $w = p\hat{w}+r$ for some non-negative integers $\hat{n}, \hat{w}$ and $0 \leq r, R \leq p-1$. If $n \in G(p, s, w)$, then we have $\hat{n} \in G(p, s, \hat{w})$ for any positive integer s. In particular, if $w < p$, then $G(p, s, \hat{w}) = J(p, s)$.*

Proof. The idea is similar to the proof of Lemma 1. Using (12), we write

$$H_n^{(s)} - H_w^{(s)} = H_{p\hat{n}+R}^{(s)} - H_{p\hat{w}+r}^{(s)} \equiv \frac{1}{p^s}\left(H_{\hat{n}}^{(s)} - H_{\hat{w}}^{(s)}\right) + \left(H_R^{(s)} - H_r^{(s)}\right) \equiv 0 \pmod p$$

where

$$\nu_p\left(H_R^{(s)} - H_r^{(s)}\right) \geq 0$$

as both $r, R \leq p-1$. Thus, we have $\nu_p\left(\frac{1}{p^s}\left(H_{\hat{n}}^{(s)} - H_{\hat{w}}^{(s)}\right)\right) \geq 0$. Therefore, $\hat{n}$ lies in the set $G(p, s, \hat{w})$ with

$$\left(H_{\hat{n}}^{(s)} - H_{\hat{w}}^{(s)}\right) \equiv 0 \pmod{p^s}.$$

Moreover, if $w < p$, then $\hat{w} = \lfloor \frac{w}{p} \rfloor = 0$ and we are done. □

We can also generalize Lemma 2 as follows, which we state without the proof as the process is similar.

Lemma 6. *Let p be an odd prime and w, s be positive integers. If $G(p, s, w)$ is finite, then $G(p, s, pw + r)$ is also finite for any integer $0 \leq r \leq p-1$.*

Now, let us define the counting function for $G(p, s, w)$.

Definition 6. *For any real number $x \geq 1$, a prime number p and a positive integer w, we define*

$$G(p, s, w)(x) = G_{p,s,w}(x) = |G(p, s, w) \cap [1, x]|.$$

We are ready to prove Theorem B.

Proof of Theorem B. To begin with, our first step is to divide the difference set into smaller sets.

Definition 7. *For any prime number p and a positive integer w, we define*

$$G_{p,s,w}^{(1)} = G_{p,s,w} \cap [1, p-1] \text{ and } G_{p,s,w}^{(k+1)} = \{pn + r \in G_{p,s,w} : n \in G_{p,s,\hat{w}}^{(k)}, r \in [0, p-1]\}$$

where $\hat{w} = \lfloor \frac{w}{p} \rfloor$, $k \in \mathbb{Z}^{>0}$.

Recall by Proposition 6 that

$$D_{p,w}^{(k)} = D(p, w) \cap [p^{k-1}, p^k - 1]$$

for any prime number p and positive integer k. By extending this result, we obtain the following proposition which we present without proof.

Proposition 10. *The equality*

$$G^{(k)}_{p,s,w} = G_{p,s,w} \cap [p^{k-1}, p^k - 1]$$

holds for any prime number p *and positive integers* s, k *and* w.

Hence, we have

$$G(p, s, w) = \bigcup_{k=1}^{\infty} G^{(k)}_{p,s,w}.$$

Now, in order to count the elements of $G(p, s, w)$, we can consider the intersection of the set with intervals of length at most p. That is, one may first bound the set

$$G(p, s, w) \cap [x, x + y]$$

for some positive real numbers x, y with $1 \le y < p$. Therefore, we may consider to generalize Lemma 3.

Given two positive integers $n_1, n_2 \in G(p, s, w) \cap [x, x + y]$, with for some prime p, positive integers s, w and real numbers x, y with $1 \le y < p$, the equivalences

$$H^{(s)}_{n_1} - H^{(s)}_{w} \equiv 0 \pmod p \text{ and } H^{(s)}_{n_2} - H^{(s)}_{w} \equiv 0 \pmod p$$

imply that

$$H^{(s)}_{n_2} - H^{(s)}_{n_1} \equiv 0 \pmod p. \tag{13}$$

On the other hand, if we have $n_1, n_2 \in J(p, s) \cap [x, x+y]$ under the same conditions above, we end up with (13). This fact is valid for any finite number of elements inside $G(p, s, w) \cap [x, x + y]$. Consequently, the counting of $G(p, s, w) \cap [x, x + y]$ is essentially equivalent to the counting of $J(p, s) \cap [x, x + y]$, similar to the argument in the proof of Lemma 3. The process was covered broadly in [5, Lemma 3.3, Lemma 3.4] by the author.

Now, as we observed the fact that counting $J(p, s)$ is equivalent to the counting of the difference set, we rely on the proof of bounding $J(p, s)$ given by the author as below.

Theorem ([5, Theorem A]). *Suppose that* p *is a prime number,* s *is any positive integer and* $x \ge 1$ *is any real number. Then,*

$$J_{p,s}(x) \le 3x^{\frac{2}{3} + \frac{1}{25 \log p} + \frac{\log s}{3 \log p} + \frac{\log s}{3 \log x}}$$

holds. Moreover, whenever $p > se^{\frac{3}{25}}$ *holds, we have*

$$J_{p,s}(x) = o(x).$$

Hence, when our setup becomes identical with the cited theorems proof, we are done. Eventually, we need the following lemma from [5].

Lemma 7 ([5, Lemma 3.5]). *Let p be a prime number and x, y be real numbers with $\frac{8}{3} \leq y < p$. Then, the inequality*

$$|J(p,s) \cap [x, x+y]| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}$$

holds for any positive integer s.

Now, if we set

$$A = \left(\frac{9}{8}\right)^{\frac{1}{3}} (p-1)^{\frac{2}{3}} s^{\frac{1}{3}},$$

we obtain that

$$|G^{(1)}_{p,s,w}| = |G_{p,s,w} \cap [1, p-1]| \leq A.$$

Moreover, we have

$$|G^{(k+1)}_{p,s,w}| = \sum_{n \in G^{(k)}_{p,s,\hat{w}}} |G(p,s,w) \cap [pn, pn+p-1]| \leq |G^{(k)}_{p,s,\hat{w}}| A$$

so that

$$|G^{(k)}_{p,s,\hat{w}}| \leq A^k$$

holds for any $k \in \mathbb{Z}^{>0}$. Finally, as the upper bounds do not contain w, our setup is now complete. Hence, the upper bound for $J(p,s)$ is also valid for $G(p,s,w)$.

For the last part of the theorem, namely, to obtain the equality

$$G_{p,s,w}(x) = o(x),$$

we only need to work on the inequality

$$\frac{1}{25 \log p} + \frac{\log s}{3 \log p} + \frac{\log s}{3 \log x} < \frac{1}{3}$$

and end up with the condition $p > se^{\frac{3}{25}}$, which can be easily shown. The proof is now complete. □

Now, we prove our last result, Theorem C, which is a direct consequence of [5, Theorem B.(i)].

Theorem C. *Let p be a prime number, s, w be positive integers with $s \geq 2$ and $p-1 \nmid s$. If the inequality*

$$\nu_p\left(H_k^{(s)}\right) \leq s-1$$

holds for any $k \in \{1, 2, \dots p-1\}$, then $G(p,s,w)$ is finite. Moreover, if

$$p^m \leq w < p^{m+1}$$

for some integer $m \geq 0$, then we have $G(p,s,w) \subseteq \{1, \dots, p^{m+1}-1\}$.

Proof. Using Fact 9, we first obtain that $J(p,s)$ is finite for any p,s as in the statement, by showing that $J^{(2)}_{p,s}=\emptyset$. Suppose that $\nu_p\left(H^{(s)}_k\right)\le s-1$ holds for any integer $1\le k\le p-1$, for some prime number p, and a positive integer $s\ge 2$ with $p-1\nmid s$. Assume also that $pn+r\in J^{(2)}_{p,s}\neq\emptyset$ for some integers n and $0\le r\le p-1$. Note that we have $n\in[1,p-1]$ as $pn+r\in J^{(2)}_{p,s}=J(p,s)\cap[p,p^2-1]$ via [5, Lemma 3.1]. Now,

$$H^{(s)}_{pn+r}\equiv\frac{1}{p^s}H^{(s)}_n+H^{(s)}_r\equiv 0 \pmod p$$

implies that

$$\nu_p\left(H^{(s)}_n\right)\ge s$$

by the Archimedean property as $\nu_p\left(H^{(s)}_r\right)\ge 0$. On the other hand, the inequality $\nu_p\left(H^{(s)}_n\right)\ge s$ contradicts with our assumption, as $n\in J(p,s)$ with $1\le n\le p-1$. Thus, $J^{(2)}_{p,s}=\emptyset$ and we have

$$J(p,s)=J^{(1)}_{p,s}=J(p,s)\cap[1,p-1].$$

Next, let us take any positive integer w. By Lemmas 5 and 6, if we show that $G(p,s,\hat{w})$ is finite, then we are done. We can bound w as $p^m\le w<p^{m+1}$ for some integer $m\ge 0$. Now, since $J(p,s)$ is finite, the set

$$G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big)$$

is also finite, since

$$1\le\left\lfloor\frac{w}{p^m}\right\rfloor\le p-1$$

and $J(p,s)=G(p,s,\lfloor\lfloor\frac{w}{p^m}\rfloor/p\rfloor)$. Also, as $G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big)$ is finite, $G\big(p,s,\lfloor\frac{w}{p^{m-1}}\rfloor\big)$ is also finite. Continuing the process, we end up with the finiteness of $G(p,s,w)$ and the first part of the theorem is done.

Now, let us obtain the upper bound for the set $G(p,s,w)$. Take any $n\in G(p,s,w)$ so that $p^m\le w\le n$. Again by Lemma 5, we have

$$\left\lfloor\frac{n}{p^m}\right\rfloor\in G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big)$$

where $1\le\lfloor\frac{w}{p^m}\rfloor\le p-1$. Now, assume that $\lfloor\frac{n}{p^m}\rfloor\ge p$ holds. Then, let us write $\lfloor\frac{n}{p^m}\rfloor=pk+r$ for some k,r with $k\ge 1$ and $0\le r\le p-1$. As we have

$$\left\lfloor\frac{n}{p^m}\right\rfloor=pk+r\in G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big),$$

we may write

$$H^{(s)}_{pk+r}\equiv\frac{1}{p^s}H^{(s)}_k+\left(H^{(s)}_r-H^{(s)}_{\lfloor\frac{w}{p^m}\rfloor}\right)\equiv 0 \pmod p$$

such that $\nu_p\big(H_k^{(s)}\big) \geq s$, thus $k \in J(p,s)$. However, $J(p,s)$ is bounded above by $p-1$ and for any $k \in J(p,s)$, we have

$$\nu_p\big(H_k^{(s)}\big) \leq s-1.$$

Hence the assumption $\lfloor \frac{n}{p^m} \rfloor \geq p$ fails and

$$\frac{n}{p^m} - 1 < \lfloor \frac{n}{p^m} \rfloor \leq p-1$$

yields that $p^m \leq w \leq n < p^{m+1}$. The proof is now complete. □

5. Computations

In this section, we begin by computing the difference sets $D(p,w)$ for some prime p and positive integers w.

Example 1. $p=5, w=2$. *To compute* $D(5,2)$, *recall that we have* $D_{5,2}^{(1)} = D(5,2) \cap [1,4]$. *Next, as* $\hat{w} = \hat{2} = \lfloor \frac{2}{5} \rfloor = 0$, *we have*

$$D_{5,2}^{(k+1)} = \{5n + r \in D(5,2) : \ \ n \in D_{5,0}^{(k)}, \ \ 0 \leq r \leq 4\}$$

for positive integers k.

Also, as

$$D(5,0) = \{n \in \mathbb{N} : \ \ p \mid H_n - H_0 = H_n\}$$

we have $D(5,0) = J(5) = J_5$. *The prime* 5 *is a harmonic prime so that*

$$J(5) = \{4, 20, 24\}$$

by [16]. Therefore, we have $J_5^{(1)} = \{4\}, J_5^{(2)} = J_5 \cap [5,24] = \{20,24\}$ *and* $J_5^{(3)} = \emptyset$. *Then, by Fact 5, we can write*

$$J_5 = J_5^{(1)} \cup J_5^{(2)}.$$

Moreover, we also have $J_5^{(k)} = \emptyset$ *for any* $k \geq 3$ *by the same fact.*

The equality yields that

$$D_{5,2}^{(k+1)} = \{5n + r \in D(5,2) : \ \ n \in J_5^{(k)}, \ \ 0 \leq r \leq 4\} = \emptyset$$

for any $k \geq 3$. *Consequently, we have*

$$D(5,2) = D_{5,2}^{(1)} \cup D_{5,2}^{(2)} \cup D_{5,2}^{(3)}.$$

Now, we can say that 2 *and* $\bar{2} = 5-1-2 = 2$ *is already in the set* $D(5,2)$ *via Remark 2. Then, with the help of [26], we see that there is not any other element in the first level so* $D_{5,2}^{(1)} = \{2\}$. *Next,*

$$D_{5,2}^{(2)} = \{5n + r \in D(5,2) : \ \ n \in J_5^{(1)}, \ \ 0 \leq r \leq 4\}$$

and as $J_5^{(1)} = \{4\}$ *we only need to check* $\{5 \cdot 4 + r\}$ *for* $r \in \{0,1,2,3,4\}$. *By Proposition 3 we already know that* $p(p-1)+w = 22 \in D(5,2)$. *Eventually, we see that* $D_{5,2}^{(2)} = \{22\}$ *using [26]. Then, for* $D_{5,2}^{(3)}$ *we consider the set*

$$\{5n + r \in D(5,2) : \ n \in J_5^{(2)} = \{20, 24\}, \ 0 \le r \le 4\}.$$

That is, we check

$$\{100 + r : r \in \{0,1,2,3,4\}\} \cap D(5,2) \text{ and } \{120 + r : r \in \{0,1,2,3,4\}\} \cap D(5,2).$$

Finally, we obtain $D_{5,2}^{(3)} = \{101, 103, 121, 123\}$ *so that*

$$D(5,2) = \{2\} \cup \{22\} \cup \{101, 103, 121, 123\} = \{2, 22, 101, 103, 121, 123\}.$$

In the next example, we will see that one do not need to compute each level

$$D_{p,w}^{(k)}$$

to determine $D(p,w)$, as long as $D(p,\hat{w})$ is known.

Example 2. $p = 5, w = 11$. *Now, let us consider the case* $w = 11 > p = 5$. *First, let us write* $\hat{w} = \hat{11} = \lfloor \frac{11}{5} \rfloor = 2$ *as we need* $D(5,2)$ *to determine* $D(5,11)$. *So, we have* $D_{5,11}^{(1)} = D(5,11) \cap [1,4]$ *and*

$$D_{5,11}^{(k+1)} = \{5n + r \in D(5,11) : \ n \in D_{5,2}^{(k)}, \ 0 \le r \le 4\}$$

for any $k \ge 1$. *By the first example, we know that* $D_{5,2}^{(k)} = \emptyset$ *for any* $k \ge 4$. *Thus,* $D_{5,11}^{(k)} = \emptyset$ *for any* $k \ge 4$ *and*

$$D(5,11) = D_{5,11}^{(1)} \cup D_{5,11}^{(2)} \cup D_{5,11}^{(3)}.$$

By following our steps in the first example, we can completely determine $D(5,11)$. *However, we can use Lemma 1 and quickly get the result:*

if n *belongs to* $D(p,w)$ *then* $\hat{n}$ *belongs to* $D(p,\hat{w})$ *for any* n *and* w.

That is,

$$D(5,11) = D(5,11) \cap \{5 \cdot n + r : n \in \{2, 22, 101, 103, 121, 123\}, \ 0 \le r \le 4\}.$$

Thus, using [26] we conclude that

$$D(5,11) = \{11, 13, 506, 508, 515, 519, 617\}.$$

Example 3. $p = 5, w = 59$. *In this case,* $D(5,59)$ *can be determined by* $D(5,11) = \{11, 13, 506, 508, 515, 519, 617\}$ *as* $\hat{w} = \lfloor \frac{w}{p} \rfloor = \lfloor \frac{59}{5} \rfloor = 11$. *Hence, using [26] again, we have that*

$$D(5,59) = \{55, 59, 65, 69, 2532, 2541, 2543, 2576, 2578, 2596, 2598, 3085, 3089\}.$$

Recall by Fact 5 that if $J_p^{(k)} = \emptyset$ for some $k \in \mathbb{Z}^{>0}$, then $J_p^{(t)} = \emptyset$ for any $t \geq k$ and we have

$$J_p = \bigcup_{t=1}^{k-1} J_p^{(t)}.$$

However, this might not be the case with the difference sets. For instance, if we choose $p = 7$, then we know by [16] that

$$J_7 = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}$$

with $J_7 = \bigcup_{t=1}^{6} J_7^{(t)}$. We have $J_7^{(7)} = \emptyset$ and hence, $J_7^{(t)} = \emptyset$ for any $t \geq 7$. Now if we pick $w = 2$, we obtain that

$$D_{7,2}^{(1)} = \{2, 4\} \text{ and } D_{7,2}^{(2)} = \{44, 46\}$$

by Remark 2 and Proposition 3. On the other hand, even if

$$D_{7,2}^{(3)} = D(7,2) \cap [7^2, 7^3 - 1] = \emptyset$$

holds, we cannot conclude that $D(7,2) = D_{7,2}^{(1)} \cup D_{7,2}^{(2)}$ as

$$D_{7,2}^{(4)} = \{2094, 2098, 2359, 2365, 2388, 2392\} \neq \emptyset.$$

On the other hand, one may observe that as $J_7^{(7)} = \emptyset$ then $D_{7,2}^{(8)}$ is also empty as

$$D_{7,2}^{(8)} = \{6n + r \in D(7,2) : \ \ n \in J_7^{(7)}, \ \ 0 \leq r \leq 6\}.$$

Hence, the number of non-empty $D_{p,w}^{(k)}$'s cannot exceed the number of non-empty $J_p^{(k)}$'s for $k \in \mathbb{Z}^{>0}$.

To sum up, given a prime number p and a positive integer w, it may be time consuming to determine $D(p,w)$ completely. However, we can find the integer m satisfying $p^m \leq w < p^{m+1}$, namely $m = \lfloor \log_p w \rfloor$. Then, $\lfloor \frac{w}{p^m} \rfloor$ yields the base step to start with. To determine $D(p, \lfloor \frac{w}{p^m} \rfloor)$ we need to determine J_p (see Example 1). This process is done by finding the integer k where $J_p^{(k)} = \emptyset$.

First, we find

$$D\left(p, \left\lfloor \frac{w}{p^m} \right\rfloor\right) \cap [1, p-1] = D^{(1)}_{p, \lfloor \frac{w}{p^m} \rfloor}.$$

Then, we check

$$D\left(p, \left\lfloor \frac{w}{p^m} \right\rfloor\right) \cap [pn, pn + p - 1]$$

for each $n \in J_p$ so that we completely obtain $D\left(p, \lfloor \frac{w}{p^m} \rfloor\right)$. Next, we determine $D\left(p, \lfloor \frac{w}{p^{m-1}} \rfloor\right)$ by proceeding as we did in the examples above. After m steps, we finally have $D(p, w)$.

TABLE 1. The number of elements in the sets $J(p)$ and $D(p, w)$ for several p, w values.

p	$\lvert J(p)\rvert$	$\lvert D(p,1)\rvert$	$\lvert D(p,2)\rvert$
3	3	3	3
5	3	4	6
7	13	10	20
13	3	10	12
17	3	6	12
23	3	4	8

TABLE 2. The elements in the sets $J(p)$ and $D(p, 1)$ for several p values.

p	$J(p)$	$D(p,1)$
3	$\{2, 7, 22\}$	$\{1, 66, 68\}$
5	$\{4, 20, 24\}$	$\{1, 3, 21, 23\}$
7	$\{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}$	$\{1, 5, 43, 47, 2067, 2069, 2362, 117120, 117148, 719099\}$
13	$\{12, 156, 168\}$	$\{1, 4, 8, 11, 157, 160, 164, 167, 2034, 2190\}$
17	$\{16, 272, 288\}$	$\{1, 15, 273, 287, 4632, 4904\}$
23	$\{22, 506, 528\}$	$\{1, 21, 507, 527\}$

TABLE 3. The elements in the sets $J(p)$ and $D(p, 2)$ for several p values.

p	$J(p)$	$D(p,2)$
3	$\{2, 7, 22\}$	$\{2, 7, 22\}$
5	$\{4, 20, 24\}$	$\{2, 22, 101, 103, 121, 123\}$
7	$\{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}$	$\{2, 4, 44, 46, 2094, 2098, 2359, 2365, 2388, 2392, 14673, 14677, 102726, 102730, 117117, 117123, 117145, 117151, 719096, 719102\}$
13	$\{12, 156, 168\}$	$\{2, 10, 158, 166, 2029, 2032, 2036, 2039, 2185, 2188, 2192, 2195\}$
17	$\{16, 272, 288\}$	$\{2, 7, 9, 14, 274, 279, 281, 286, 4624, 4640, 4896, 4912\}$
23	$\{22, 506, 528\}$	$\{2, 20, 508, 526, 11643, 11655, 12149, 12161\}$

Finally, let us check some examples for $G(p,s,w)$. If we choose $p=5$ and $s=2$, we have the following generalized harmonic numbers $H_n^{(s)}$ with the corresponding 5-adic orders:

n	$H_n^{(2)}$	$\nu_5\left(H_n^{(2)}\right)$
1	1	0
2	5/4	1
3	49/36	0
4	205/144	1

Hence via Theorem C, if we take an integer w satisfying $p^m \leq w < p^{m+1}$, we expect to get $G(p,s,w) \subseteq \{1,\ldots,p^{m+1}-1\}$. We first find $J(5,2)=\{2,4\}$ and obtained the following results via [26]:

w	$G(p,s,w)$
$2 \in [1,4]$	$\{2,4\}$
$13 \in [5,24]$	$\{13,20,22,24\}$
$66 \in [25,124]$	$\{66,120,122,124\}$
$331 \in [125,624]$	$\{331,623\}$

On the other hand, we have $H_3^{(2)} = \frac{49}{36}$ and

$$\nu_7\left(H_3^{(2)}\right) = 2 \not\leq 1,$$

such that our condition in Theorem C fails. In fact, we have

$$26 \in G(7,2,3),\ \ 27 \in G(7,2,21),\ \ 182 \in G(7,2,43).$$

Lastly, let us close the section with another counter example. One may check that the case $p=37$ and $s=3$ yields some elements in $G(p,s,w)$ that are greater than 37. That is because we have $\nu_{37}\left(H_{36}^{(3)}\right) = 3 \not\leq 3-1 = 2$. For instance, if we pick $w=10$, we obtain that $1344 \in G(37,3,10)$ and $1344 > p-1 = 37-1 = 36$.

6. Conclusion

In this section, we first present some of the generalizations of the harmonic numbers. The first one of those is the Dedekind harmonic numbers. Let K be a number field. Then, the n^{th} Dedekind harmonic number, denoted by $h_K(n)$ is defined as

$$\sum_{\substack{0\neq I\subseteq \mathcal{O}_K\\ N(I)\leq n}} \frac{1}{N(I)},$$

where the sum is ranging over all the non-zero ideals of $\mathcal{O}_K$ with norm less than or equal to n. This idea was inspired by the Dedekind zeta function $\zeta_K(s)$ for K and these are indeed an extension of harmonic numbers as taking $K = \mathbb{Q}$ yields that

$$h_K(n) = H_n$$

as $\zeta_K(s) = \zeta(s)$ in that case.

In [4], it was shown that almost all of these numbers are non-integer. Moreover, the differences of these numbers was also studied. In fact, it was proven under the Riemann hypothesis for $\zeta_K(s)$ that the difference

$$h_K(n) - h_K(m)$$

is not an integer after a while. Namely, there exist constants $\alpha, x_0 > 0$ such that $h_K(n) - h_K(m) \notin \mathbb{Z}$ for any positive integers $n > m \geq x_0$ whenever

$$n - m \geq \alpha(d_K \log m + \log \Delta_K)\sqrt{m}$$

holds, where d_K is the degree of K and Δ_K denotes the absolute value of the discriminant of K.

Euler introduced the harmonic zeta function given as

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s},$$

where $\Re(s) > 1$. He showed that the identity

$$2\zeta_H(m) = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1)$$

holds for any integers $k \geq 2$, provided that the sum vanishes if $m = 2$. In particular, if we let $m = 2$, we get

$$2\zeta_H(2) = 2\left(\sum_{n=1}^{\infty} \frac{H_n}{n^2}\right) = 4\zeta(3)$$

so that

$$\zeta_H(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$$

and for $m = 3$, we have that

$$\zeta_H(3) = \sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4) = \frac{\pi^4}{72}.$$

Consequently, one may obtain the special values of the harmonic zeta function via the special values of the Riemann zeta function. One of the applications of the special values of the harmonic zeta function is to approximate the real numbers given in [1, 2]. Moreover, $\zeta_H(s)$ is just one example of a Dirichlet series. It was shown lately that not only this function can be used for the approximation purpose but all Dirichlet series can be used [14].

Now, we point out a direction that has another generalization of the harmonic numbers and the harmonic differences for interested readers. Conway and Guy presented a generalization in their book, *The Book of Numbers* [13] called the hyperharmonic numbers. The hyperharmonic numbers were defined recursively as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

where $r \geq 2$, and that $h_n^{(1)} = h_n$. These numbers are also endowed with a variety of arithmetic and analytical features. In particular, the integerness of the difference of hyperharmonic numbers was studied in [6] and it was shown that almost all of the differences

$$h_n^{(r)} - h_m^{(s)}$$

are non-integer. However, there are also some cases that the difference is an integer, infinitely many times.

To relate the differences of harmonic numbers with hyperharmonic numbers, one may consider the following identity given by Conway and Guy. They stated that the n^{th} hyperharmonic number of order r can be written as

$$h_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}). \tag{14}$$

The identity (14) points out that in order to work on the p-adic order of harmonic differences, we may consider to work on the p-adic valuations of the binomial coefficient and the corresponding hyperharmonic number.

Now, recall the polynomial at (6)

$$f_d(x) = (x+1)(x+2)\dots(x+d)$$

for some positive integer d. Notice that the polynomial appears in the numerator of the binomial coefficient, as we have

$$\binom{n+r-1}{r-1} = \frac{(n+r-1)(n+r-2)\dots(r)}{n!}$$

so that one direction is to study this polynomial. Moreover, by feeding with the harmonic difference, we may write for (14) that

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) \\ &= \frac{(n+r-1)(n+r-2)\dots(r)}{n!}\left(\frac{1}{r} + \frac{1}{r+1} + \dots + \frac{1}{n+r-1}\right) \\ &= \frac{f_n'(r-1)}{n!} \end{aligned}$$

and the focus completely turns on to the hyperharmonic numbers.

Also, if we consider a particular case for the binomial coefficient, some fruitful relations appear, together with a conjecture on the differences [24]. Following the same notation as [24], we let

$$c_n = \binom{2n}{n},$$

to be the n^{th} central binomial coefficient, for any $n \geq 0$. Also, let

$$C_n = \frac{1}{n+1} c_n,$$

be the n^{th} Catalan number with $n \geq 0$.

The main concern of the paper was the p-adic order of the differences

$$c_{ap^{n+1}+b} - c_{ap^n+b} \text{ and } C_{ap^{n+1}+b} - C_{ap^n+b},$$

where a, b are integers with p being a prime number satisfying $(a, p) = 1$ and $n \geq n_k$ for some integer $n_k \geq 0$. Consequently, some identites involving these numbers were presented. For instance, one of the results which were given was as follows.

Fact 11 ([24, Theorem 2.2]). *The equality*

$$\nu_p\left(C_{ap^{n+1}} - C_{ap^n}\right) = n + \nu_p\left(\binom{2a}{a}\right)$$

holds for any integers $n, a \geq 1$ and any prime $p \geq 2$ with $(a, p) = 1$.

The identities yield the function

$$g(k) = 2\binom{2k}{k}(H_{2k} - H_k) \;\; k \geq 1,$$

which is needed to work on the p-adic order of those differences. Finally, a conjecture was proposed, which is still open:

Conjecture ([24, Conjecture 2.9]). *The inequality*

$$\nu_p(g(k)) \leq 2$$

holds for any prime $p \geq 5$ and $k \geq 1$.

In other words, for any prime $p \geq 5$ and $k \geq 1$,

$$\nu_p\left(H_{2k} - H_k\right) \leq 2$$

holds [24, Conjecture 2.10].

So, one may consider to pursue the above case about the differences as an another alternative. Finally, notice that if we let $r = k + 1$ in the function $g(k)$ and set $n = k$, we obtain that

$$g(k) = 2\binom{2k}{k}(H_{2k} - H_k) = 2\binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) = 2h_n^{(r)} = 2h_n^{(n+1)}$$

by (14).

Thus, we are back to the hyperharmonic numbers. Finally, let us finalize the discussion with an equivalent conjecture to those above:

Conjecture 12. *Let $p \geq 5$ be a prime number. Then,*

$$\nu_p(h_n^{(n+1)}) \leq 2$$

holds for any positive integer n.

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