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FIXED POINT THEOREMS FOR CONTRAVARIANT MAPS IN BIPOLAR $b-$METRIC SPACES WITH INTEGRATION APPLICATION

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Abstract. As a natural extension of the metric and the bipolar metric, this article introduces the new abstract bipolar $b-$ metric. The bipolar $b-$metric is a novel technique addressed in this article; it is explained by combining the well-known $b-$metric in the theory of metric spaces, as defined by Mutlu and Gürdal (2016) [10], with the description of the bipolar metric. In this new definition, well-known mathematical terms such as Cauchy and convergent sequences are utilized. In the bipolar $b-$metric, fundamental topological concepts are also defined to investigate the existence of fixed points implicated in such mappings under different contraction conditions. An example is provided to demonstrate the presented results.

1. Introduction

Fixed point theory is a fabulous blend of Topology, Analysis, and Geometry. It has been a crucial part of exploring linear and nonlinear phenomena. There are also excellent applications of fixed-point theorems to encourage mathematical inquiry, economics, game theory, computer science, and digital image embedding. Thanks to its application in mathematics and other disciplines, the Banach ([4]) contraction principle has become widely used. Two situations enable this principle to become widespread. Either the mapping space is universalized, or the map's contraction condition is extended.

One of the exciting topics of the last few decades is the theory of fixed points. In particular, the issue of changing the abstract structure of the mapping to form a fixed point has been intensively studied. The concept of a metric space has been variously revised, expanded, and generalized to ensure the existence of a fixed point for certain mappings

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defined in these new constructs. The most exciting and general concept is the $b-$metric space. Several mathematicians have considered it by different names (such as the quasi-metric [5] and the general metric), but it became famous for the publications of Bakhtin [3] and Czerwik [6].

It is exciting to get fixed point theorems for covariant and contravariant maps with different contraction maps in both expanding and non-expanding topological spaces. There are unique bivariate metric spaces like $b-$metric spaces [3, 6], extended $b-$metric spaces, and trivariate metric-type spaces like bipolar metric spaces [10].

In [10] the new distance function, the distance between the members of two different sets is different from the empty set. A successful description of generalized and improved metric spaces is called bipolar metric spaces. This study also validated new versions of Banach and Kannan Caristi's fixed point theorems (see [14]).

Recent articles on bipolar metric space refer to popular theorems of fixed point theory contained in them (see [14] and [15]). In addition, various issues related to this theory are covered (see [1, 2, 7, 11, 12, 8, 13, 16, 17]).

This study aims to combine the bipolar metric space defined in [10] 2016 with the $b-$metric space definition, which is a new approach for general metric spaces. This article discusses the existence of and gives examples of some fixed point theorems in the bipolar $b-$metric.

2. Preliminaries

In this section, the definition and theorem that will be required for the analysis will be reminded again for convenience.

Recall (see, e.g., [3, 6]) that a $b-$metric d on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$d(x,z) \leq b[d(x,y) + d(y,z)],$$

for all $x, y, z \in X$, for some fixed $b \geq 1$.

Definition 2.1. *([10]). A bipolar metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \times Y \longrightarrow \mathbb{R}^+$ is a function satisfying the following conditions:*

(i) $d(x,y) = 0 \Longleftrightarrow x = y$,

(ii) if $x, y \in X \cap Y$, then $d(x,y) = d(y,x)$,

(iii) $d(x_1,y_2) \leq d(x_1,y_1) + d(x_2,y_1) + d(x_2,y_2)$

for all $(x,y), (x_1,y_1), (x_2,y_2) \in X \times Y$. Then d is called a bipolar metric on the pair (X, Y).

Example 2.1. *Let $X = \{(a, 2a) |\ a \in \mathbb{R}\}$, $Y = \{(d, c) |\ d, c \in \mathbb{R}\} = \mathbb{R}^2$ and*

$$d(x,y) = \mid a - d \mid + \mid 2a - c \mid,$$

for every $x = (a, 2a) \in X$ and $y = (d, c) \in Y$. Obviously $X \cap Y = X$ and conditions (i) and (ii) of Definition 2.1 are satisfied.

For each $x = (a, 2a), x' = (a', 2a') \in X$ and $y = (d, c), y' = (d', c') \in Y$, we have

$$\begin{aligned} d(x,y) &= \mid a - d \mid + \mid 2a - c \mid \\ &\leq \mid a - d' \mid + \mid 2a - c' \mid + \mid a' - d' \mid + \mid 2a' - c' \mid + \mid a' - d \mid + \mid 2a' - c \mid \\ &= d(x,y') + d(x',y') + d(x',y). \end{aligned}$$

So, condition (iii) of Definition 2.1 is also satisfied and d is a bipolar metric.

Definition 2.2. *([10]). Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets.*

(a) *Let $f : X_1 \cup Y_1 \longrightarrow X_2 \cup Y_2$ be a given function.*

If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, f is said a covariant map from (X_1, Y_1) to (X_2, Y_2) and write $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$.

(b) *$f : X_1 \cup Y_1 \longrightarrow X_2 \cup Y_2$ be a given function.*
If $f(X_1) \subseteq Y_2$, and $f(Y_1) \subseteq X_2$, f is said a contravariant map from (X_1, Y_1) to (X_2, Y_2) and $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ is written in this paper.

Example 2.2. *Let $X = \{(a, 0)|\ a \in \mathbb{R}\}$, $Y = \{(b, c)|\ b, c \in \mathbb{R}\} = \mathbb{R}^2$ and $f : X \cup Y \longrightarrow X \cup Y$ defined by $f(x, y) = (x^2, xy)$ for every $(x, y) \in X \cup Y$. Obviously $f(X) \subseteq X$ and $f(Y) \subseteq Y$. Therefore, f is a covariant map from (X, Y) to (X, Y), that is $f : (X, Y) \rightrightarrows (X, Y)$.*

It is superimposed on the $b-$metric with the bipolar metric, just as different previously defined metrics are combined in one definition. An example of these is the metric structure in defining bipolar and ultrametric,and the description presented is the definition of bipolar $b-$metric.

3. MAIN RESULTS

It is superimposed on the $b-$metric with the bipolar metric, just as different previously defined metrics are combined in one definition. An example of these is the metric structure in defining bipolar and ultrametric [7], and the description presented is the definition of bipolar $b-$metric.

Definition 3.1. *A bipolar $b-$metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \times Y \longrightarrow \mathbb{R}^+$ is a function satisfying the following conditions:*
(i) $d(x, y) = 0 \Longleftrightarrow x = y$,
(ii) if $x, y \in X \cap Y$, then $d(x, y) = d(y, x)$,
(iii) $d(x_1, y_2) \leq b[d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)]$ for all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$ and $b \geq 1$. We say d is a bipolar $b-$metric on the pair (X, Y).

Example 3.1. *Let $X = \{(a, 0)|\ a \in \mathbb{R}\}$, $Y = \{(d, c)|\ d, c \in \mathbb{R}\} = \mathbb{R}^2$ and*

$$d(x, y) = (a - d)^2 + \mid c \mid,$$

for every $x = (a, 0) \in X$ and $y = (d, c) \in Y$. Obviously $X \cap Y = X$ and conditions (i) and (ii) of Definition 3.1 are satisfied.
For each $x = (a, 0), x' = (a', 0) \in X$ and $y = (d, c), y' = (d', c') \in Y$ and $b = 3$, we have

$$\begin{aligned} d(x, y) &= (a - d)^2 + \mid c \mid = [(a - d') + (d' - a') + (a' - d)]^2 + \mid c \mid \\ &\leq 3(a - d')^2 + \mid c' \mid + 3(a' - d')^2 + \mid c' \mid + 3(a' - d)^2 + \mid c \mid \\ &= 3[d(x, y') + d(x', y') + d(x', y)]. \end{aligned}$$

So, condition (iii) of Definition 3.1 is also satisfied and d is a bipolar $b-$metric for $b = 3$.

It is useful to state the following here. If $b = 1$ in a bipolar $b-$metric, a bipolar metric is obtained (see [10]).
It should be noted that in the preceding example, if (X, Y, d) is a bipolar $b-$metric space, then (X, Y, d) is not necessarily a bipolar metric space, because the triangle inequality does not hold.

Let Ψ denote a family of mappings such that for each $\psi \in \Psi$,
$\psi : [0, \infty) \longrightarrow (0, \infty)$ and

(1) $\psi(t)$ is continuous and it is decreasing for every $t \in [0, \infty)$,

(2) $\int_0^{ms} \psi(t)dt \leq m \int_0^s \psi(t)dt$ for every $s > 0$ and $m \geq 1$.

For example, if $\psi : [0, \infty) \longrightarrow (0, \infty)$ defined by $\psi(t) = e^{-t}$, $\psi(t) = \frac{1}{1+t}$, then it is easy to see that $\psi \in \Psi$.

Example 3.2. *Let (X, Y, d) be a bipolar $b-$metric space. If it is defined with*

$$\rho(x, y) = \int_0^{d(x,y)} \psi(t)dt, \ for\ every\ (x, y) \in X \times Y\ and\ \psi \in \Psi,$$

then (X, Y, ρ) is a bipolar $b-$metric space.

Proof. Obviously conditions (i) and (ii) of Definition 3.1 are satisfied. Now, since d is bipolar $b-$metric hence for all $(x, y), (x_1, y_1) \in X \times Y$ and $b \geq 1$, we have $d(x, y) \leq b[d(x, y_1) + d(x_1, y_1) + d(x_1, y)]$. Since ψ is positive we get:

$$\begin{aligned}
\rho(x, y) &= \int_0^{d(x,y)} \psi(t)dt \\
&\leq \int_0^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]} \psi(t)dt \\
&= \int_0^{b\,d(x,y_1)} \psi(t)dt + \int_{b\,d(x,y_1)}^{b[d(x,y_1)+d(x_1,y_1)]} \psi(t)dt \\
&+ \int_{b[d(x,y_1)+d(x_1,y_1)]}^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]} \psi(t)dt.
\end{aligned}$$

If set $t = b\,d(x, y_1) + s$, since ψ is decreasing then we get:

$$\begin{aligned}
\int_{b\,d(x,y_1)}^{b[d(x,y_1)+d(x_1,y_1)]} \psi(t)dt &= \int_0^{b\,d(x_1,y_1)} \psi(b\,d(x, y_1) + s)ds \leq \int_0^{b\,d(x_1,y_1)} \psi(s)ds \\
&\leq b \int_0^{d(x_1,y_1)} \psi(s)ds.
\end{aligned}$$

Similarly, if set $t = b[d(x, y_1) + d(x_1, y_1)] + s$, then

$$\begin{aligned}
\int_{b[d(x,y_1)+d(x_1,y_1)]}^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]} \psi(t)dt &= \int_0^{b\,d(x_1,y)} \psi(b[d(x, y_1) + d(x_1, y_1)] + s)ds \\
&\leq \int_0^{b\,d(x_1,y)} \psi(s)ds \leq b \int_0^{d(x_1,y)} \psi(s)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(x, y) &= \int_0^{d(x,y)} \psi(t)dt \\
&\leq \int_0^{b\,d(x,y_1)} \psi(t)dt + \int_{b\,d(x,y_1)}^{b[d(x,y_1)+d(x_1,y_1)]} \psi(t)dt \\
&+ \int_{b[d(x,y_1)+d(x_1,y_1)]}^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]} \psi(t)dt \\
&\leq b \int_0^{d(x,y_1)} \psi(t)dt + b \int_0^{d(x_1,y_1)} \psi(t)dt + b \int_0^{d(x_1,y)} \psi(t)dt \\
&= b[\rho(x, y_1) + \rho(x_1, y_1) + \rho(x_1, y)].
\end{aligned}$$

So, condition (iii) of Definition 3.1 is also satisfied and ρ is a bipolar $b-$metric. $\square$

Remark. *If d_1, d_2 are bipolar $b-$metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we shall sometimes write $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.*

Definition 3.2. *Let (X, Y, d) be a bipolar $b-$metric space*

(a) The set X is called the left pole, Y is called the right pole and $X \cap Y$ is called the center of (X, Y, d). Especially, the points in the left pole are called left points, the points in the right pole are called right points, and the points in the center are called central points.

(b) A sequence $\{x_n\} \subseteq X$ is called a left sequence, and a sequence $\{y_n\} \subseteq Y$ is called a right sequence. In a bipolar $b-$metric space, a left or right sequence is simply called a sequence.

(c) A sequence $\{u_n\}$ is said to be convergent to a point u, if and only if $\{u_n\}$ is a left sequence, u is a right point and $\lim\limits_{n\longrightarrow\infty} d(u_n, u) = 0$, or $\{u_n\}$ is a right sequence, u is a left point and $\lim\limits_{n\longrightarrow\infty} d(u, u_n) = 0$.

(d) A bi-sequence $\{(x_n, y_n)\}$ on (X, Y, d) is a sequence on the set $X \times Y$. If the sequence $\{x_n\}$ and $\{y_n\}$ are convergent, then the bi-sequence $\{(x_n, y_n)\}$ is said to be convergent, and if $\{x_n\}$ and $\{y_n\}$ converge to a common fixed point, then $\{(x_n, y_n)\}$ is said to be bi-convergent.

(e) $\{(x_n, y_n)\}$ is called a Cauchy bi-sequence if $\lim\limits_{n,m\longrightarrow\infty} d(x_n, y_m) = 0$.

(f) A bipolar $b-$metric space is called complete, if every Cauchy bi-sequence is convergent, hence bi-convergent.

Definition 3.3. *Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar $b-$metric spaces*

(a) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x_0, y) < \delta$ implies $d_2(fx_0, fy) < \varepsilon$ for all $y \in Y_1$.

(b) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y_0) < \delta$ implies $d_2(fx, fy_0) < \varepsilon$ for all $x \in X_1$.

(c) A map f is called continuous if it is left-continuous at each point $x \in X_1$ and right-continuous at each point $y \in Y_1$.

(d) A contravariant map $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ is continuous if and only if it is continuous as a covariant map $f : (X_1, Y_1) \rightrightarrows (Y_2, X_2)$.

Remark. *([10]). A covariant or contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous if and only if $\{u_n\} \longrightarrow v$ on (X_1, Y_1, d_1) implies $\{f(u_n)\} \longrightarrow f(v)$ on (X_2, Y_2, d_2).*

In bipolar $b-$metric space we have the following Lemma.

Lemma 3.1. *Let (X, Y, d) be a bipolar $b-$metric space with $b \geq 1$, and suppose that $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ are convergent to y, x respectively, where $y \in Y$ and $x \in X$. Then we have*

$$\frac{1}{b} d(x, y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq b d(x, y).$$

In particular if $b = 1$, then we have $\lim\limits_{n\longrightarrow\infty} d(x_n, y_n) = d(x, y)$.

Proof. By property (iii) of Definition 3.1, we have

$$d(x, y) \quad \leq \quad b[d(x, y_n) + d(x_n, y_n) + d(x_n, y)].$$

Taking the lower limit as $n \to \infty$ we obtain

$$\frac{1}{b}d(x,y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n).$$

On the other hand

$$d(x_n, y_n) \quad \leq \quad b[d(x_n, y) + d(x, y) + d(x, y_n)].$$

And taking the upper limit as $n \to \infty$ we obtain

$$\limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq bd(x, y).$$

So we obtain the desired result. □

Definition 3.4. *Let (X, Y, d) be a bipolar $b-$metric space and assume that $f, g : (X, Y, d) \rightrightarrows (X, Y, d)$. Then the pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\longrightarrow\infty} d(gfx_n, fgy_n) = 0$, whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in X and Y respectively, such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gy_n = t$ for some $t \in X \cap Y$.*

Lemma 3.2. *Let (X, Y, d) be a bipolar $b-$metric space and $f, g : (X, Y, d) \rightrightarrows (X, Y, d)$ such that the pair $\{f, g\}$ be compatible and g is continuous. Suppose that $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gy_n = u$ for some $u \in X \cap Y$. Then $\lim_{n\longrightarrow\infty} fgy_n = gu$.*

Proof. Since f and g are compatible, hence $\lim_{n\longrightarrow\infty} d(gfx_n, fgy_n) = 0$. Since g is continuous it follows that

$$\lim_{n\to\infty} gfx_n = gu.$$

By property (*iii*) of Definition 3.1, we have

$$d(gu, fgy_n) \quad \leq \quad b[d(gu, gu) + d(gfx_n, gu) + d(gfx_n, fgy_n)].$$

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\longrightarrow\infty} d(gu, fgy_n) \quad \leq \quad b[\lim_{n\longrightarrow\infty} d(gu, gu) + \lim_{n\longrightarrow\infty} d(gfx_n, gu) + \lim_{n\longrightarrow\infty} d(gfx_n, fgy_n)] = 0.$$

Therefore, $\lim_{n\longrightarrow\infty} d(gu, fgy_n) = 0$, so we obtain the desired result. □

In this section, we first express and prove some different extensions and generalizations of the Banach contraction principle [4] on bipolar $b-$metric spaces.

Theorem 3.3. *Let (X, Y, d) be a complete bipolar $b-$metric space and $f, g : (X, Y, d) \rightrightarrows (X, Y, d)$ with:*

(i) *$f(X) \subseteq g(X)$, $f(Y) \subseteq g(Y)$ and g is continuous,*

(ii)

$$d(f(x), f(y)) \leq \frac{\lambda}{b^2} d(g(x), g(y)),$$

for all $(x, y) \in X \times Y$ and $0 < \lambda < b$,

(iii) *the pair (f, g) is compatible.*

Then the functions $f, g : X \cup Y \longrightarrow X \cup Y$ have a unique common fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = g(u) = u$.

Proof. Let $x_0 \in X$ and $y_0 \in Y$. For each $n \in \mathbb{N}$, define $f(x_n) = g(x_{n+1}) = a_n$ and $f(y_n) = g(y_{n+1}) = b_n$. Then $\{(a_n, b_n)\}$ is a bisequence on (X, Y, d). For each positive integer n, we have

$$\begin{aligned}
d(a_n, b_n) &= d(f(x_n), f(y_n)) \\
&\leq \frac{\lambda}{b^2} d(g(x_n), g(y_n)) = \frac{\lambda}{b^2} d(a_{n-1}, b_{n-1}) = \frac{\lambda}{b^2} d(f(x_{n-1}), f(y_{n-1})) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_{n-1}), g(y_{n-1})) = \frac{\lambda^2}{b^4} d(f(x_{n-2}), f(y_{n-2})) \\
&\vdots \\
&\leq \frac{\lambda^n}{b^{2n}} d(a_0, b_0).
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_n, b_{n+1}) &= d(f(x_n), f(y_{n+1})) \\
&\leq \frac{\lambda}{b^2} d(g(x_n), g(y_{n+1})) = \frac{\lambda}{b^2} d(a_{n-1}, b_n) = \frac{\lambda}{b^2} d(f(x_{n-1}), f(y_n)) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_{n-1}), g(y_n)) = \frac{\lambda^2}{b^4} d(f(x_{n-2}), f(y_{n-1})) \\
&\vdots \\
&\leq \frac{\lambda^n}{b^{2n}} d(a_0, b_1).
\end{aligned}$$

Hence for $m \geq n$ we get

$$\begin{aligned}
d(a_m, b_n) &\leq b[d(a_m, b_{n+1}) + d(a_n, b_{n+1}) + d(a_n, b_n)] \\
&\leq bd(a_m, b_{n+1}) + \frac{\lambda^n}{b^{2n-1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_m, b_{n+1}) &\leq b[d(a_m, b_{n+2}) + d(a_{n+1}, b_{n+2}) + d(a_{n+1}, b_{n+1})] \\
&\leq bd(a_m, b_{n+2}) + \frac{\lambda^{n+1}}{b^{2n+1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(a_m, b_{m-1}) &\leq b[d(a_m, b_m) + d(a_{m-1}, b_m) + d(a_{m-1}, b_{m-1})] \\
&\leq bd(a_m, b_m) + \frac{\lambda^{m-1}}{b^{2m-3}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Therefore, if set $d(a_0, b_1) + d(a_0, b_0) = \alpha$ then we have:

$$\begin{aligned}
d(a_m, b_n) &\leq bd(a_m, b_{n+1}) + \frac{\lambda^n}{b^{2n-1}} \alpha \\
&\leq b^2 d(a_m, b_{n+2}) + \frac{\lambda^n}{b^{2n-1}} \alpha + \frac{\lambda^{n+1}}{b^{2n+1}} \alpha \\
&\vdots \\
&\leq b^{m-n} d(a_m, b_m) + \frac{\lambda^n}{b^{2n-1}} \alpha + \frac{\lambda^{n+1}}{b^{2n+1}} \alpha + \cdots + \frac{\lambda^{m-1}}{b^{2m-3}} \alpha \\
&\leq b^{m-n} \frac{\lambda^m}{b^{2m}} \alpha + \frac{\lambda^n}{b^{2n-1}} \alpha + \frac{\lambda^{n+1}}{b^{2n+1}} \alpha + \cdots + \frac{\lambda^{m-1}}{b^{2m-3}} \alpha
\end{aligned}$$

$$\begin{aligned} &\leq \frac{\lambda^n}{b^{2n-1}}\alpha + \frac{\lambda^{n+1}}{b^{2n+1}}\alpha + \cdots + \frac{\lambda^{m-1}}{b^{2m-3}}\alpha + \frac{\lambda^m}{b^{m+n}}\alpha \\ &\leq (\frac{\lambda}{b})^n\alpha + (\frac{\lambda}{b})^{n+1}\alpha + \cdots + (\frac{\lambda}{b})^{m-1}\alpha + (\frac{\lambda}{b})^m\alpha \\ &\leq \frac{(\frac{\lambda}{b})^n\alpha}{1-\frac{\lambda}{b}} \longrightarrow 0. \end{aligned}$$

Therefore, (a_n, b_n) is a Cauchy bisequence. Since (X, Y, d) is complete, (a_n, b_n) converges, and thus biconverges to a point $u \in X \cap Y$ and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} a_n = u,$$

and

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_{n+1}) = \lim_{n\to\infty} b_n = u.$$

We show that u is a common fixed point of f and g.

Let g be continuous it follows that

$$\lim_{n\to\infty} gf(x_n) = g(u), \qquad \lim_{n\to\infty} gg(x_n) = g(u).$$

Since f and g are compatible, so by Lemma 3.2 $\lim_{n\longrightarrow\infty} fg(y_n) = g(u)$. Putting $x = gx_n$ and $y = u$ in inequality (ii) of Theorem 3.3 we obtain

$$d(fg(x_n), f(u)) \leq \frac{\lambda}{b^2}\, d(gg(x_n), g(u)). \tag{3.1}$$

Now, by taking the upper limit when $n \to \infty$ in (3.1) and using Lemma 3.1 we get

$$\begin{aligned} \frac{1}{b}d(g(u), f(u)) &\leq \limsup_{n\longrightarrow\infty} d(fg(x_n), g(u)) \\ &\leq \frac{\lambda}{b^2} \limsup_{n\longrightarrow\infty} d(gg(x_n), g(u)) \\ &= \frac{\lambda}{b^2} bd(g(u), g(u)) = 0. \end{aligned}$$

Consequently $d(g(u), f(u)) = 0$, it follows that $f(u) = g(u)$. Now, we show that $f(u) = u$. Putting $x = u$ and $y = y_n$ in inequality (ii) of Theorem 3.3 we obtain

$$d(f(u), f(y_n)) \leq \frac{\lambda}{b^2} d(g(u), g(y_n)). \tag{3.2}$$

Similarly by taking the upper limit when $n \to \infty$ in (3.2) and using Lemma 3.1 we obtain

$$\begin{aligned} \frac{1}{b}d(f(u), u) &\leq \limsup_{n\longrightarrow\infty} d(f(u), f(y_n)) \leq \frac{\lambda}{b^2} \limsup_{n\longrightarrow\infty} d(g(u), g(y_n)) \\ &\leq \frac{\lambda}{b^2} bd(g(u), u)) = \frac{\lambda}{b} d(f(u), u)) \\ &< \frac{1}{b} d(f(u), u), \end{aligned}$$

it follows that $g(u) = f(u) = u$. If there exists another common fixed point v in $X \cap Y$ of f and g, then

$$\begin{aligned} d(u, v) &= d(f(u), f(v)) \leq \frac{\lambda}{b^2} d(g(u), g(v)) = \frac{\lambda}{b^2} d(u, v) \\ &< d(u, v), \end{aligned}$$

which implies that $d(u,v)=0$ and $u=v$. Thus u is a unique common fixed point of f and g. The proof of the theorem is completed. □

Now we give an example to support our result.

Example 3.3. *Let $X=\{(a,0)\mid a\in\mathbb{R}\}$ and $Y=\{(d,c)\mid d\in\mathbb{R},\ c\in[0,\infty)\}$ be endowed with bipolar $b-$metric $d(x,y)=(a-d)^2+c$, where $x=(a,0)\in X$ and $y=(d,c)\in Y$. By Example 3.1, (X,Y,d) is a bipolar $b-$metric for $b=3$. For every $(x,y)\in X\cup Y$, define $f,g: X\cup Y\longrightarrow X\cup Y$ by $f(x,y)=\frac{1}{3}(\sin(x),\ln(1+\frac{y}{3}))$ and $g(x,y)=(x,y^2+y)$. $f,g:(X,Y)\rightrightarrows(X,Y)$, that is f and g are two covariant maps from (X,Y) to (X,Y). It is easy to see that the pairs $\{f,g\}$ are compatible mappings.*

Also for each $x\in X$ and $y\in Y$ we have

$$\begin{aligned}
d(fx,fy) &= d(f(a,0),f(d,c))\\
&= \frac{1}{9}(\sin(a)-\sin(d))^2+\frac{1}{3}\ln(1+\frac{c}{3})\\
&\le \frac{1}{9}(\sin(a)-\sin(d))^2+\frac{1}{9}c\\
&\le \frac{1}{9}(a-d)^2+\frac{1}{9}(c^2+c)\\
&= \frac{1}{9}d(g(a,0),g(d,c))=\frac{1}{9}d(gx,gy)\\
&\le \frac{\lambda}{b^2}d(gx,gy),
\end{aligned}$$

where $1\le\lambda<3$ and $b=3$. Thus f and g satisfy the conditions given in Theorem 3.3 and $(0,0)\in X\cap Y$ is the unique common fixed point of f and g.

Now we get the special cases of Theorem 3.3 as follows:

Corollary 3.4. *Let (X,Y,d) be a complete bipolar $b-$metric space and $f:(X,Y,d)\rightrightarrows(X,Y,d)$ be a mapping such that*

$$d(fx,fy)\le\frac{\lambda}{b^2}d(x,y),\quad \text{for all } (x,y)\in X\times Y\quad \text{with } 0<\lambda<b.$$

Then f has a unique fixed point in $X\cap Y$.

Proof. If we take g as identity map on X and on Y, then from Theorem 3.3 follows that f has a unique fixed point. □

Corollary 3.5. *Let (X,Y,d) be a complete bipolar metric space and $f,g:(X,Y,d)\rightrightarrows(X,Y,d)$ with:*

(i) *$f(X)\subseteq g(X)$, $f(Y)\subseteq g(Y)$ and g is continuous,*

(ii)

$$d(f(x),f(y))\le\lambda d(g(x),g(y)),$$

for all $(x,y)\in X\times Y$ and $0<\lambda<1$,

(iii) *the pair (f,g) is compatible.*

Then the functions $f,g:X\cup Y\longrightarrow X\cup Y$ have a unique the common fixed point in $X\cap Y$. There exists a unique point $u\in X\cap Y$ such that $f(u)=g(u)=u$.

Proof. It is enough to set $b=1$ in Theorem 3.3. □

The following corollary gives the Theorem of Mutlu, Gürdal [10].

Corollary 3.6. *Let (X, Y, d) be a complete bipolar $b-$metric space and let $f : (X, Y, d) \rightrightarrows (X, Y, d)$ with:*

$$d(f(x), f(y)) \leq \lambda d(x, y), \ \ for\ all\ (x, y) \in X \times Y \ and\ 0 < \lambda < 1.$$

Then the function $f : X \cup Y \longrightarrow X \cup Y$ has a unique fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = u$.

Proof. If we take g as identity map on X and on Y, then from Corollary 3.5 follows that f has a unique fixed point. □

4. Fixed point theorem for contravariant maps

Below we prove a similar result for contravariant maps.

Definition 4.1. *Let (X, Y, d) be a bipolar $b-$metric space and $f : (X, Y, d) \rightleftarrows (X, Y, d)$ and $g : (X, Y, d) \rightrightarrows (X, Y, d)$. Then the pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\longrightarrow\infty} d(gfy_n, fgx_n) = 0$, whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in X and Y respectively, such that $\lim_{n\longrightarrow\infty} fy_n = \lim_{n\longrightarrow\infty} gx_n = t$ for some $t \in X \cap Y$.*

Lemma 4.1. *Let (X, Y, d) be a bipolar $b-$metric space and let $f : (X, Y, d) \rightleftarrows (X, Y, d)$ and $g : (X, Y, d) \rightrightarrows (X, Y, d)$ such that the pair $\{f, g\}$ be compatible and g is continuous. Suppose that $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that $\lim_{n\longrightarrow\infty} fy_n = \lim_{n\longrightarrow\infty} gx_n = u$ for some $u \in X \cap Y$. Then $\lim_{n\longrightarrow\infty} fgx_n = gu$.*

Proof. Since f and g are compatible, hence $\lim_{n\longrightarrow\infty} d(gfy_n, fgx_n) = 0$. Since g is continuous it follows that

$$\lim_{n\to\infty} gfy_n = gu.$$

By property (iii) of Definition 3.1, we have

$$d(gu, fgx_n) \leq b[d(gu, gu) + d(gfy_n, gu) + d(gfy_n, fgx_n)].$$

Taking the limit as $n \to \infty$ we obtain

$$\begin{aligned} \lim_{n\longrightarrow\infty} d(gu, fgx_n) \leq \ & b[\lim_{n\longrightarrow\infty} d(gu, gu) + \lim_{n\longrightarrow\infty} d(gfy_n, gu) \\ & + \lim_{n\longrightarrow\infty} d(gfy_n, fgx_n)] = 0. \end{aligned}$$

Therefore, $\lim_{n\longrightarrow\infty} d(gu, fgx_n) = 0$, so we obtain the desired result. □

Theorem 4.2. *Let (X, Y, d) be a complete bipolar $b-$metric space and $f : (X, Y, d) \rightleftarrows (X, Y, d)$ and $g : (X, Y, d) \rightrightarrows (X, Y, d)$ with:*

(i) *$f(X) \subseteq g(Y)$, $f(Y) \subseteq g(X)$ and g is continuous,*

(ii)

$$d(f(y), f(x)) \leq \frac{\lambda}{b^2} d(g(x), g(y)),$$

for all $(x, y) \in X \times Y$ and $0 < \lambda < b^2$,

(iii) *the pair (f, g) is compatible.*

Then the functions $f, g : X \cup Y \longrightarrow X \cup Y$ have a unique common fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = g(u) = u$.

Proof. Let $x_0 \in X$ and $y_0 \in Y$. For each $n \in \mathbb{N}$, define $f(x_n) = g(y_n) = b_n$ and $f(y_n) = g(x_{n+1}) = a_n$. Then $\{(a_n, b_n)\}$ is a bisequence on (X, Y, d). For each positive integer n, we have

$$\begin{aligned}
d(a_n, b_n) &= d(f(y_n), f(x_n)) \\
&\leq \frac{\lambda}{b^2} d(g(x_n), g(y_n)) = \frac{\lambda}{b^2} d(a_{n-1}, b_n) = \frac{\lambda}{b^2} d(f(y_{n-1}), f(x_n)) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_n), g(y_{n-1})) = \frac{\lambda^2}{b^4} d(a_{n-1}, b_{n-1}) \\
&\vdots \\
&\leq \frac{\lambda^{2n}}{b^{4n}} d(a_0, b_0).
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_n, b_{n+1}) &= d(f(y_n), f(x_{n+1})) \\
&\leq \frac{\lambda}{b^2} d(g(x_{n+1}), g(y_n)) = \frac{\lambda}{b^2} d(a_n, b_n) = \frac{\lambda}{b^2} d(f(y_n), f(x_n)) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_n), g(y_n)) = \frac{\lambda^2}{b^4} d(a_{n-1}, b_n) \\
&\vdots \\
&\leq \frac{\lambda^{2n}}{b^{4n}} d(a_0, b_1).
\end{aligned}$$

Hence for $m \geq n$ we get

$$\begin{aligned}
d(a_m, b_n) &\leq b[d(a_m, b_{n+1}) + d(a_n, b_{n+1}) + d(a_n, b_n)] \\
&\leq bd(a_m, b_{n+1}) + \frac{\lambda^{2n}}{b^{4n-1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_m, b_{n+1}) &\leq b[d(a_m, b_{n+2}) + d(a_{n+1}, b_{n+2}) + d(a_{n+1}, b_{n+1})] \\
&\leq bd(a_m, b_{n+2}) + \frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(a_m, b_{m-1}) &\leq b[d(a_m, b_m) + d(a_{m-1}, b_m) + d(a_{m-1}, b_{m-1})] \\
&\leq bd(a_m, b_{m-2}) + \frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Therefore, if set $d(a_0, b_1) + d(a_0, b_0) = \alpha$ then we have:

$$\begin{aligned}
d(a_m, b_n) &\leq bd(a_m, b_{n+1}) + \frac{\lambda^{2n}}{b^{4n-1}} \alpha \\
&\leq b^2 d(a_m, b_{n+2}) + \frac{\lambda^{2n}}{b^{4n-1}} \alpha + \frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}} \alpha \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
&\leq b^{m-n}d(a_m,b_m)+\frac{\lambda^{2n}}{b^{4n-1}}\alpha+\frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha+\cdots+\frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}\alpha\\
&\leq b^{m-n}\frac{\lambda^{2m}}{b^{4m}}\alpha+\frac{\lambda^{2n}}{b^{4n-1}}\alpha+\frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha+\cdots+\frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}\alpha\\
&\leq \frac{\lambda^{2n}}{b^{4n-1}}\alpha+\frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha+\cdots+\frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}\alpha+\frac{\lambda^{2m}}{b^{3m+n}}\alpha\\
&\leq \frac{1}{b}[(\frac{\lambda^2}{b^4})^n\alpha+(\frac{\lambda^2}{b^4})^{n+1}\alpha+\cdots+(\frac{\lambda^2}{b^4})^{m-1}\alpha+(\frac{\lambda^2}{b^4})^{m}\alpha]\\
&\leq \frac{1}{b}[\frac{(\frac{\lambda^2}{b^4})^n\alpha}{1-\frac{\lambda^2}{b^4}}]\longrightarrow 0.
\end{aligned}
$$

Therefore, (a_n,b_n) is a Cauchy bisequence. Since (X,Y,d) is complete, (a_n,b_n) converges, and thus biconverges to a point $u\in X\cap Y$ and

$$\lim_{n\to\infty} f(y_n)=\lim_{n\to\infty} g(x_{n+1})=\lim_{n\to\infty} a_n=u,$$

and

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(y_n)=\lim_{n\to\infty} b_n=u.$$

We show that u is a common fixed point of f and g.

Since g is continuous it follows that

$$\lim_{n\to\infty} gf(y_n)=g(u),\qquad \lim_{n\to\infty} gg(x_{n+1})=g(u).$$

Since f and g are compatible, so by Lemma 4.1 $\lim_{n\longrightarrow\infty} fg(x_n)=g(u)$. Putting $x=gx_n$ and $y=u$ in inequality (ii) of Theorem 4.2 we obtain

$$d(f(u),fg(x_n)) \leq \frac{\lambda}{b^2}\, d(gg(x_n),g(u)). \tag{4.1}$$

Now, by taking the upper limit when $n\to\infty$ in (4.1) and using Lemma 3.1 we get

$$
\begin{aligned}
\frac{1}{b}d(f(u),g(u)) &\leq \limsup_{n\longrightarrow\infty} d(f(u),fg(x_n))\\
&\leq \frac{\lambda}{b^2}\limsup_{n\longrightarrow\infty} d(gg(x_n),g(u))\\
&= \frac{\lambda}{b^2}bd(g(u),g(u))=0.
\end{aligned}
$$

Consequently $d(f(u),g(u))=0$, it follows that $f(u)=g(u)$. Now, we show that $f(u)=u$. Putting $x=u$ and $y=y_n$ in inequality (ii) of Theorem 4.2 we obtain

$$d(f(y_n),f(u)) \leq \frac{\lambda}{b^2}d(g(u),g(y_n)). \tag{4.2}$$

Similarly by taking the upper limit when $n\to\infty$ in (4.2) and using Lemma 3.1 we obtain

$$
\begin{aligned}
\frac{1}{b}d(u,f(u)) &\leq \limsup_{n\longrightarrow\infty} d(f(y_n),f(u))\leq\frac{\lambda}{b^2}d(g(u),g(y_n))\\
&\leq \frac{\lambda}{b^2}bd(g(u),u)=\frac{\lambda}{b}d(f(u),u)\\
&< \frac{1}{b}d(u,f(u)),
\end{aligned}
$$

it follows that $g(u) = f(u) = u$. If there exists another common fixed point v in $X \cap Y$ of f and g, then

$$\begin{aligned} d(u, v) &= d(f(u), f(v)) \leq \frac{\lambda}{b^2} d(g(v), g(u)) = \frac{\lambda}{b^2} d(u, v) \\ &< d(u, v), \end{aligned}$$

which implies that $d(u, v) = 0$ and $u = v$. Thus u is a unique common fixed point of f and g. The proof of the theorem is completed. □

Example 4.1. *Let*

$$X = \left\{(x_n) \in \mathbb{R}^\infty |\ x_n \leq 0 \text{ for each } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \sqrt{-x_n} < \infty\right\}$$

and

$$Y = \left\{(y_n) \in \mathbb{R}^\infty |\ y_n \geq \text{ for each } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \sqrt{y_n} < \infty\right\}.$$

Defined $d : X \times Y \to \mathbb{R}$, *by* $d(x, y) = \left(\sum_{n=1}^{\infty} \sqrt{y_n - x_n}\right)^2$, *where* $x = (x_n) \in X$ *and* $y = (y_n) \in Y$.

Then (X, Y, d) *is a bipolar* $b-$*metric space with the constant* $b = 4$.

$$f : (X, Y, d) \rightleftarrows (X, Y, d),\ f(u_n) = (-u_n)$$
$$g : (X, Y, d) \rightrightarrows (X, Y, d),\ g(u_n) = (u_n^3) + 2u_n$$

These are compatible as $f \circ g = g \circ f$. *Note that*

$$d(f(y_n), f(x_n)) \leq \frac{8}{4^2} d(g(x_n), g(y_n)),$$

and also g *is continuous and* $f(X) \subseteq g(Y)$, $f(Y) \subseteq g(X)$. *Then by Theorem 4.2.* f *and* g *must have a unique common fixed point. Indeed, the only point* $(u_n) = (0, 0, 0, \ldots) \in X \cap Y$ *is a common fixed point of* f *and* g.

Now we get the special cases of Theorem 4.2 as follows:

Corollary 4.3. *Let* (X, Y, d) *be a complete bipolar* $b-$*metric space and* $f : (X, Y, d) \rightleftarrows (X, Y, d)$ *be a mapping such that*

$$d(fy, fx) \leq \frac{\lambda}{b^2} d(x, y), \quad \text{for all} \quad (x, y) \in X \times Y, \quad \text{with} \quad 0 < \lambda < b^2.$$

Then f *has a unique fixed point in* $X \cap Y$.

Proof. If we take g as identity map on X and on Y, then from Theorem 4.2 follows that f has a unique fixed point. □

Corollary 4.4. *Let* (X, Y, d) *be a complete bipolar metric space and* $f : (X, Y, d) \rightleftarrows (X, Y, d)$ *and* $g : (X, Y, d) \rightrightarrows (X, Y, d)$ *with:*

(i) $f(X) \subseteq g(Y)$, $f(Y) \subseteq g(X)$ *and* g *is continuous,*

(ii)

$$d(f(y), f(x)) \leq \lambda d(g(x), g(y)),$$

for all $(x, y) \in X \times Y$ *and* $0 < \lambda < 1$,

(iii) *the pair* (f, g) *is compatible.*

Then the functions $f, g : X \cup Y \longrightarrow X \cup Y$ *have a unique common fixed point in* $X \cap Y$. *There exists a unique point* $u \in X \cap Y$ *such that* $f(u) = g(u) = u$.

Proof. It is enough to set $b = 1$ in Theorem 4.2. □

The following corollary gives the Theorem of Mutlu, Gürdal [10].

Corollary 4.5. *Let (X, Y, d) be a complete bipolar metric space and let $f : (X, Y, d) \rightleftarrows (X, Y, d)$ with:*

$$d(f(y), f(x)) \leq \lambda d(x, y), \ for\ all\ (x, y) \in X \times Y\ and\ 0 < \lambda < 1.$$

Then the function $f : X \cup Y \longrightarrow X \cup Y$ has a unique fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = u$.

Proof. If we take g as identity map on X and on Y, then from Corollary 4.4 follows that f has a unique fixed point. □

Remark. *In Fixed point theory for readers' interest well known of Meir-Keeler type contraction, Čirič type of quasi contraction, Nadler type of contraction Sehgal-Guseman type of contraction, the completion of bipolar $b-$metric space, Suzuki-Berinde type of contraction, etc., that is these structures can be proven in bipolar $b-$metric space exploration.*

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