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On Strong (i,j)-Semi*-Γ-Open Sets in Ideal Bitopological Space

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Research Article

Abstract — In this study, we introduce the concepts of (i,j)-semi*-Γ-open sets within the context of ideal bitopological spaces. This concept is demonstrated to be weaker than the established the notion of (i,j)-semi-Γ-open sets. Subsequently, we define strong (i,j)-semi*-Γ-open sets in ideal bitopological spaces, elucidating some of their essential characteristics. Furthermore, leveraging this newly introduced concept, we establish the notions of strong (i,j)-semi*-Γ-interior and strong (i,j)-semi*-Γ-closure.

Keywords *Ideal bitopological spaces, generalized open sets, (i,j)-Γ-open sets, (i,j)-semi*-Γ-open sets*

Mathematics Subject Classification (2020) 54E55, 54A05

1. Introduction

Recent studies have focused on bitopological spaces $(X,\mathcal{V}_1,\mathcal{V}_2)$, a nonempty set X endowed with two topologies $\mathcal{V}_1$ and $\mathcal{V}_2$ [1–5]. In 2006, Noiri and Rajesh studied the generalized closed sets concerning an ideal in bitopological spaces [6]. An ideal on a topological space $(X,\mathcal{V})$ is a collection of subsets of X with the hereditary properties:

i. if $U\in\Gamma$ and $V\subset U$, then $V\in\Gamma$

ii. if $U\in\Gamma$ and $V\in\Gamma$, then $U\cup V\in\Gamma$

Let Γ be an ideal on X. Then, $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is termed an ideal bitopological space.

For $P(X)$ being the entire set of subsets of X, and for $i\in\{1,2\}$, an operator $(.)^*_i : P(X)\to P(X)$, referred to as the local function of $\mathcal{U}$ with respect to $\mathcal{V}_i$ and Γ, is defined as follows: for $U\subset X$, $U^*_i(\mathcal{V}_i,\Gamma)=\{x\in X \mid V\cap U\notin\Gamma, \text{ for every } V\in\mathcal{V}_i(x)\}$, where $\mathcal{V}_i(x)=\{V\in\mathcal{V}_i \mid x\in V\}$ [7].

For each ideal topological space $(X,\mathcal{V},\Gamma)$, there exists a topology $\mathcal{V}^*(\Gamma)$ that is more refined than $\mathcal{V}$, generated by the base $\mathcal{B}(\Gamma,\mathcal{V})=\{V-I \mid V\in\mathcal{V} \text{ and } I\in\Gamma\}$. However, it is worth noting that $\mathcal{B}(\Gamma,\mathcal{V})$ is not always a topology [8]. Moreover, we can observe that $\mathrm{Cl}^*_i(U)=U\cup U^*_i(\mathcal{V}_i,\Gamma)$ defines a Kuratowski closure operator for $\mathcal{V}^*_i(\Gamma)$.

Ekici and Noiri [9] introduced the notion of semi-Γ-open sets in ideal topological spaces. Çaldaş et al. [10] introduced the notion of (i,j)-semi-Γ-open sets in ideal bitopological spaces. Finally, Aqeel and Bin-Kuddah [11] established the concept of strong semi*-Γ-open sets in ideal topological spaces.

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Throughout this paper, we use the notation U_i^* for $U_i^*(\mathcal{V}_i,\Gamma)$. Moreover, $\mathrm{Int}_i(U)$ ($\mathrm{Cl}_i(U)$) and $\mathrm{Int}_i^*(U)$ ($\mathrm{Cl}_i^*(U)$) denote the interior (closure) of U respect to $\mathcal{V}_i$ and $\mathcal{V}_i^*$, respectively.

2. Preliminaries

In this paper, we consistently refer to $(X,\mathcal{V}_1,\mathcal{V}_2)$ as a bitopological space without assuming any separation axioms. Additionally, $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is considered as an ideal bitopological space, denoted by the abbreviation IBS. We use OS and CS as abbreviations for open sets and closed sets, respectively.

Definition 2.1. [11] A subset U of an ideal topological space $(X,\mathcal{V},\Gamma)$ is called as:

i. Semi-$\Gamma-$OS if $U\subset \mathrm{Cl}^*(\mathrm{Int}(U))$

ii. Semi*-$\Gamma-$OS if $U\subset \mathrm{Cl}(\mathrm{Int}^*(U))$

iii. Strong semi*-$\Gamma-$OS if $U\subset \mathrm{Cl}^*(\mathrm{Int}^*(U))$

Definition 2.2. [12] For any IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$, Γ is called as codense if $\mathcal{V}_i\cap\Gamma=\{\emptyset\}$, for $i\in\{1,2\}$.

Lemma 2.3. [11] For any IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$, if Γ is a codense ideal, then the following hold:

i. $\mathrm{Cl}_j(U)=\mathrm{Cl}_j^*(U)$, for all j-open set $U\subset X$

ii. $\mathrm{Int}_i(F)=\mathrm{Int}_i^*(F)$, for all j-closed set $F\subset X$

Theorem 2.4. [12] Let U be a subset of IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. For $i,j\in\{1,2\}$ and $i\neq j$,

i. If $\Gamma=\emptyset$, then $U_j^*(\Gamma)=\mathrm{Cl}_j(U)$

ii. If $\Gamma=P(X)$, then $U_j^*(\Gamma)=\emptyset$

iii. $U_j^*\subset \mathrm{Cl}_j(U)$

Lemma 2.5. [12] Let U be an (i,j)-Γ-OS in $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, $U_j^*=\left(\mathrm{Int}_i(U_j^*)\right)_j^*$.

Lemma 2.6. [12] Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ and $V\in\mathcal{V}_j$. Then, $V\cap \mathrm{Cl}_j^*(U)\subset \mathrm{Cl}_j^*(V\cap U)$.

Definition 2.7. [12–14] A subset U of an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is called as:

i. (i,j)-Γ-OS if $U\subset \mathrm{Int}_i(U_j^*)$

ii. (i,j)-semi-Γ-OS if $U\subset \mathrm{Cl}_j^*(\mathrm{Int}_i(U))$

iii. (i,j)-semi-Γ-CS if $\mathrm{Int}_j^*(\mathrm{Cl}_i(U))\subset U$

iv. (i,j)-α-Γ-OS if $U\subset \mathrm{Int}_i(\mathrm{Cl}_j^*(\mathrm{Int}_i(U)))$

v. (i,j)-pre-Γ-OS if $U\subset \mathrm{Int}_i(\mathrm{Cl}_j^*(U))$

vi. (i,j)-pre-Γ-CS if $\mathrm{Cl}_j(\mathrm{Int}_i^*(U))\subset U$

3. On (i,j)-Semi*-Γ-Open Set

This section defines the concept of (i,j)-semi*-Γ-open sets in an ideal bitopological space and presents some associated properties.

Definition 3.1. A subset U in an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is called as an (i,j)-semi*-Γ-OS if $U\subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$, for $i,j\in\{1,2\}$ and $i\neq j$. The set of all the (i,j)-semi*-Γ-open sets in X is denoted by $S_{ij}^*\Gamma O(X)$.

Example 3.2. Consider an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ where $X=\{\alpha,\beta,\gamma\}$, $\Gamma=\{\emptyset,\{\beta\}\}$, $\mathcal{V}_1=\{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\},X\}$, and $\mathcal{V}_2=\{\emptyset,\{\beta\},X\}$. Then,

$$\mathcal{V}_1^*=\{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\},\{\alpha,\gamma\},X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\beta\}, \{\alpha,\gamma\}, X\}$$

Therefore, $\{\beta,\gamma\}$ is a $(1,2)$-semi*-Γ-OS.

Proposition 3.3. Following properties are valid for any IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$:

i. Every (i,j)-semi-Γ-OS is an (i,j)-semi*-Γ-OS.

ii. Every (i,j)-α-Γ-OS is an (i,j)-semi*-Γ-OS.

Proof. *i.* Assume $U \subset X$ is an (i,j)-semi-Γ-OS. Since $\mathcal{V}_i \subset \mathcal{V}_i^*$, then $U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i(U)) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$.

ii. The proof follows from (i).

□

Lemma 3.4. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $U \subset X$ be an (i,j)-semi*-Γ-OS. Then, $\mathrm{Cl}_j(U)$ is (i,j)-semi*-Γ-open set.

Proof. Since U is an (i,j)-semi*-Γ-OS, then $U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. Thus,

$$\mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Cl}_j(\mathrm{Int}_i^*(U))) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U)) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(\mathrm{Cl}_j(U)))$$

□

Theorem 3.5. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $U \subset X$. Then, U is an (i,j)-semi*-Γ-OS if and only if $\mathrm{Cl}_j(U) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$.

Proof. Let U be an (i,j)-semi*-Γ-OS. Then, according to Lemma 3.4, it follows that $\mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. Suppose $\mathrm{Cl}_j(U) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. This implies $U \subset \mathrm{Cl}_j(U)$. Using the hypothesis, we obtain $U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$. □

Theorem 3.6. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $V \subset X$. Then, V is an (i,j)-semi*-Γ-OS if and only if, there exists an (i,j)-semi*-Γ-OS U such that $U \subset V \subset \mathrm{Cl}_j(U)$.

Proof. Let V is an (i,j)-semi*-Γ-OS. Then, $V \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(V))$. Let $U = \mathrm{Int}_i^*(V)$ be (i,j)-$\Gamma-$OS. In other words, U is an (i,j)-semi*-Γ-OS and we have $U = \mathrm{Int}_i^*(V) \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(V)) = \mathrm{Cl}_j(U)$. In contrast, if U is an (i,j)-semi*-Γ-OS such that $U \subset V \subset \mathrm{Cl}_j(U)$, then since $U \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U))$, it follows that $V \subset \mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Cl}_j(\mathrm{Int}_i^*(U))) = \mathrm{Cl}_j(\mathrm{Int}_i^*(U)) = \mathrm{Cl}_j(\mathrm{Int}_i^*(V))$. Hence, V is an (i,j)-semi*-Γ-OS. □

Definition 3.7. A subset U in $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ is called as an (i,j)-semi*-Γ-CS if complement of U is an (i,j)-semi*-Γ-OS. The set of all the (i,j)-semi*-Γ-closed sets in X is denoted by $S_{ij}^*\Gamma C(X)$.

Theorem 3.8. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $\{U_\alpha \mid \alpha \in \Lambda\}$ be a family of subsets of X where Λ is an index set. Then, if $U_\alpha \in S_{ij}^*\Gamma O(X)$, for every $\alpha \in \Lambda$, then $\bigcup\limits_{\alpha\in\Lambda} U \in S_{ij}^*\Gamma O(X)$.

Proof. Let $U_\alpha \in S_{ij}^*\Gamma O(X)$, for every $\alpha \in \Lambda$. Then, $U_\alpha \subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U_\alpha))$. Therefore,

$$\bigcup_{\alpha\in\Delta} U_\alpha \subset \bigcup_{\alpha\in\Lambda} \mathrm{Cl}_j(\mathrm{Int}_i^*(U_\alpha)) \subset \mathrm{Cl}_j\left(\bigcup_{\alpha\in\Lambda} \mathrm{Int}_i^*(U_\alpha)\right)$$
$$\subset \mathrm{Cl}_j\left(\mathrm{Int}_i^*\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right)\right)$$

□

A finite intersection of (i,j)-semi*-Γ-open sets need not to be in $S_{ij}^*\Gamma O(X)$ in general as demonstrated by the following example.

Example 3.9. Let $X = \{\alpha, \beta, \gamma, \eta\}$, $\mathcal{V}_1 = \{\emptyset, X, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$, and $\mathcal{V}_2 = \{\emptyset, X\}$. Let $\Gamma = \{\emptyset, \{\gamma\}, \{\eta\}, \{\gamma, \eta\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \eta\}, X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \eta\}, X\}$$

Therefore, $\{\alpha, \eta\}$ and $\{\beta, \eta\}$ are $(1, 2)$-semi*-Γ open sets. However, $\{\alpha, \eta\} \cap \{\beta, \eta\} = \{\eta\}$ is not $(1, 2)$-semi*-Γ open.

Theorem 3.10. A subset U in $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ is an (i, j)-semi*-Γ-CS if and only if $\text{Int}_j^*(\text{Cl}_i(U)) \subset U$.

Proof. Let U is an (i, j)-semi*-Γ-CS. Then, $X - U$ is an (i, j)-semi*-Γ-OS. Therefore, $X - U \subset \text{Cl}_j(\text{Int}_i^*(X - U)) = X - \text{Int}_j^*(\text{Cl}_i(U))$. Consequently, $\text{Int}_j^*(\text{Cl}_i(U)) \subset U$.

In contrast, if $\text{Int}_j^*(\text{Cl}_i(U)) \subset U$, then $X - U \subset X - \text{Int}_j^*(\text{Cl}_i(U)) \subset \text{Cl}_j(\text{Int}_i^*(X - U))$. Therefore, $X - U$ is an (i, j)-semi*-Γ-OS. Thus, U is an (i, j)-semi*-Γ-CS. □

Theorem 3.11. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and Γ is codense. Then, any subset U is an (i, j)-semi*-Γ-CS if and only if $\text{Int}_i(\text{Cl}_j(U)) \subset U$.

Proof. Let U be an (i, j)-semi*-Γ-CS. Then, $\text{Int}_i^*(\text{Cl}_j(U)) \subset U$. Since $\text{Int}(U) \subset \text{Int}^*(U)$, then $\text{Int}_i(\text{Cl}_j(U)) \subset U$.

In contrast, let $U \subset X$ and $\text{Int}_i(\text{Cl}_j(U)) \subset U$. Since Γ is codense, this implies that $\text{Int}_i^*(\text{Cl}_j(U)) \subset U$. Therefore, U is an (i, j)-semi*-Γ-CS. □

4. On Strong (i, j)-semi*-Γ-Open Set

This section suggests strong (i, j)-semi*-Γ-open sets in ideal bitopological spaces.

Definition 4.1. A subset U of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ is named as a strong (i, j)-semi*-Γ-open set if $U \subset \text{Cl}_j^*(\text{Int}_i^*(U))$. The collection comprised of all the strong (i, j)-semi*-Γ-open sets in X is denoted by $SS_{ij}^*\Gamma O(X)$.

Example 4.2. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS such that $X = \{\alpha, \beta, \gamma, d\}$, $\mathcal{V}_1 = \{\emptyset, \{\beta\}, \{\alpha, \gamma, \eta\}, X\}$, $\mathcal{V}_2 = \{\emptyset, \{\alpha, \beta\}, X\}$, and $\Gamma = \{\emptyset, \{\gamma\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset, \{\beta\}, \{\alpha, \eta\}, \{\alpha, \beta, \eta\}, \{\alpha, \gamma, \eta\}, X\}$$

and

$$\mathcal{V}_2^* = \{\emptyset, \{\alpha, \beta\}, \{\alpha, \beta, \eta\}, X\}$$

Therefore, $\{\beta, \gamma\}$ is a strong $(1, 2)$-semi*-Γ-OS but $\{\alpha, \gamma\}$ is not.

Proposition 4.3. Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$.

i. Every (i, j)-semi-Γ-OS is a strong (i, j)-semi*-Γ-OS.

ii. Every (i, j)-α-Γ-OS is a strong (i, j)-semi*-Γ-OS.

iii. Every strong (i, j)-semi*-Γ-OS is an (i, j)-semi*-Γ-OS.

The evidences come from Proposition 3.3 and Definition 4.1.

Let $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$ be an IBS and $U \subset X$. Then, we get this diagram:

$$(i, j)\text{-}\alpha\text{-}\Gamma\text{-OS} \rightarrow (i, j)\text{-semi-}\Gamma\text{-OS} \rightarrow \text{strong } (i, j)\text{-semi}^*\text{-}\Gamma\text{-}OS \rightarrow (i, j)\text{-semi}^*\text{-}\Gamma\text{-}OS$$

Generally, the opposites of Proposition 4.3 are inaccurate, as demonstrated by the next example.

Example 4.4. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS such that $X=\{\alpha,\beta,\gamma,\eta\}$, $\mathcal{V}_1=\{\emptyset,\{\beta\},\{\alpha,\gamma,\},\{\alpha,\beta,\gamma\},X\}$, $\mathcal{V}_2=\{\emptyset,\{\gamma\},X\}$, and $\Gamma=\{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\}\}$. Then,

$$\mathcal{V}_1^*=\{\emptyset,\{\beta\},\{\gamma\},\{\alpha,\gamma\},\{\gamma,\eta\},\{\alpha,\beta,\gamma\},\{\alpha,\gamma,\eta\},\{\beta,\gamma,\eta\},X\}$$

and

$$\mathcal{V}_2^*=\{\emptyset,\{\gamma\},\{\gamma,\eta\},\{\alpha,\gamma,\eta\},\{\beta,\gamma,\eta\},X\}$$

Therefore, $\{\alpha,\beta,\eta\}$ is a $(1,2)$-semi*-Γ-OS but it is not a strong $(1,2)$-semi*-Γ-open.

Example 4.5. In Example 4.2, $\{\alpha,\eta\}$ is a strong $(1,2)$-semi*-Γ-OS but it is not a $(1,2)$-semi-Γ-open.

Proposition 4.6. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $U\subset X$. Then, U is a strong (i,j)-semi*-Γ-OS if and only if $\mathrm{Cl}_j^*(U)=\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$.

PROOF. Assume U is a strong (i,j)-semi*-Γ-OS, then $U\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. This implies that $\mathrm{Cl}_j^*(U)\subset \mathrm{Cl}_j^*(\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)))=\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. Thus, $\mathrm{Cl}_j^*(U)\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. In contrast, assume $\mathrm{Cl}_j^*(U)=\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. Since $U\subset \mathrm{Cl}_j^*(U)$, then $U\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$. □

Proposition 4.7. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $V\subset X$. Then, V is a strong (i,j)-semi*-Γ-OS if and only if there exists a strong (i,j)-semi*-Γ-OS U such that $U\subset V\subset \mathrm{Cl}_j^*(U)$.

PROOF. Assume V is a strong (i,j)-semi*-Γ-OS, then $V\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V))$. Let $U=\mathrm{Int}_i^*(V)$. Then, $U\subset V\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V))=\mathrm{Cl}_j^*(U)$. Moreover,

$$U\subset V\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V))=\mathrm{Cl}_j^*(\mathrm{Int}_i^*(\mathrm{Int}_i^*(V))=\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$$

Therefore, U is a strong (i,j)-semi*-Γ-OS.

In contrast, if U is a strong (i,j)-semi*-Γ-OS such that $U\subset V\subset \mathrm{Cl}_j^*(U)$, then $\mathrm{Cl}_j^*(U)=\mathrm{Cl}_j(V)$ and $\mathrm{Int}_i^*(U)\subset \mathrm{Int}_i^*(V)$. Besides, $U\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$ and hence

$$V\subset \mathrm{Cl}_j^*(U)\subset \mathrm{Cl}_j^*(\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)))=\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(V))$$

which V is a strong (i,j)-semi*-Γ-OS. □

Theorem 4.8. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $\{U_\alpha\subset X:\alpha\in\Delta\}$ be a family of subsets of X where Δ is an arbitrary index set.

i. If $U_\alpha\in SS_{ij}^*\Gamma O(X)$, for every $\alpha\in\Delta$, then $\bigcup_{\alpha\in\Delta}\{U_\alpha:\alpha\in\Delta\}\in SS_{ij}^*\Gamma O(X)$.

ii. If $U\in SS_{ij}^*\Gamma O(X)$ and $V\in\mathcal{V}_j$, then $U\cap V\in SS_{ij}^*\Gamma O(X)$.

PROOF. *i.* Since $U_\alpha\in SS_{ij}^*\Gamma O(X)$, for every $\alpha\in\Delta$, it follows that $U_\alpha\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U_\alpha))$. Consequently,

$$\bigcup_{\alpha\in\Delta}U_\alpha\subset\bigcup_{\alpha\in\Delta}\mathrm{Cl}_j^*(\mathrm{Int}_i^*(U_\alpha))\subset \mathrm{Cl}_j^*\left(\bigcup_{\alpha\in\Delta}\mathrm{Int}_i^*(U_\alpha)\right)\subset \mathrm{Cl}_j^*\left(\mathrm{Int}_i^*\left(\bigcup_{\alpha\in\Delta}U_\alpha\right)\right)$$

ii. Let $U\in SS_{ij}^*\Gamma O(X)$ and $V\in\mathcal{V}_j$. Since $U\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$, applying Lemma 2.6 yields:

$$U\cap V\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))\cap V\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)\cap V)$$

□

Generally, the intersection of strong (i,j)-semi-Γ-open sets need not be in $SS_{ij}^*\Gamma O(X)$ as demonstrated by the next example.

Example 4.9. Let $X = \{\alpha,\beta,\eta,\gamma\}$, $\mathcal{V}_1 = \{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\},X\}$, and $\mathcal{V}_2 = \{\emptyset, X\}$. If $\Gamma = \{\emptyset,\{\eta\},\{\gamma\},\{\eta,\gamma\}\}$. Then,

$$\mathcal{V}_1^* = \{\emptyset,\{\alpha\},\{\beta\},\{\alpha,\beta\},\{\alpha,\beta,\gamma\},\{\alpha,\beta,\eta\}\}$$

and

$$\mathcal{V}_2^* = \{\emptyset,\{\alpha,\beta\},\{\alpha,\beta,\gamma\},\{\alpha,\beta,\eta\}\}$$

Therefore, $\{\alpha,\eta,\gamma\}$ and $\{\beta,\eta,\gamma\}$ are strong $(1,2)$-semi*-Γ-OS; however, $\{\alpha,\eta,\gamma\}\cap\{\beta,\eta,\gamma\} = \{\eta,\gamma\}$, which is not a strong $(1,2)$-semi*-Γ-OS.

Theorem 4.10. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS. The union of (i,j)-semi*-Γ-OS and a strong (i,j)-semi*-Γ-OS is an (i,j)-semi*-Γ-open.

Proof. Let $U \in SS^*_{ij}\Gamma O(X)$ and V is an (i,j)-semi-Γ-OS, then

$$\begin{aligned} U\cup V &\subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \cup (\mathrm{Cl}_j(\mathrm{Int}_i^*(V)) \\ &\subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U)) \cup \mathrm{Cl}_j(\mathrm{Int}_i^*(V)) \\ &= \mathrm{Cl}_j(\mathrm{Int}_i^*(U) \cup \mathrm{Int}_i^*(V)) \\ &\subset \mathrm{Cl}_j(\mathrm{Int}_i^*(U\cup V)) \end{aligned}$$

Hence, $U\cup V$ is an (i,j)-semi*-Γ-OS. □

Definition 4.11. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $U\subset X$. Then, U is called as a strong (i,j)-semi*-Γ-CS if its complement is a strong (i,j)-semi*-Γ-open. The set of all the strong (i,j)-semi*-Γ-closed sets in X is denoted by $SS^*_{ij}C(X)$.

Theorem 4.12. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $U\subset X$. Then, U is a strong (i,j)-semi*-Γ-CS if and only if $\mathrm{Int}_j^*(\mathrm{Cl}_i^*(U))\subset U$.

Proof. Assume U is a strong (i,j)-semi*-Γ-CS of X. Then,

$$X-U \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(X-U)) = X - \mathrm{Int}_j^*(\mathrm{Cl}_i^*(U))$$

Thus, $\mathrm{Int}_j^*(\mathrm{Cl}_i^*(U))\subset U$. In contrast, assume U is any subset of X such that $\mathrm{Int}_j^*(\mathrm{Cl}_i^*(U))\subset U$. This gives that $X-U \subset X-\mathrm{Int}_j^*(\mathrm{Cl}_i(U)) \subset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(X-U))$. Therefore, $X-U$ is a strong (i,j)-semi*-Γ-OS. □

Theorem 4.13. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and Γ is codense. Then, U is a strong (i,j)-semi*-Γ-CS if and only if $\mathrm{Int}_i(\mathrm{Cl}_j^*(U))\subset U$.

Proof. Assume U is a strong (i,j)-semi*-Γ-CS of X. Then, $\mathrm{Int}_i^*(\mathrm{Cl}_j^*(U))\subset U$. Thus, $\mathrm{Int}_i(\mathrm{Cl}_j^*(U))\subset U$. In contrast, assume U is any subset of X such that $\mathrm{Int}_i(\mathrm{Cl}_j^*(U))\subset U$. This suggests that $\mathrm{Int}_i^*(\mathrm{Cl}_j^*(U))\subset U$, which gives that U is a strong (i,j)-semi*-Γ-CS. □

Theorem 4.14. Let $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$ be an IBS and $U,V\in SS^*_{ij}\Gamma C(X)$. Then, $U\cap V$ is a strong (i,j)-semi*-Γ-CS.

Proof. Assume $U,V\in SS^*_{ij}\Gamma C(X)$. Then,

$$\begin{aligned} U\cap V &\supset \mathrm{Int}_i^*(\mathrm{Cl}_j^*(U)) \cap \mathrm{Int}_i^*(\mathrm{Cl}_j^*(V)) \\ &\supset \mathrm{Int}_i^*(\mathrm{Cl}_j^*(U)\cap \mathrm{Cl}_j^*(V)) \\ &\supset \mathrm{Int}_i^*(\mathrm{Cl}_j^*(U\cap V)) \end{aligned}$$

Therefore, $U\cap V\in SS^*_{ij}\Gamma C(X)$. □

5. The Strong (i,j)-Semi*-Γ-Interior and Strong (i,j)-Semi*-Γ-Closure

This section defines the concept of strong (i,j)-semi*-Γ-interior and strong (i,j)-semi*-Γ-closure in an ideal bitopological space and establishes their varied characteristics.

Definition 5.1. Let U be a subset of an IBS $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. The strong (i,j)-semi*-Γ-interior of U, denoted by $ss^*_{i,j}\Gamma\text{-Int}(U)$, is defined as the union of all the strong (i,j)-semi*-Γ-open sets of X that are contained within U. In other words,

$$ss^*_{i,j}\Gamma\text{- Int}(U) = \bigcup\{V \subset U \mid V \in SS^*_{ij}\Gamma O(X)\}$$

Theorem 5.2. Let U is a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then,

$$ss^*_{i,j}\Gamma\text{- Int}(U) = U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U))$$

Proof. Let $U \subset X$. Then,

$$\begin{aligned} U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U)) &\subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U)) \\ &\subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(\mathrm{Int}^*_i(U))) \\ &= \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U \cap \mathrm{Int}^*_i(U)) \\ &\subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U)))) \end{aligned}$$

Thus, $U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U))$ is a strong (i,j)-semi*-Γ-OS contained in U, which means $U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U)) \subset ss^*_{ij}\Gamma\text{-Int}(U)$. Furthermore, since $ss^*_{i,j}\Gamma\text{-Int}(U)$ is a strong (i,j)-semi*-Γ-open, then

$$\mathrm{ss}^*_{i,j}\Gamma\text{- Int}(U) \subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(\mathrm{ss}^*_{i,j}\Gamma\text{- Int}(U))) \subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U))$$

Consequently, $ss^*_{i,j}\Gamma\text{-Int}(U) \subset U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U))$. □

Lemma 5.3. Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, U is a strong (i,j)-semi*-Γ-OS if and only if $ss^*_{i,j}\Gamma\text{-Int}(U) = U$.

Proof. Assume U is a strong (i,j)-semi*-Γ-OS. Then, $U \subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U))$. Hence, $ss^*_{i,j}\Gamma\text{-Int}(U) = U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U)) = U$. In contrast, let $ss^*_{i,j}\Gamma\text{-Int}(U) = U$. Since $ss^*_{i,j}\Gamma\text{-Int}(U) = U \cap \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U)) = U$, then $U \subset \mathrm{Cl}^*_j(\mathrm{Int}^*_i(U))$. Hence, U is a strong (i,j)-semi*-Γ-OS. □

Definition 5.4. Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, the strong (i,j)-semi*-Γ-closure of U, denoted by $ss^*_{i,j}\Gamma\text{-Cl}(U)$, defined by the intersection of all the strong (i,j)-semi*-Γ-closed sets of X containing U. In other words,

$$ss^*_{ij}\Gamma - \mathrm{Cl}(U) = \bigcap\{V \subset X : U \subset V, V \in SS^*_{ij}\Gamma C(X)\}$$

Theorem 5.5. Let U be a subset of $(X,\mathcal{V}_1,\mathcal{V}_2,\Gamma)$. Then, $ss^*_{ij}\Gamma - \mathrm{Cl}(U) = U \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U))$.

Proof. Let $U \subset X$. Then,

$$\begin{aligned} U \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U)) &= \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathrm{Cl}^*_j(U))) \\ &= \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U \cup \mathrm{Cl}^*_j(U))) \\ &\supset \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U)))) \end{aligned}$$

Thus, $U \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U))$ is as strong (i,j)-semi*-Γ-CS containing U. Therefore, $ss^*_{ij}\Gamma\text{- Cl}(U) \subset U \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U))$.

In contrast, let $ss^*_{ij}\Gamma\text{- Cl}(U) = U \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U))$. Since $ss^*_{i,j}\Gamma\text{- Cl}(U)$ is a strong (i,j)-semi*-Γ-CS, then

$$ss^*_{i,j}\Gamma\text{- Cl}(U) \supset \mathrm{Int}^*_i(\mathrm{Cl}^*_j(ss^*_{ij}\Gamma\text{- Cl}(U))) \supset \mathrm{Int}^*_i(\mathrm{Cl}^*_j(U))$$

Therefore, $ss^*_{ij}\Gamma\text{-Cl}(U)) \supset U \cup \text{Int}^*_i(\text{Cl}^*_j(U))$. Consequently, $ss^*_{ij}\Gamma\text{-Cl}(U)) = U \cup \text{Int}^*_i(\text{Cl}^*_j(U))$. □

Lemma 5.6. Let U be a subset of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$. Then, U is a strong (i,j)-semi*-Γ-CS if and only if $ss^*_{ij}\Gamma\text{-Cl}(U) = U$.

Proof. Assume U is a strong (i,j)-semi*-Γ-CS. This implies that $U \supset \text{Int}^*_i(\text{Cl}^*_j(U))$. Therefore, $ss^*_{ij}\Gamma\text{-Cl}(U) = U \cup \text{Int}^*_i(\text{Cl}^*_j(U)) = U$. In contrast, let $ss^*_{ij}\Gamma\text{-Cl}(U) = U$. Given that $ss^*_{ij}\Gamma\text{-Cl}(U) = U \cup \text{Int}^*_*(\text{Cl}^*_j(U))$, it follows that $U \supset \text{Int}^*_i(\text{Cl}^*_j(U))$. Consequently, U is a strong (i,j)-semi*-Γ-CS. □

Theorem 5.7. Let U be a subset of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$. Then, the following properties are held:

i. If U is an (i,j)-preΓ-OS, then $ss^*_{ij}\Gamma\text{-Cl}(U) = \text{Int}^*_i(\text{Cl}^*_j(U))$.

ii. If U is an(i,j)-preΓ-CS, then $ss^*_{ij}\Gamma\text{-Int}(U) = \text{Cl}^*_j(\text{Int}^*_i(U))$.

Proof. *i.* Let U be an (i,j)-preΓ-OS. Then, $U \subset \text{Int}_i(\text{Cl}^*_j(U)) \subset \text{Int}^*_i(\text{Cl}^*_j(U))$. This gives that

$$ss^*_{ij}\Gamma\text{- Cl}(U) = U \cup \text{Int}^*_i(\text{Cl}^*_j(U)) = \text{Int}^*_i(\text{Cl}^*_j(U))$$

ii. Let U be an (i,j)-preΓ-CS. Then, $\text{Cl}^*_j(\text{Int}^*_i(U)) \subset \text{Cl}_j(\text{Int}^*_i(U)) \subset U$. This suggests that

$$ss^*_{ij}\Gamma\text{- Int}(U)) = U \cap \text{Cl}^*_j(\text{Int}^*_i(U)) = \text{Cl}^*_j(\text{Int}^*_i(U))$$

□

Theorem 5.8. Let U be a subset of $(X, \mathcal{V}_1, \mathcal{V}_2, \Gamma)$. The following properties are held:

i. $\text{Int}^*_i(ss^*_{ij}\Gamma\text{-Cl}(U)) = \text{Int}^*_i(\text{Cl}^*_j(U))$

ii. $\text{Cl}^*_j(ss^*_{ij}\Gamma\text{-Int}(U)) = \text{Cl}^*_j(\text{Int}^*_i(U))$

Proof. *i.*

$$\begin{aligned}
\text{Int}^*_i(ss^*_{ij}\Gamma\text{- Cl}(U)) &= \text{Int}^*_i(U \cup \text{Int}^*_i(\text{Cl}^*_j(U))) \\
&\supset \text{Int}^*_i(U) \cup \text{Int}^*_i(\text{Int}^*_i(\text{Cl}^*_j(U))) \\
&= \text{Int}^*_i(U) \cup \text{Int}^*_i(\text{Cl}^*_j(U)) \\
&= \text{Int}^*_i(\text{Cl}^*_j(U))
\end{aligned}$$

In contrast,

$$\begin{aligned}
\text{Int}^*_i(ss^*_{ij}\Gamma\text{- Cl}(U)) &= \text{Int}^*_i(U \cup \text{Int}^*_i(\text{Cl}^*_j(U))) \\
&\subset \text{Int}^*_i(\text{Cl}^*_j(U)) \cup \text{Int}^*_i(\text{Cl}^*_j(U)) \\
&= \text{Int}^*_i(\text{Cl}^*_j(U))
\end{aligned}$$

This indicates that $\text{Int}^*_i(ss^*_{ij}\Gamma\text{-Cl}(U)) = \text{Int}^*_i(\text{Cl}^*_j(U))$.

ii.

$$\begin{aligned}
\text{Cl}^*_j(ss^*_{ij}\Gamma\text{- Int}(U)) &= \text{Cl}^*_j(U \cap \text{Cl}^*_j(\text{Int}^*_i(U)) \\
&\subset \text{Cl}^*_j(\text{Cl}^*_j(U) \cap \text{Cl}^*_j(\text{Int}^*_i(U))) \\
&\subset \text{Cl}^*_j(\text{Cl}^*_j(\text{Int}^*_i(U))) \\
&= \text{Cl}^*_j(\text{Int}^*_i(U))
\end{aligned}$$

In contrast,

$$\begin{aligned}\mathrm{Cl}_j^*(ss_{ij}^*\Gamma\text{- Int}(U)) &= \mathrm{Cl}_j^*(U \cap \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))\\ &\supset \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U)) \cap \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))\\ &= \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))\end{aligned}$$

This suggests that $\mathrm{Cl}_j^*(ss_{ij}^*\Gamma\text{-Int}(U)) = \mathrm{Cl}_j^*(\mathrm{Int}_i^*(U))$.

□

6. Conclusion

In this paper, we introduced the notions of (i,j)-semi*-Γ-open sets and strong (i,j)-semi*-Γ-open sets in ideal bitopological spaces. We demonstrated that the concept of (i,j)-semi*-Γ-open set is weaker than (i,j)-open sets in ideal bitopological spaces. We discussed and proved several properties and relationships of (i,j)-semi*-Γ-open sets and strong (i,j)-semi*-Γ-open sets. Additionally, we introduced the notions of strong (i,j)-semi*-Γ-interior and strong (i,j)-semi*-Γ-closure, providing proofs for their properties.

In future studies, researchers can investigate more applications of (i,j)-semi*-Γ-open sets and strong (i,j)-semi*-Γ-open sets in ideal bitopological spaces. Furthermore, the concept of continuity can be studied in the light of the newly defined generalized open sets.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

References

[1] A. Ghareeb, T. Noiri, Λ-*Generalized closed sets with respect to an ideal bitopological space*, Afrika Matematika 24 (2013) 97–101.

[2] D. J. Sarma, S. Acharjee, *Some results on almost b-continuous functions in a bitopological space*, Boletim da Sociedade Paranaense de Matemática 37 (2) (2019) 167–177.

[3] I. Bukhatwa, S. Demiralp, *On generalized β-open sets in ideal bitopological space*, European Journal of Pure and Applied Mathematics 13 (2) (2020) 269–279.

[4] I. Bukhatwa, S. Demiralp, *On some generalized open sets in ieal bitopological spaces*, Journal of New Theory (32) (2020) 1–10.

[5] J. Oudetallah, R. Alharbi, I. M. Batiha, *On r-compactness in topological and bitopological spaces*, Axioms 12 (2) (2023) 210 13 pages.

[6] T. Noiri, N. Rajesh, *Generalized closed sets with respect to an ideal in bitopological spaces*, Acta Mathematica Hungarica 125 (1-2) (2009) 17–20.

[7] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proceedings of the Indian Academy of Sciences 20 (1945) 51–61.

[8] D. Janković, T. R. Hamlett, *New topologies from old via ideals*, The American Mathematical Monthly 97 (4) (1990) 295–310.

[9] E. Ekici, T. Noiri, **-Hyperconnected ideal topological spaces*, Analele Sale Uviversitatii Alexandru Ioan Cuza Din Iasi-Seria Noua-Matematica 58 (1) (2012) 121–129.

[10] M. Çaldaş, S. Jafari, N. Rajesh, *Some fundamental properties of β-open sets in ideal bitopological spaces*, European Journal of Pure and Applied Mathematics 6 (2) (2013) 247–255.

[11] R. M. S.Aqeel, A. A. S. Bin-Kuddah, *On strong semi*-I-open sets in ideal topological spaces*, University of Aden Journal of Natural and Applied Sciences 23 (2) (2019) 489–499.

[12] M. Çaldaş, S. Jafari, N. Rajesh, F. Smarandache, *On I-open sets and I-continuous functions in ideal bitopological spaces*, in: F. Smarandache (Ed.), Collected Papers (on Neutrosophics and Other Topics) Volume XIV, Global Knowledge Publishing House, Miami, 2022, 338–349.

[13] A. I. El-Maghrabi, M. Çaldaş, S. Jafari, R. M. Latif, A. Nasef, N. Rajesh, S. Shanthi, *Properties of ideal bitopological α-open sets*, Scientific Studies and Research Series Mathematics and Informatics 27 (2) (2017) 15–36.

[14] N. Rajesh, M. Çaldaş, S. Jafari, *Preopen sets in ideal bitopological spaces*, Boletim da Sociedade Paranaense de Matemática 29 (2) (2011) 61–68.