Communications in Advanced Mathematical Sciences
Vol. 7, No. 3, 157-167, 2024
Research Article
e-ISSN: 2651-4001
https://doi.org/10.33434/cams.1442676

Topological Degree Method for a Coupled System of ψ-fractional Semilinear Differential Equations with non Local Conditions

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Abstract
This paper explores the existence of solutions for non-local coupled semi-linear differential equations involving ψ-Caputo differential derivatives for an arbitrary $l \in (0,1)$. We use topological degree theory to condense maps and establish the existence of solutions. This theory allows us to relax the criteria of strong compactness, making it applicable to semilinear equations, which is uncommon. Additionally, we provide an example to demonstrate the practical application of our theoretical result.

Keywords: ψ-Caputo differential derivatives, Coupled semilinear differential equations, Topological degree method
2010 AMS: 26A33, 34A08, 34A12, 3AD20, 47H08

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Received: 25 February 2024, **Accepted:** 21 August 2024, **Available online:** 29 September 2024
How to cite this article: A. Baihi, A. Kajouni, K. Hilal, H. Lmou, *Topological degree method for a coupled system of ψ-fractional semilinear differential equations with non local conditions*, Commun. Adv. Math. Sci., 7(3) (2024), 157-167.

1. Introduction

Freshly, fractional differential equations aroused the fascination of numerous mathematicians, because it may depict a wide range of occurrences across numerous scientific domains and has been demonstrated to be a successful model for scientific fields such as physics, mechanics, biology, chemistry, control theory, and other domains for exemple, see [1]-[14]. There are several approaches to defining fractional integrals and derivatives, yet the most well-known are the Riemann-Liouville and the Caputo fractional integrals and derivatives, in [15], Almeida presents the extension of these derivatives by considering the Caputo fractional derivative of a function concerning another function ψ and studied some useful properties of fractional calculus.
The advantage of this revised definition of the fractional derivative is that greater model accuracy could be accomplished by selecting the proper function ψ. To look for more information about ψ-Caputo and Caputo fractional derivative, We point readers toward the documents [15]-[17].
Since coupled systems of fractional differential equations arise in many practical issues and may be closely examined to determine if they exist, it is crucial to research coupled systems of fractional differential equations. This can be done using

topological degree theory; for instance [18]-[24].
Yang He in [25] studied the existence of mild solution for the following problem

$$\begin{cases}\mathfrak{D}^q u(t)+Au(t)=f(t,u(t),Gu(t)), \quad t\in\Lambda,\\ u(0)=u_0\end{cases}$$

where $\mathfrak{D}^q$ is the Caputo fractional derivative of order $q\in(0,1)$ $-A$ is the infinitesimal generator of a compact analytic semigroup.

The topological degree is a powerful tool for proving the existence of solutions to semilinear differential equations. It provides a rigorous framework for establishing solution existence under specific conditions and can be used to associate an "algebraic count" with the number of solutions within a given domain. This count remains unchanged under continuous deformations of the problem, guaranteeing the existence of at least one solution if the degree is non-zero.

The latest use of the semilinear differential equation with the integral operator was by S.Zorlu and all in [26] when he studied to build new sufficient conditions for the approximate controllability of special classes of abstract fractional evolution control equations with the following form:

$$\begin{cases}{}_0^C D_\psi^q x(t)=-Ax(t)+Bu(t)+f(t,x(t),Gx(t)), \quad t\in\Lambda,\\ x(0)=x_0.\end{cases}$$

where x is the state variable defined on the Hilbert space X; ${}_0^C D_\psi^q$ describes the ψ-Caputo fractional derivative with order $0<q<1$; A is the infinitesimal generator of a strongly continuous semigroup $\mu(t)$ of bounded operators on X. Furthermore, u, the control function, is defined in $L^2([0,T],U)$, where U is a Hilbert space. $L^2([0,T],U)$, is the space of square integrable functions with respect to the Lebesgue measure on the interval $[0,T]$, taking values in the Hilbert space U, f is a given non-linear term such that $f(t,.,.):[0,T]\times X_\alpha\times X_\alpha\to\times X_\beta$, $\beta\in[\alpha,1]$, $0<\alpha<1$, $T<\infty$, and the Volterra integral operator is as follows:

$$Gx(t)=\int_0^t K(t,s)x(s)ds,$$

with a kernel of $K\in C(\Delta,(0,\infty))$, $\Delta=\{(t,s):0\le s\le t\le T\}$.

From the works mentioned above we examine the following coupled system of ψ-Caputo fractional semilinear differential equation:

$$\begin{cases}{}^C\mathfrak{D}_{0^+}^{l,\psi}\chi(\iota)+\mathfrak{A}_1\chi(\iota)=\hbar(\iota,\zeta(\iota),\mathfrak{R}\zeta(\iota)),\iota\in\Lambda,\\ {}^C\mathfrak{D}_{0^+}^{l,\psi}\zeta(\iota)+\mathfrak{A}_2\zeta(\iota)=\hbar(\iota,\chi(\iota),\mathfrak{R}'\chi(\iota)),\iota\in\Lambda,\\ \chi(0)+\rho(\chi)=\chi_0\\ \zeta(0)+\rho(\zeta)=\zeta_0\end{cases} \tag{1.1}$$

where $\nu>0$, with $-\mathfrak{A}_1$ and $-\mathfrak{A}_2$ generates an analytic compact semigroup $(V(\iota))_{\iota\ge0}$ and $(U(\iota))_{\iota\ge0}$ respectively. The terms $\mathfrak{R}\chi(\iota)$, $\mathfrak{R}'\chi(\iota)$ provided by:

$$\mathfrak{R}\chi(\iota)=\int_0^\iota I(\iota,\tau)\chi(\tau)d\tau,\ I\in C(S,\mathbb{R}^+)\ \mathit{and}\ \mathfrak{R}'\chi(\iota)=\int_0^\iota J(\iota,\tau)\chi(\tau)d\tau,\ J\in C(Q_0,\mathbb{R}^+),$$

with $Q=\{(\iota,\varsigma)\in\mathbb{R}^2:0\le\varsigma\le\iota\le\nu\}$, $\qquad Q_0=\{(\iota,\varsigma)\in\mathbb{R}^2:0\le\varsigma\le\iota\le\nu\}$.
The fractional derivative ${}^C D_{0^+}^{l,\psi}$ is the ψ-Caputo fractional derivative of order $l\in(0,1)$, $\nu>0$, $\chi_0\in\mathscr{X}$. the non local condition $\rho(\chi)=\sum_{w=1}^{\alpha}\rho_w\chi(\iota_w)$; where ρ_w, $w=1,...\alpha$ are constants and $0<\iota_1<\iota_2<...<\iota_n\le\nu$, and $\hbar:\Lambda\times\mathscr{X}_\wp\times\mathscr{X}_\wp\to\mathscr{X}$ is a specific function. This derivative provides a broader context for the findings reported in the literature.
Especially motivated by [11], we study the existence of the solution of our coupled problem using topological degree.

2. Preliminaries

We proceed by setting $\Lambda=[0,\nu]$. We denote by $\mathscr{X}$ a Banach space with norm $\|.\|$ and $-\mathfrak{A}:\mathscr{D}(\mathfrak{A})\to\mathscr{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $(V(\iota))_{\iota\ge0}$. We will present some basic properties in the theorem below.

Theorem 2.1. *[27]*

1. $\mathscr{X}_{\wp} = \mathscr{D}(\mathfrak{A}^{\wp})$ *is a Banach space with the norm* $\|\chi\|_{\wp} = \|\mathfrak{A}^{\wp}\chi\|$ *for each* $\chi \in \mathscr{D}(\mathfrak{A}^{\wp})$.
2. $\mathfrak{A}^{-\wp}$ *is bounded linear operator for* $0 \leq \wp \leq 1$, *there exists* $C_{\wp}$ *such that* $\|\mathfrak{A}^{-\wp}\| \leq C_{\wp}$.
3. *If* $< \wp \leq \gamma$, *then* $\mathscr{D}(\mathfrak{A}^{\gamma}) \hookrightarrow \mathscr{D}(\mathfrak{A}^{\wp})$.

Definition 2.2. *[20] We define the Kuratowski measure of noncompactness* $\kappa : \mathbb{B} \to \mathbb{R}_+$ *as follow*

$$\kappa(B) = \inf\{d > 0,\ where\ B \in \mathbb{B}\ admits\ a\ finite\ cover\ by\ sets\ of\ diameter \leq d\}.$$

Definition 2.3. *[20] Let* $\mathscr{H} : \Omega \to \mathscr{X}$ *be a continuous bounded map. Then* $\mathscr{H}$ *is*

1. κ*-lipshitz for all bounded* $P \subseteq \Omega$ *there exist* $m \geq 0$ *such that* $\kappa(\mathscr{H}(P)) \leq m\kappa(S)$.
2. κ*-contraction for all bounded* $P \subseteq \Omega$, *there exist* $0 \leq m < 1$ *such that* $\kappa(\mathscr{H}(P)) \leq m\kappa(S)$.
3. κ*-condensing for all bounded* $P \subseteq \Omega$, *with* $\kappa(P) > 0$ *if* $\kappa(\mathscr{H}(P)) < \kappa(S)$.

Let's consider for $\mathscr{H} : \Omega \to \mathscr{X}$; $\Theta C_{\kappa}(\Omega)$: the class of all strict κ-contractions and $C_{\kappa}(\Omega)$: the class of all κ-condensing. Moreover, $\mathscr{H}$ is Lipshitz if there exists $m > 0$ such that $\|\mathscr{H}(\chi) - \mathscr{H}(\zeta)\| \leq m\|\chi - \zeta\|$, for all $\chi, \zeta \in \Omega$ and that if $m < 1$, then $\mathscr{H}$ is a strict contraction.

Proposition 2.4. *[20] If* $\mathscr{H}_1, \mathscr{H}_2 : \Omega \to \mathscr{X}$ *is lipshitz with constant* m_1 *and* m_2 *respectively, then* $\mathscr{H}_1 + \mathscr{H}_2$ *is* κ*-lipshitz with constant* $m_1 + m_2$.

Proposition 2.5. *[20] If* $\mathscr{H}_1 : \Omega \to \mathscr{X}$ *is lipshitz then it's* κ*-lipshitz with the same constant.*

Proposition 2.6. *[20] If* $\mathscr{H}_1 : \Omega \to \mathscr{X}$ *is compact, then* $\mathscr{H}_1$ *is* κ*-lipshitz with constant* $m = 0$.

Theorem 2.7. *[20] Let* $\mathscr{H}_1 : \mathscr{X} \to \mathscr{X}$ *be* κ*-condensing and*

$$\Theta = \{\chi \in \mathscr{X} :\ there\ exists\ 0 \leq \beta \leq 1\ such\ that\ \chi = \beta\mathscr{H}_1\chi\}.$$

If Θ is a bounded set in $\mathscr{X}$, so there exists $r > 0$ such that $\Theta \subset B_r(0)$, then

$$\mathfrak{D}(I - \beta\mathscr{H}_1,\ B_r(0), 0) = 1,\ \ for\ all\ \beta \in [0,1].$$

As a result, $\mathscr{H}_1$ has a minimum of one fixed point, and $B_r(0)$ contains the set of fixed points for $\mathscr{H}_1$. We will also provide the required data and resources on ψ-fractional derivatives and ψ-fractional integrals,

Definition 2.8. *[17] Let* $l > 0$, $\chi \in L^1(\Lambda, \mathbb{R})$ *and* $\psi \in C^n(\Lambda, \mathbb{R})$ *such that* $\psi'(\iota) > 0$ *for all* $\iota \in \Lambda$. *The* ψ*-Riemann Liouville fractional integral of order* l *of the function* χ *is given by*

$$^{C}I_{0^+}^{l,\psi}\chi(\iota) = \frac{1}{\Gamma(l)}\int_0^{\iota}\psi'(\kappa)(\psi(\iota) - \psi(\kappa))^{l-1}\chi(\kappa)d\kappa.$$

Definition 2.9. *[17] Let* $l > 0$, $\chi \in C^{n-1}(\Lambda, \mathbb{R})$ *and* $\psi \in C^n(D, \mathbb{R})$ *such that* $\psi'(\iota) > 0$ *for all* $\iota \in \Lambda$. *The* ψ*-Caputo fractional derivative of order* l *of the function* χ *is given by*

$$^{C}D_{0^+}^{l,\psi}\chi(\iota) = \frac{1}{\Gamma(n-l)}\int_0^{\iota}\psi'(\kappa)(\psi(\iota) - \psi(\kappa))^{n-l-1}\chi_{[n]}^{\psi}(\kappa)d\kappa,$$

where

$$\chi_{[n]}^{\psi}(\kappa) = \left(\frac{1}{\psi'(\iota)}\frac{d}{d\kappa}\right)^n\chi(\kappa)\ \ and\ \ n = [l] + 1,$$

and $[l]$ *denotes the integer part of the real number* l.

Proposition 2.10. *[17] Let* $l > 0$, $\chi \in C^{n-1}(\Lambda, \mathbb{R})$, *then we have the following propositions*

1. $^{C}D_{0^+}^{l,\psi}I_{0^+}^{l,\psi}\chi(\iota) = \chi(\iota)$.
2. $I_{0^+}^{l,\psi C}D_{0^+}^{l,\psi}\chi(\iota) = \chi(\iota) - \sum\limits_{k=0}^{n-1}\frac{\chi_{[k]}^{\psi}(0)}{k!}(\psi(\iota) - \psi(0))^k$.
3. $I_{0^+}^{l,\psi}$ *is linear and bounded from* $C(\Lambda, \mathbb{R})$ *to* $C(\Lambda, \mathbb{R})$.

3. Representation of Mild Solution

Let's first take a look at this problem

$$\begin{cases} {}^{C}\mathfrak{D}_{0^+}^{l,\psi}\chi(\iota)+\mathfrak{A}\chi(\iota)=\hbar(\iota,\chi(\iota),\Re\chi(\iota)), \iota\in\Lambda, \\ \chi(0)=\chi_0 \end{cases} \tag{3.1}$$

with $-\mathfrak{A}$ generates an analytic compact semigroup $(V(\iota))_{\iota\geq 0}$ of uniformly bounded linear operators on a Banach space $\mathscr{X}$, $\hbar:\Lambda\times\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}\to\mathscr{X}$ is a given function and ${}^{C}\mathfrak{D}_{0^+}^{l,\psi}$ is the ψ-Caputo fractional derivative of order $l\in(0,1)$, $\nu>0$, $x_0\in\mathscr{X}$. The definition of $\Re\chi(\iota)$, which could be seen as a system control, is given earlier.

According to the definition 2.3 and the proposition 2.4, it is necessary to rewrite the Cauchy problem (3.1) in the equivalent integral equation

$$\chi(\iota)=\chi_0+\frac{1}{\Gamma(l)}\int_0^{\iota}(\psi(\iota)-\psi(\kappa))^{l-1}\psi'(\kappa)(\mathfrak{A}\chi(\kappa)+\hbar(\kappa,\chi(\kappa),\Re\chi(\kappa)))d\kappa. \tag{3.2}$$

Lemma 3.1. *If (3.2) holds, then we have*

$$\begin{aligned}\chi(\iota)=&\int_0^{\infty}\Phi_l(\rho)V((\psi(\iota)-\psi(0))^l\rho)\chi_0 d\rho\\ &+l\int_0^{\iota}\int_0^{\infty}\rho\Phi_l(\rho)(\psi(\iota)-\psi(\kappa))^{l-1}\psi'(\kappa)V((\psi(\iota)-\psi(0))^l\rho)[\hbar(\kappa,\chi(\kappa),\Re\chi(\kappa))]d\rho d\kappa,\end{aligned}$$

where

$$\Phi_l(\rho)=\frac{1}{l}\rho^{-1-\frac{1}{l}}\omega_l\left(\rho^{-\frac{1}{l}}\right)\geq 0,$$

$$\omega_l=\frac{1}{\pi}\sum_{p=1}^{\infty}\left((-1)^{p-1}\rho^{-lp-1}\frac{\Gamma(pl+1)}{p!}sin(pl\pi)\right),\ \ \rho\in(0,\infty).$$

For $\chi\in\mathscr{X}$ and $0<l<1$, two families $\{X_{\psi}^{l}(\iota,\varsigma):0\leq\varsigma\leq\iota\leq\nu\}$ and $\{Y_{\psi}^{l}(\iota,\varsigma):0\leq\varsigma\leq\iota\leq\nu\}$ of operators are as follows:

$$X_{\psi}^{l}(\iota,\varsigma)\chi=\int_0^{\infty}\Phi_l(\rho)V((\psi(\iota)-\psi(\varsigma))^l\rho)\chi d\rho:\mathscr{X}\times\mathscr{X}\to\mathscr{X}_{\wp}$$

and

$$Y_{\psi}^{l}(\iota,\varsigma)\chi=l\int_0^{\infty}\rho\Phi_l(\rho)V((\psi(\iota)-\psi(\varsigma))^l\rho)\chi d\rho:\mathscr{X}\times\mathscr{X}\to\mathscr{X}_{\wp}$$

respectively.

Lemma 3.2. *[14] The operators $X_{\psi}^{l}(\iota,\varsigma)$ and $Y_{\psi}^{l}(\iota,\varsigma)$ meet the following requirements:*

1. *$\forall\ \iota\geq\varsigma\geq 0$ and $\chi\in\mathscr{X}_{\wp}$, the operators $X_{\psi}^{l}(\iota,\varsigma)$ and $Y_{\psi}^{l}(\iota,\varsigma)$ are bounded linear operators, i.e. $\forall\chi\in\mathscr{X}_{\wp}$*

$$\left\|X_{\psi}^{l}(\iota,\varsigma)\chi\right\|_{\wp}\leq M_V\left\|\chi\right\|_{\wp}\ \ \textit{and}\ \ \left\|Y_{\psi}^{l}(\iota,\varsigma)\chi\right\|_{\wp}\leq\frac{M_V}{\Gamma(l)}\left\|\chi\right\|_{\wp}.$$

2. *The operators $X_{\psi}^{l}(\iota,\varsigma)$ and $Y_{\psi}^{l}(\iota,\varsigma)$ are strongly continuous $\forall\iota\geq\varsigma\geq 0$. That is, $\chi\in\mathscr{X}_{\wp}$ and $0\leq\varsigma\leq\iota_1\leq\iota_2\leq T$, we have the following:*

$$\left\|X_{\psi}^{l}(\iota_2,\varsigma)\chi-X_{\psi}^{l}(\iota_1,\varsigma)\chi\right\|_{\wp}\to 0\ \ \textit{and}\ \ \left\|Y_{\psi}^{l}(\iota_2,\varsigma)\chi-Y_{\psi}^{l}(\iota_1,\varsigma)\chi\right\|_{\wp}\to 0,$$

as $\iota_2\to\iota_1$.

3. *The operators $X_{\psi}^{l}(\iota,\varsigma)$ and $Y_{\psi}^{l}(\iota,\varsigma)$ are compact operators $\forall\iota,\varsigma>0$.*

4. *If $X^l_\psi(\iota,\varsigma)$ and $Y^l_\psi(\iota,\varsigma)$ are strongly continuous compact semigroups of linear bounded operators.*
5. *If $0<l<1$, then*

$$^{C}D^{l,\psi}_{0^+}\{X^l_\psi(\iota,0)\chi_0\} = \mathfrak{A}\{X^l_\psi(\iota,0)\chi_0\},$$

and

$$^{C}D^{l,\psi}_{0^+}\{P^\iota_0\{w(\iota)\}\} + w(\iota) = \mathfrak{A}\{P^\iota_0\{w(\iota)\}\} + w(\iota),$$

where

$$P^\iota_0\{\chi(\iota)\} := \int_{\iota_1}^{\iota_2} (\psi(\iota_2)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota_2,\varsigma)\chi(\varsigma)d\varsigma,$$

such that $\iota_1,\iota_2\in\Lambda$; $\chi, w\in\mathscr{C}_\wp$.

Definition 3.3. *A solution $\chi(.;\chi_0;u)\in\mathscr{C}_\wp$ is called an $\wp$-mild solution of (3.1) if it satisfies the integral equation*

$$\chi(\iota) = X^l_\psi(\iota,0)\chi_0 + \int_0^\iota (\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma$$

4. Main Results

The Banach space $C(\Lambda,\mathscr{X}_\wp)$ is denoted by $\mathscr{C}_\wp$ with $\wp\in(0,\nu)$ it supnorm: $\|\chi\|_\infty = \sup_{\iota\in\Lambda}\|\chi\|_\wp$, $for\ \chi\in\mathscr{C}_\wp$, and the product space $\mathscr{C}_\wp\times\mathscr{C}_\wp$ is a banach space under the norms $\|(\chi,\zeta)\|_\wp = \|\chi\|_\wp + \|\zeta\|_\wp$ and $\|(\chi,\zeta)\|_\wp = \max\{\|\chi\|_\wp, \|\zeta\|_\wp\}$

To ensure that, we make the following assumptions.

($\mathscr{C}_1$) $V(\kappa)$ *and* $W(\kappa)$ $\kappa>0$ are compact analytic semigroups and $\lim_{\kappa\to 0^+} V(\kappa) = W(\kappa) = I$ (the identity operator).

($\mathscr{C}_2$) $\exists\gamma\in[\wp,1]$ such that the function $\hbar:\Lambda\times\mathscr{X}_\wp\times\mathscr{X}_\wp\to\mathscr{X}_\gamma$ satisfies the following properties:

(a) $\forall(\chi,\varsigma)\in\mathscr{X}_\wp\times\mathscr{X}_\wp$, the function $\hbar(.,\chi,\varsigma)$ is strongly measurable.

(b) $\forall\iota\in\Lambda$, the function $\hbar(\iota,.):\mathscr{X}_\wp\times\mathscr{X}_\wp\to\mathscr{X}_\gamma$ is continuous.

($\mathscr{C}_3$) For every $(\iota,\chi)\in\Lambda\times\mathscr{C}_\wp$, there exist positive constant $\ell_{\hbar 1}$, $\ell_{\hbar 2}$ and $\alpha_\hbar$ such that

$$\|\hbar(\iota,\chi(\iota),\Re\chi(\iota))\|_\wp \le \ell_{\hbar 1}\|\chi\|^{a_1} + \ell_{\hbar 1}\|\Re\chi\|^{a_1} + \alpha_\hbar,$$

denote $\ell_\hbar = \sup_{\nu\in\Lambda}\{\ell_{\hbar 1},\ell_{\hbar 2}\}$.

($\mathscr{C}_4$) For every χ and $\zeta\in\mathscr{C}_\wp$, there exist a constant L_ρ such that

$$|\rho(\chi)-\rho(\zeta)| \le L_\rho|\chi-\zeta|.$$

($\mathscr{C}_5$) For every χ and $\zeta\in\mathscr{C}_\wp$, there exist positive constant ρ and γ such that

$$\|\rho(\chi)\| \le \rho\|\chi\|^{a_2} + \gamma.$$

$$\|\rho(\zeta)\| \le \rho\|\zeta\|^{a_2} + \gamma.$$

We define the operator $\mathscr{H}_i:\mathscr{C}_\wp\times\mathscr{C}_\wp\to\mathscr{C}_\wp\times\mathscr{C}_\wp$, by $i=1,2$

$$\mathscr{H}_1(\chi,\zeta)(\iota) = (\mathscr{F}_1\chi(\iota),\mathscr{F}_2\zeta(\iota)),\quad \mathscr{H}_2(\chi,\zeta)(\iota) = (\mathscr{G}_1\chi(\iota),\mathscr{G}_2\zeta(\iota)),$$

with

$$\mathscr{F}_1\chi(\iota) = (\chi_0-\rho(\chi))X^l_\psi(\iota,0),\quad \mathscr{F}_2\zeta(\iota) = (\zeta_0-\rho(\zeta))W^l_\psi(\iota,0),$$

$$\mathscr{G}_1\chi(\iota) = \int_0^\iota (\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma,$$

$$\mathscr{G}_2\zeta(\iota) = \int_0^\iota (\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Z^l_\psi(\iota,\varsigma)\hbar(\varsigma,\zeta(\varsigma),\Re\zeta(\varsigma))d\varsigma,$$

where $\mathscr{F}_1,\mathscr{F}_2,\mathscr{G}_1,\mathscr{G}_2:\mathscr{C}_\wp\to\mathscr{C}_\wp$ then $\mathscr{H}(\chi,\zeta)(\iota) = \mathscr{H}_1(\chi,\zeta)(\iota) + \mathscr{H}_2(\chi,\zeta)(\iota)$ are solutions of the (1.1).

Theorem 4.1. *The operator $\mathscr{H}_1$ is Lipshitz with constant L_ρ. Consequently $\mathscr{H}_1$ is λ-lipshitz with the same constant and satisfies the growth condition:*

$$\|\mathscr{H}_1(\chi,\zeta)\|_{\wp} \leq C\|(\chi,\zeta)\|_{\wp}^{a_2}+D. \tag{4.1}$$

Proof. Let $\chi_1,\chi_2 \in \mathscr{X}_{\wp}$, then we have

$$|\mathscr{F}_1\chi_1(\iota)-\mathscr{F}_1\chi_2(\iota)| \leq |\rho(\chi_1)-\rho(\chi_2)||X_\psi^l(\iota,0)| \leq M|\rho(\chi_1)-\rho(\chi_2)| \leq ML_\rho|\chi_1-\chi_2|,$$

and

$$|\mathscr{F}_2\zeta_1(\iota)-\mathscr{F}_2\zeta_2(\iota)| \leq |\rho(\zeta_1)-\rho(\zeta_2)||W_\psi^l(\iota,0)| \leq M|\rho(\zeta_1)-\rho(\zeta_2)| \leq ML_\rho|\zeta_1-\zeta_2|,$$

Taking supremum over ι, we obtain

$$\|\mathscr{F}_1\chi_1-\mathscr{F}_1\chi_2\| \leq ML_\rho\|\chi_1-\chi_2\|, \quad \|\mathscr{F}_2\zeta_1-\mathscr{F}_2\zeta_2\| \leq ML_\rho\|\zeta_1-\zeta_2\|.$$

Now

$$\begin{aligned}\|\mathscr{H}_1(\chi_1,\zeta_1)-\mathscr{H}_1(\chi_2,\zeta_2)\| &= \max\{\|\mathscr{F}_1\chi_1(\iota)-\mathscr{F}_1\chi_2(\iota)\|,\|\mathscr{F}_2\zeta_1(\iota)-\mathscr{F}_2\zeta_2(\iota)\|\}\\ &\leq \max\{ML_\rho\|\chi_1-\chi_2\|,ML_\rho\|\zeta_1-\zeta_2\|\}\\ &\leq ML_\rho\max\{\|\chi_1-\chi_2\|,\|\zeta_1-\zeta_2\|\}\\ &\leq ML_\rho\max\{\|(\chi_1,\zeta_1)-(\chi_2,\zeta_2)\|\}.\end{aligned}$$

By proposition, $\mathscr{H}_1$ is κ-Lipshitz with constant ML_ρ. Also, in terms of growing conditions, we have

$$\|\mathscr{F}_1\chi(\iota)\|_{\wp} = \|\chi_0-\rho(\chi)\|_{\wp}\left\|X_\psi^l(\iota,0)\right\|_{\wp} \leq M(\|\chi_0\|_{\wp}+\rho\|\chi\|^{a_2}+\gamma)$$

and

$$\|\mathscr{F}_2\zeta(\iota)\|_{\wp} = \|\zeta_0-\rho(\zeta)\|_{\wp}\left\|W_\psi^l(\iota,0)\right\|_{\wp} \leq M(\|\zeta_0\|_{\wp}+\rho\|\zeta\|^{a_2}+\gamma).$$

Therefore, it may be inferred that

$$\|\mathscr{H}_1(\chi,\zeta)\|_{\wp} = \|(\mathscr{F}_1(\chi),\mathscr{F}_2(\zeta))\|_{\wp} = \|\mathscr{F}_1(\chi)\|_{\wp}+\|\mathscr{F}_2(\zeta)\|_{\wp} \leq M(\|(\chi_0,\zeta_0)\|_{\wp}+\rho\|(\chi,\zeta)\|_{\wp}^{a_2}+2\gamma),$$

then

$$\|\mathscr{H}_1(\chi,\zeta)\|_{\wp} \leq C\|(\chi,\zeta)\|_{\wp}^{a_2}+D,$$

with $C=M\rho$ and $D=M(\|(\chi_0,\zeta_0)\|_{\wp}+2\gamma)$. □

Theorem 4.2. *The operator $\mathscr{H}_2$ is continuous and satisfies the growth condition:*

$$\|\mathscr{H}_2(\chi,\zeta)\|_{\wp} \leq E\|(\chi,\zeta)\|_{\wp}^{a_1}+F. \tag{4.2}$$

Proof. Consider a bounded subset of $\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}$,

$$S=\{(\chi,\zeta)\in\mathscr{X}_{\wp}\times\mathscr{X}_{\wp};\|(\chi,\zeta)\|_{\wp}\leq L\}\subset\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}.$$

Consider the sequence $\{(\chi_w,\zeta_w)\}$ in Banach such that $(\chi_w,\zeta_w)\to(\chi,\zeta)\in\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}$ as $w\to\infty$, we demonstrate that $\mathscr{H}_2$ is continuous. For this, take into account

$$\begin{aligned}\|\mathscr{G}_1(\chi_w)-\mathscr{G}_1(\chi)\|_{\wp} &\leq \int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|Y_\psi^l(\iota,\varsigma)\hbar(\varsigma,\chi_w(\varsigma),\Re\chi_w(\varsigma))-\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))\right\|_{\wp}d\varsigma\\ &\leq \frac{M}{\Gamma(l)}\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\|\hbar(\varsigma,\chi_w(\varsigma),\Re\chi_w(\varsigma))-\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))\|_{\wp}d\varsigma,\end{aligned}$$

from condition $(\mathscr{C}_2)$ (b) $\forall \varsigma \in [0,\nu]$, $\hbar(\varsigma,\chi_w(\varsigma),\mathfrak{R}\chi_w(\varsigma)) - \hbar(\varsigma,\chi(\varsigma),\mathfrak{R}\chi(\varsigma)) \to 0$ and using the condition $(\mathscr{C}_3)$ next, we obtain

$$\frac{(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)}{\Gamma(l)}\|\hbar(\varsigma,\chi_w(\varsigma),\mathfrak{R}\chi_w(\varsigma)) - \hbar(\varsigma,\chi(\varsigma),\mathfrak{R}\chi(\varsigma))\|_{\wp} \leq \frac{(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)}{\Gamma(l)}((1+i^*)\ell_{\hbar 1}\|\chi\|^{a_1}+\alpha_{\hbar}),$$

where $\ell_h = \sup\limits_{\iota\in\Lambda}\{\ell_{h1},\ell_{h2}\}$, which implies that the term on the left is integrable. Lebesque Dominated Convergent Theorem allows us to arrive at

$$\int_0^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\|\hbar(\varsigma,\chi_w(\varsigma),\mathfrak{R}\chi_w(\varsigma)) - \hbar(\varsigma,\chi(\varsigma),\mathfrak{R}\chi(\varsigma))\|_{\wp}d\varsigma \quad as \quad w\to\infty.$$

Consequently

$$\|\mathscr{G}_1(\chi_w)-\mathscr{G}_1(\chi)\|_{\wp} \quad as \quad w\to\infty,$$

and

$$\|\mathscr{G}_2(\zeta_w)-\mathscr{G}_2(\zeta)\|_{\wp} \quad as \quad w\to\infty,$$

that by following the same steps as above, finally:

$$\|\mathscr{H}_2(\chi_w,\zeta_w)-\mathscr{H}_2(\chi,\zeta)\|_{\wp} \quad as \quad w\to\infty.$$

$\mathscr{H}_2$ is continuous, as demonstrated by this: We now have the following growth conditions for $\mathscr{H}_2$.

$$\begin{aligned}
\|\mathscr{G}_1\chi(\iota)\|_{\wp} &\leq \frac{M}{\Gamma(l)}\int_0^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\|\hbar(\varsigma,\chi(\varsigma),\mathfrak{R}\chi(\varsigma))\|_{\wp}d\varsigma \\
&\leq \frac{M}{\Gamma(l)}\int_0^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)[(1+i^*)\ell_{\hbar}\|\chi\|^{a_1}+\alpha_{\hbar}] \\
&\leq \frac{M}{\Gamma(l+1)}(\psi(\iota)-\psi(0))^{l}[(1+i^*)\ell_{\hbar}\|\chi\|^{a_1}+\alpha_{\hbar}],
\end{aligned}$$

where $\ell_h = \sup\limits_{\iota\in\Lambda}\{\ell_{h1},\ell_{h2}\}$. Similarly

$$\|\mathscr{G}_2\zeta(\iota)\|_{\wp} \leq \frac{M}{\Gamma(l+1)}(\psi(\iota)-\psi(0))^{l}[(1+j^*)\ell_{\hbar}\|\zeta\|^{a_1}+\alpha_{\hbar}].$$

This implies that

$$\|\mathscr{H}_2(\chi(\iota),\zeta(\iota))\|_{\wp} \leq \|\mathscr{G}_1\chi(\iota),\mathscr{G}_2\zeta(\iota)\|_{\wp} \leq \frac{M(\psi(\iota)-\psi(0))^{l}}{\Gamma(l+1)}[(1+k^*)\ell_{\hbar}\|(\chi,\zeta)\|^{a_1}+\alpha_{\hbar}],$$

where $k^* = \sup\{i^*,j^*\}$ then

$$\|\mathscr{H}_2(\chi,\zeta)\|_{\wp} \leq E\|(\chi,\zeta)\|_{\wp}^{a_1}+F$$

where $\quad E = \frac{M(\psi(\iota)-\psi(0))^{l}}{\Gamma(l+1)}(1+k^*)\ell_{\hbar} \quad$ and $\quad F = \frac{M(\psi(\iota)-\psi(0))^{l}}{\Gamma(l+1)}\alpha_{\hbar}$.

□

Theorem 4.3. *$\mathscr{H}_2$ is a compact operator. As such, $\mathscr{H}_2$ is κ-Lipshitz, where the constant is zero.*

Proof. Let Ω be a bounded subset of $\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}$, and $\{(\chi_w,\zeta_w)\}$ be a sequence located in Ω. The growth condition will then enable us to write

$$\|\mathscr{H}_2(\chi_w(\iota),\zeta_w(\iota))\|_{\wp} \leq EL^{a_1}+F.$$

It demonstrates the uniform boundedness of $\mathscr{H}_2(\Omega)$.

Now, consider $0 \leq \tau \leq \iota \leq 1$, to achieve equi-continuity. Using $(\mathscr{C}_2)$ it is obvious that

$$\|\mathscr{G}_2\chi_w(\iota)-\mathscr{G}_2\chi_w(\tau)\|_{\wp}\to 0 \ \ and \ \ \|\mathscr{G}_2\zeta_w(\iota)-\mathscr{G}_2\zeta_w(\tau)\|_{\wp}\to 0,$$

there after

$$\|\mathscr{H}_2(\chi_w,\zeta_w)(\iota)-\mathscr{H}_2(\chi_w,\zeta_w)(\tau)\|_{\wp}\to 0,$$

$\mathscr{H}_2(\chi,\zeta)$ is equicontinuous, in other words. The Arzela Ascoli Theorem states that $\mathscr{H}_2(\chi,\zeta)$ is compact. Therefore, $\mathscr{H}_2$ is κ-lipshitz with constant 0. according to proposition 2.6. $\square$

Theorem 4.4. *Assume that $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ hold, then (1.1) has at least one solution.*

Proof. According to theorems 4.1, 4.3 $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, are κ-Lipshitz with constant ML_ρ and κ-lipshitz with constant 0. Let's consider this set

$$T=\{(\chi,\zeta)\in\mathscr{X}_{\wp}\times\mathscr{X}_{\wp};\exists\,\alpha\in[0,1]\ni(\chi,\zeta)=\alpha\mathscr{H}(\chi,\zeta)\}.$$

Let $(\chi,\zeta)\in T$, then using (4.1) and (4.2), we have

$$(\chi,\zeta)=\alpha(\mathscr{H}_1(\chi,\zeta)+\mathscr{H}_2(\chi,\zeta)),$$

which implies that

$$\|(\chi,\zeta)\|_{\wp}\leq\alpha(C\|(\chi,\zeta)\|_{\wp}^{a_2}+D+E\|(\chi,\zeta)\|_{\wp}^{a_1}+F).$$

where $a_1,a_2\in[0,1)$. Therefore If there is at least one fixed point in $\mathscr{H}$, then there is at least one solution in (1.1). The set of solutions is limited in $\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}$, and $(\chi,\zeta)\in\mathscr{X}_{\wp}\times\mathscr{X}_{\wp}$. $\square$

5. Example

The example that follows is examined in the final section to back up our main result.
Examine the following equation for a fractional partial differential:

$$\begin{cases}\mathfrak{D}^{\frac{1}{2}}\chi(\iota,y)+\mathfrak{A}_1\chi(\iota,y)=\dfrac{\sin(\iota)}{9+cos\iota}\exp\left(\zeta(\iota,y)+\displaystyle\int_0^{\iota}\exp(\iota\varsigma)\zeta(\varsigma,y)d\varsigma\right),\\ \mathfrak{D}^{\frac{1}{2}}\zeta(\iota,y)+\mathfrak{A}_2\zeta(\iota,y)=\dfrac{\sin(\iota)}{9+cos\iota}\exp\left(\chi(\iota,y)+\displaystyle\int_0^{\iota}\exp(\iota\varsigma)\chi(\varsigma,y)d\varsigma\right),\\ \chi(0)=\sum\limits_{w=1}^{10}\rho_w|\chi(\iota_w)|,\ \rho_w>0,\ 0<\iota_w<1,\ w=1,...,10,\\ \zeta(0)=\sum\limits_{w=1}^{10}\rho_w|\zeta(\iota_w)|,\ \rho_w>0,\ 0<\iota_w<1,\ w=1,...,10.\end{cases}\tag{5.1}$$

where $l=\frac{1}{2}$, $\nu=1$, $\chi(\iota)=\chi(\iota,y)$, $\zeta(\iota)=\mathfrak{y}(\iota,y)$, $\mathfrak{A}_1\chi(\iota,y)=\dfrac{\partial^2}{\partial y^2}\mathfrak{y}(\iota,y)$, $\mathfrak{A}_2\zeta(\iota,y)=\dfrac{\partial^2}{\partial y^2}\chi(\iota,y)$, $\rho(\chi)=\sum\limits_{w=1}^{10}\rho_w|\chi(\iota_w)|$, $\rho(\zeta)=\sum\limits_{w=1}^{10}\rho_w|\zeta(\iota_w)|$, with $\sum\limits_{w=1}^{10}\rho_w<1$,

$$\hbar(\iota,\zeta(\iota),\mathfrak{R}\zeta(\iota))=\frac{\sin(\iota)}{9+cos\iota}\exp\left(\mathfrak{y}(\iota,y)+\int_0^{\iota}\exp(\iota\varsigma)\zeta(\varsigma,y)d\varsigma\right),$$

$$\hbar(\iota,\chi(\iota),\mathfrak{R}'\chi(\iota))=\frac{\sin(\iota)}{9+cos\iota}\exp\left(\chi(\iota,y)+\int_0^{\iota}\exp(\iota\varsigma)\chi(\varsigma,y)d\varsigma\right).$$

Let $\mathscr{X}$ be defined as $\mathscr{X}=L^2[0,\pi]$ and $\mathfrak{A}_i$ for $i=1,2$ by $\mathfrak{A}_i\upsilon=-\upsilon''$ on the domain
$\mathscr{D}(\mathfrak{A}_i)=\{\upsilon(.)\in L^2[0,\pi],\ \upsilon,\ \upsilon',\ are\ absolutely\ continuous,\ \upsilon''\in L^2[0,\pi],\ \upsilon(0)=\upsilon(\pi)=0\}$, for $i=1,2$. It is noticeable that $\mathfrak{A}$ has a discrete spectrum and the eigenvalues are $\{-n^2:\ n\in\mathbb{N}$ with the corresponding normalized eigenvectors $e_n(y)=\sqrt{\frac{2}{\pi}}\sin ny$. Consequently,

$$\mathfrak{A}_i\upsilon=-\sum_{n=1}^{\infty}n^2\langle\upsilon,e_n\rangle e_n,\quad \upsilon\in\mathscr{D}(\mathfrak{A}_i),\ for\ i=1,2.$$

In addition, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the infinitesimals generators of a bounded analytic semigroup $(T_1(\iota))_{\iota\geq 0}$, $(T_2(\iota))_{\iota\geq 0}$, respectively where

$$T_i(\iota)\upsilon = \sum_{n=1}^{\infty} e^{-n^2\iota}\langle \upsilon, e_n\rangle e_n, \quad \upsilon \in \mathscr{X}, \; for\; i=1,2.$$

Surely, $\forall \iota \geq 0$, $\|T_i(\iota)\| \leq e^{-\iota}$, $for\; i=1,2$. Hence, we take $M=1$, which implies that $\sup\limits_{\iota\in(0,\infty)} \|T_i(\iota)\| = 1$, $for\; i=1,2$ and $(\mathscr{C}_1)$ are satisfied.
For $q=\frac{1}{2}$, the operator $\mathfrak{A}^{\frac{1}{2}}$ is given by the following:

$$\mathfrak{A}_i^{\frac{1}{2}}\upsilon = -\sum_{n=1}^{\infty} n\langle \upsilon, e_n\rangle e_n, \quad \upsilon \in \mathscr{D}(\mathfrak{A}_{\mathrm{i}}^{\frac{1}{2}}), \; for\; i=1,2.$$

where $\mathscr{D}(\mathfrak{A}_i^{\frac{1}{2}}) = \{\upsilon \in \mathscr{X} : \sum\limits_{n=1}^{\infty} n\langle \upsilon, e_n\rangle e_n \in \mathscr{X}\}$, $for\; i=1,2$ and $\left\|\mathfrak{A}_i^{-\frac{1}{2}}\right\| = 1$. Let $\mathscr{X}_{\frac{1}{2}} = \left(\mathscr{D}(\mathfrak{A}_i^{\frac{1}{2}}), \|\;\|_{\frac{1}{2}}\right)$, $for\; i=1,2$, where $\|\chi\|_{\frac{1}{2}} = \left\|\mathfrak{A}_1^{\frac{1}{2}}\chi\right\|_{\mathscr{X}}$, $\|\mathfrak{y}\|_{\frac{1}{2}} = \left\|\mathfrak{A}_2^{\frac{1}{2}}\mathfrak{y}\right\|_{\mathscr{X}}$ for $\chi \in \mathscr{D}(\mathfrak{A}_1^{\frac{1}{2}})$ and $\mathfrak{y} \in \mathscr{D}(\mathfrak{A}_2^{\frac{1}{2}})$. The conditions $(\mathscr{C}_4)$ and $(\mathscr{C}_5)$ are satisfied with $L_\rho = \rho = \sum\limits_{w=1}^{10}\rho_w$, $\alpha=0$ and $\omega=1$, indeed, we have

$$|\rho(\chi(\iota))| = |\sum_{w=1}^{10}\rho_w|\chi(\iota_w)||, \qquad |\rho(\mathfrak{y}(\iota))| = |\sum_{w=1}^{10}\rho_w|\mathfrak{y}(\iota_w)||,$$

it follows that

$$|\rho(\chi)| \leq \sum_{w=1}^{10}\rho_w\|\chi\|, \qquad |\rho(\zeta)| \leq \sum_{w=1}^{10}\rho_w\|\zeta\|.$$

On the other hand

$$|\rho(\chi_1(\iota)) - \rho(\chi_2(\iota))| \leq |\sum_{w=1}^{10}|\rho_w\chi_1(\iota_w)| - \sum_{w=1}^{10}\rho_w|\chi_2(\iota_w)||$$

and

$$|\rho(\zeta_1(\iota)) - \rho(\zeta_2(\iota))| \leq |\sum_{w=1}^{10}|\rho_w\zeta_1(\iota_w)| - \sum_{w=1}^{10}\rho_w|\zeta_2(\iota_w)||,$$

from which, we have

$$|\rho(\chi_1) - \rho(\chi_2)| \leq \sum_{w=1}^{10}\rho_w|\chi_1 - \chi_2|, \quad |\rho(\zeta_1) - \rho(\zeta_2)| \leq \sum_{w=1}^{10}\rho_w|\zeta_1 - \zeta y_2|.$$

To prove the condition $(\mathscr{C}_3)$. Let $\iota \in \Lambda$, $\chi, \in \mathbb{R}$, then we have

$$\begin{aligned}
&|\hbar(\iota, \chi(\iota), \mathfrak{R}\chi(\iota))| \\
&\leq \left|\frac{\sin(\iota)}{9+cos\iota}\exp\left(\chi(\iota,y) + \int_0^\iota \exp(\iota\varsigma)\chi(\varsigma,y)d\varsigma\right)\right| \\
&\leq \left|\frac{\sin\iota}{9+e^\iota}\right||\chi(\iota,y)+1| \\
&\leq \frac{1}{10}(|\chi|+1).
\end{aligned}$$

Then the conditions hold, with $\ell_1 = \ell_2 = \frac{1}{10}$ and $\alpha_f = 1$. Ultimately, the fractional problem (5.1) has a unique solution on $[0,1]$ since all the conditions of Theorem 4.4 are met.

Conclusion

In this article, we have studied and investigated the existence and solutions of a new class of ψ-Caputo-type fractional semilinear differential equations. The significance of this work lies in its generality, surpassing existing works based on evolution equations. We have established the existence results for the problem (1.1) using the topological degree method. Finally, we have presented an example to illustrate the obtained result. In the future, we plan to incorporate the control operator $Bu(t)$ to explore the controllability of the system. Additionally, we aim to expand our analysis by utilizing the Hilfer derivative.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private, or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

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