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Research Article

Exploring a new two-parameter Archimedean copula: the Gumbel-Joe copula

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Abstract

In this article, we present a novel Archimedean copula constructed from a unique strict generator function. It can be described as a two-parameter unification of the well-established Gumbel-Barnett and Joe copulas. The first part is devoted to its formulation, as well as those of the corresponding density, the conditional copula, and the Kendall distribution function. Graphs are also included to illustrate their shape behavior under different parameter configurations. In a second part, we examine some of its notable properties, with emphasis on the correlation properties. Practical applications are discussed in the final part. In particular, we use the maximum likelihood estimation method to determine the unknown parameters involved from the data and perform a simulation study to demonstrate the effectiveness of this approach. We also analyze a dataset to provide practical illustrations of copula behavior and potential.

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1. Introduction

Copulas are fundamental tools in multivariate data analysis and modelling, particularly in finance, risk management, hydrology, environmental science, insurance, reliability engineering, telecommunications, and medicine. Their primary function is to identify the dependence structure between random variables, providing a versatile approach to modelling complex interdependencies that go far beyond the scope of linear correlation. In particular, copulas are instrumental in capturing non-linear relationships and various forms of association between random variables, allowing a more accurate representation of real-world scenarios. For more information on this topic, two important copula references among many valuable resources are [22] written by Roger B. Nelsen and [18] written by H. Joe. These two books provide comprehensive information, theoretical foundations and practical applications in the field of copula theory, allowing a deeper understanding and implementation in various analytical contexts.

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On the other hand, the construction of copulas has been the subject of much research in the field of statistics. In particular, Plackett[23] made significant contributions by generating copulas via joint distributions that are combined through a sophisticated ratio form. Marshall and Olkin[20] introduced the Marshall-Olkin copula, which is derived from a bivariate exponential distribution. Genest and MacKay[11] proposed the Archimedean copulas, a category of copulas centered around the notion of generator function. A generator function can be described as a function that satisfies certain properties: it is non-negative, decreasing, and convex. Extending this, Marshall and Olkin[21] used inverse Laplace transformations to derive copulas within the Archimedean class. In addition, Alhadlaq and Alzaid[1] furthered the understanding by employing cumulative distribution functions and probability generating functions that obey the specific conditions of the generator functions, allowing the creation of Archimedean copulas. These different approaches highlight the evolution and diversity of copula construction, proposing different mathematical foundations and techniques to provide these essential statistical tools.

One of the most widely used Archimedean copulas in the field of statistics is the Gumbel-Barnett copula, originally introduced and described in [13] and [15]. This copula is characterized by a specific strict generating function, which is defined as follows:

$$\varphi(u) = \log\left[1 - a\log(u)\right],$$

for $u \in [0, 1]$. The parameter a is contained in the unit interval $[0, 1]$. It controls various dependence characteristics. Specifically, as a approaches 0, the Gumbel-Barnett copula tends towards independence of the underlying random variable, while higher values of a indicate moderate negative correlation dependence between them. In fact, Kendall's tau is contained in $[-0.361, 0]$. Furthermore, the corresponding upper and lower tail dependence parameters are equal to 0. For more information, we refer to [22] and [9]. Thus, despite its usefulness, a major drawback of the Gumbel-Barnett copula is its inability to capture highly negative or positive correlations and tail dependence. These limitations have motivated the development of alternative copulas that provide more accurate tools for a wide range of applications. These alternative copulas have enriched the field by providing greater flexibility and a more complete representation of dependence structures than the Gumbel-Barnett copula.

The Joe copula, another famous Archimedean copula, was first introduced and discussed by [16] and [17]. Its associated strict generator is expressed as follows:

$$\varphi(u) = -\log\left[1 - (1-u)^b\right],$$

for $u \in [0, 1]$. The parameter b is contained in the interval $[1, \infty)$. Unlike the Gumbel-Barnett copula, the corresponding upper and lower tail dependence parameters are not equal to 0; they are computed as 2 and $2 - 2^{\frac{1}{b}}$, respectively. Thus, the Joe copula is tail dependent, with an upper tail modulated by b. In addition, Kendall's tau is contained in $[0, 1]$. Furthermore, the Joe copula exhibits stochastic monotonicity and has a density that is totally positive of order two. Thus, a major limitation of the Joe copula is its inability to capture negative correlation. In this sense, it can be seen as antagonistic to the Gumbel-Barnett copula.

With the idea of increasing the flexibility of dependence modelling, extensions to two-parameter copulas have been explored in the literature. In particular, Nelsen[22] presented several methods for constructing two-parameter copulas. One method is to create a composite of two (Archimedean) generator functions, while another approach is to introduce a shape parameter into the copulas themselves. Joe[18] developed various methods of adding parameters to existing copulas, including the use of mixtures of copulas. These extensions have continued to evolve. For example, Amblard and Girard[3] introduced a second parameter into the FGM copula, which greatly extended its dependence range. Chesneau[5] proposed a weighted version of the Gumbel-Barnett copula, adding an extra level of flexibility. In addition, in more recent work, Chesneau[6] and Chesneau[7] generalized two well-known one-parameter copulas by incorporating a second parameter, thus enriching the toolbox of available copula models for various applications. Another advantage of considering two-parameter copulas is that we can reduce them to one-parameter copulas by choosing one parameter as a function of the other in an original way. Thus, we can innovate by creating one-parameter copulas based on two-parameter copulas. This point of view is taken into account in this study.

The motivation for this research can be summarized in two fundamental points. Primarily, it should be noted that strict two-parameter Archimedean copulas are relatively rare, given that the majority of existing copulas have only one parameter. Moreover, well-known copula extensions, such as the Gumbel-Barnett and Joe copulas are exceptionally rare in the current literature. Therefore, in this article, we propose a new two-parameter Archimedean copula. This copula can be seen as an extension of the Gumbel-Barnett or Joe copulas, thus addressing the rarity of such two-parameter constructions in existing copula theory.

The coming sections are structured as follows: Section 2 introduces the extended Gumbel-Barnett (or Joe) copula, examining its basic functions and properties. Section 3 examines its dependence structure from different perspectives. This includes the evaluation of Spearman's rho, Kendall's tau, and Blomqvist's correlation coefficient. In addition, an investigation of the upper and lower tail dependence parameters is presented. A small-scale simulation study is carried out in Section 4. A practical application is discussed in Section 5. Finally, Section 6 contains our conclusions drawn from the results and discussions presented throughout the article.

2. The two-parameter Gumbel(-Barnett)-Joe copula

2.1. Known facts about Archimedean copula

The classical definition of a strict generator function is recalled below (see [22]).

Definition 2.1. A function $\varphi(u)$ for $u \in [0,1]$ is said to be a strict generator function if and only if it satisfies the following conditions:

C1: $\varphi(1) = 0$,
C2: $\lim_{u\to 0} \varphi(u) = +\infty$,
C3: for any $u \in [0,1)$, $\varphi'(u) < 0$ and $\varphi'(1) \le 0$,
C4: for any $u \in [0,1]$, $\varphi''(u) \ge 0$.

Based on a specific strict generator function, a copula is associated, as described in the next definition.

Definition 2.2. The strict Archimedean copula associated to a strict generator function $\varphi(u)$ is defined by

$$C(x,y) = \varphi^{-1}\left[\varphi(x) + \varphi(y)\right],$$

for $(x,y) \in [0,1]^2$.

This copula definition will be the core of the newly proposed copula. For more information on the above definitions, all the necessary details on Archimedean copulas can be found in [22] and [18].

2.2. New strict generator function

The next result presents our generalized version of the strict generator function of the Gumbel-Barnett copula, also known as Nelsen's strict Archimedean copula number 9. We will discuss its connection with that of the Joe copula later.

Proposition 2.3. *The following function defines a new strict generator function:*

$$\varphi(u) = \log\left\{1 - a\log\left[1-(1-u)^b\right]\right\}, \tag{2.1}$$

for $u \in [0,1]$, $b \geq 1$ *and* $a \in (0, 1/b]$.

Proof. Let us prove each item in Definition 2.1 in turn.

For C1: We clearly have

$$\varphi(1) = \log\left\{1 - a\log\left[1-(1-1)^b\right]\right\} = \log(1) = 0.$$

For C2: Since $a > 0$ and $b > 0$, we have $1-(1-u)^b \sim_{u\sim 0} bu$ and $\lim_{u\to 0}[-a\log(u)] = +\infty$. It follows from these limit results that

$$\lim_{u\to 0}\varphi(u) = \lim_{u\to 0}\log\left\{1 - a\log\left[1-(1-u)^b\right]\right\}$$
$$= \log\left\{1 - a\log(b) + \lim_{u\to 0}[-a\log(u)]\right\} = \lim_{v\to+\infty}\log(v) = +\infty.$$

For C3: For any $u \in [0,1)$, we have

$$\varphi'(u) = -\frac{ab(1-u)^{b-1}}{[1-(1-u)^b]\{1-a\log[1-(1-u)^b]\}}.$$

Since $b \geq 1$, $a > 0$ and $-\log[1-(1-u)^b] \geq 0$, we have $\varphi'(u) < 0$. It is clear that $\varphi'(1) = 0 \leq 0$.

For C4: For any $u \in [0,1]$, we have

$$\varphi''(u) = ab(1-u)^{b-2}\frac{(1-ab)(1-u)^b - a[(1-u)^b + b - 1]\log[1-(1-u)^b] + b - 1}{[1-(1-u)^b]^2\{1-a\log[1-(1-u)^b]\}^2}.$$

The denominator term is clearly non-negative. For the numerator term, since $b \geq 1$ and $a \in (0, 1/b]$, we have $ab \geq 0$, $1 - ab \geq 0$, $(1-u)^{b-2} \geq 0$, $(1-u)^b \geq 0$, $-\log[1-(1-u)^b] \geq 0$ and $b - 1 \geq 0$; all the sub-terms are non-negative. We thus have $\varphi''(u) \geq 0$.

This completes the proof; the considered function $\varphi(u)$ is validated as a strict generator function. □

To our knowledge, this is the first result that validates the function of Equation (2.1) as a strict generating function. We can also see that the conditions on a and b are interdependent. If this is a problem in an applied scenario, one solution is to merge the parameters such as $a = 1/b$ with $b \geq 1$.

2.3. Corresponding copula

By Definition 2.2, the copula corresponding to the strict generator function described in Equation (2.1) has a closed-form expression. It will be determined in the next result.

Proposition 2.4. *The following function defines a valid strict Archimedean copula:*

$$C(x,y) =$$
$$1 - \left\{1 - \left[1-(1-x)^b\right]\left[1-(1-y)^b\right]\exp\left\{-a\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\}\right\}^{1/b}, \tag{2.2}$$

for $(x,y) \in [0,1]^2$, $b \geq 1$ *and* $a \in (0, 1/b]$.

Proof. Let us consider the strict generator function described in Equation (2.1). Then its inverse function is obtained by solving the following equation: $\varphi^{-1}[\varphi(u)] = u$ for any $u \in [0,1]$, which yields

$$\varphi^{-1}(v) = 1 - \left\{1 - \exp\left[\frac{1}{a}(1 - e^{v})\right]\right\}^{1/b},$$

for $v \geq 0$. Therefore, based on Definition 2.2, the corresponding Archimedean copula is obtained as

$$\begin{aligned} C(x,y) &= \varphi^{-1}\left[\varphi(x) + \varphi(y)\right] \\ &= 1 - \left\{1 - \exp\left[\frac{1}{a}\left(1 - e^{\log\left\{1 - a\log\left[1-(1-x)^b\right]\right\} + \log\left\{1 - a\log\left[1-(1-y)^b\right]\right\}}\right)\right]\right\}^{1/b} \\ &= 1 - \left\{1 - \exp\left[\frac{1}{a}\left(1 - \left\{1 - a\log\left[1 - (1-x)^b\right]\right\}\left\{1 - a\log\left[1 - (1-y)^b\right]\right\}\right)\right]\right\}^{1/b} \\ &= 1 - \left\{1 - \left[1 - (1-x)^b\right]\left[1 - (1-y)^b\right]\exp\left\{-a\log\left[1 - (1-x)^b\right]\log\left[1 - (1-y)^b\right]\right\}\right\}^{1/b}. \end{aligned}$$

This ends the proof. □

We can note that the main term involving the exponential function in Equation (2.2) can be also rewritten as a variable-power function under the following forms:

$$\begin{aligned} &\left[1 - (1-x)^b\right]\left[1 - (1-y)^b\right]\exp\left\{-a\log\left[1 - (1-x)^b\right]\log\left[1 - (1-y)^b\right]\right\} \\ &= \left[1 - (1-x)^b\right]\left[1 - (1-y)^b\right]^{1 - a\log\left[1-(1-x)^b\right]} \end{aligned}$$

or

$$\begin{aligned} &\left[1 - (1-x)^b\right]\left[1 - (1-y)^b\right]\exp\left\{-a\log\left[1 - (1-x)^b\right]\log\left[1 - (1-y)^b\right]\right\} \\ &= \left[1 - (1-y)^b\right]\left[1 - (1-x)^b\right]^{1 - a\log\left[1-(1-y)^b\right]}. \end{aligned}$$

So we see that a can be viewed as a parameter that modulates (or activates) a functional power of $1 - (1-x)^b$ or $1 - (1-y)^b$, depending on the expression being considered.

On the basis of the copula described in Equation (2.2), the following connections with existing copulas are valid:

- By taking $b = 1$, we get the Gumbel-Barnett copula with the exact admissible range of values: $a \in [0,1]$, that is

$$C(x,y) = xy\exp\left[-a\log(x)\log(y)\right].$$

  It is worth noting that the independence copula is a special case of the Gumbel-Barnett copula, which is obtained by taking $a = 0$.
- By taking $a = 0$ and $b \geq 1$, we obtain the Joe copula indicated as

$$C(x,y) = 1 - \left[(1-x)^b + (1-y)^b - (1-x)^b(1-y)^b\right]^{1/b}.$$

  Two special cases of the Joe copula are the independence copula obtained by taking $b = 1$, and the Fréchet-Hoeffding upper bound (or comonotonicity) copula obtained by applying $b \to \infty$.

In this sense, the copula in Equation (2.2) is a generalization of the copula above, with an additional parameter that provides more flexibility. To the best of our knowledge, it is new in the literature and not experienced from an applied point of view.

Based on these remarks, it is logical to call the proposed copulas the Gumbel-(Barnett)-Joe (GJ) copula. Figure 1 exhibits some contour plots of the GJ copula for different values of the parameters with respect to the conditions , $b \geq 1$ and $a \in (0, 1/b]$. From this figure, we can see that the GJ copula tends to increase with smaller values of the parameter a

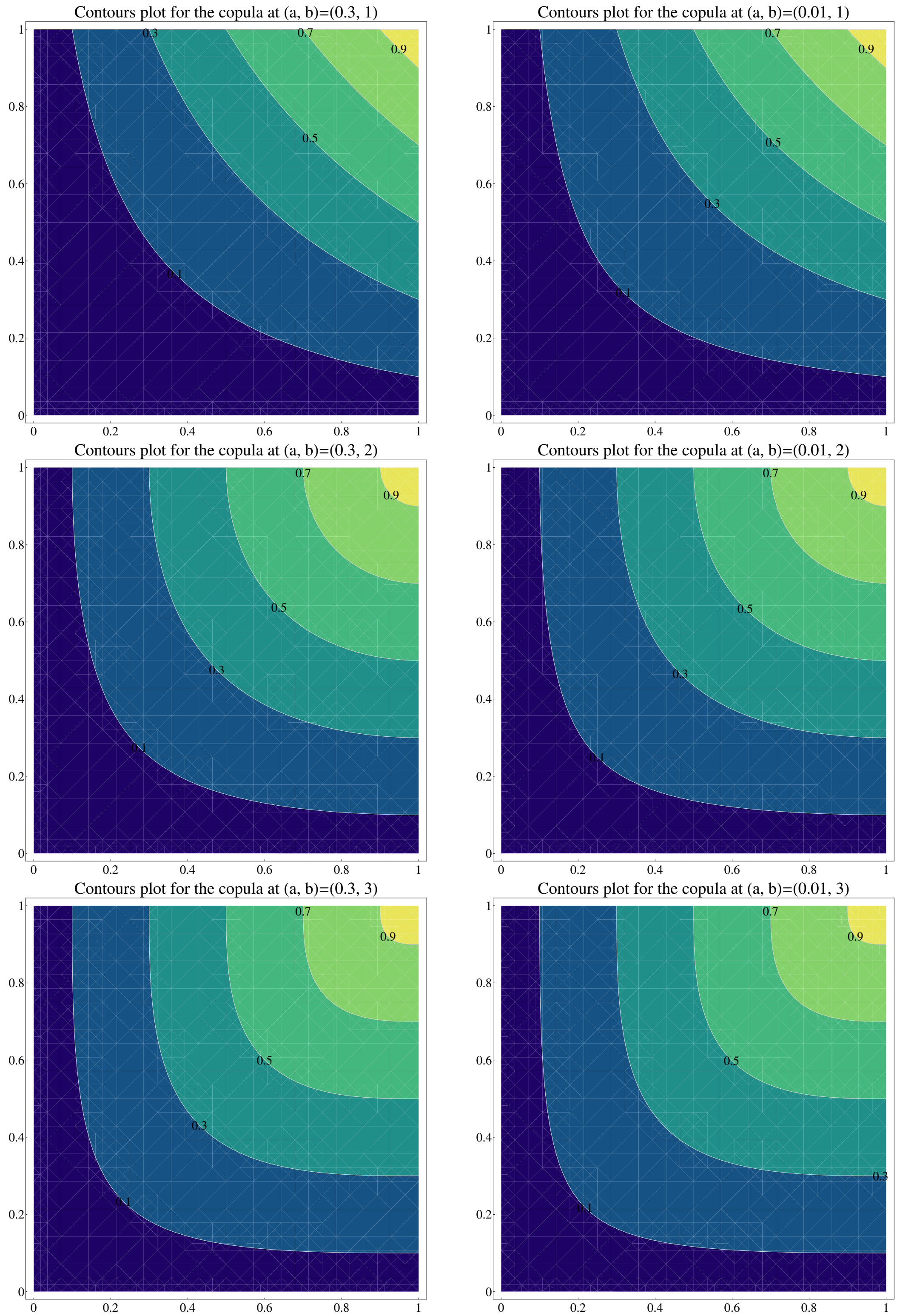


Figure 1. Contour plots for the GJ copula at different values of the parameters.

while b remains fixed, or with larger values of b while a remains fixed. In other words, the GJ copula changes rapidly for smaller values of the parameters a and b.

The conditional version of the GJ copula is obtained by differentiation as follows:

$$\begin{aligned}C(x|y) &= \frac{\partial}{\partial y}C(x,y)\\ &= (1-y)^{b-1}\left[(1-x)^b-1\right]\left\{a\log\left[1-(1-x)^b\right]-1\right\}\\ &\left\{1-\left[1-(1-x)^b\right]\left[1-(1-y)^b\right]\right.\\ &\left.\exp\left\{-a\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\}\right\}^{1/b}\\ &\left\{(1-x)^b+(1-y)^b-(1-x)^b(1-y)^b\right.\\ &\left.+\exp\left\{a\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\}-1\right\}^{-1},\end{aligned}$$

for $(x,y)\in[0,1]^2$, $b\geq 1$ and $a\in(0,1/b]$.

The corresponding Kendall distribution function is given as

$$\begin{aligned}K(t) &= t-\frac{\varphi(t)}{\varphi'(t)}\\ &= t+\frac{\left[1-(1-t)^b\right]\left\{1-a\log[1-(1-t)^b]\right\}\log\left\{1-a\log\left[1-(1-t)^b\right]\right\}}{ab(1-t)^{b-1}},\end{aligned}$$

for $t\in[0,1]$, $b\geq 1$ and $a\in(0,1/b]$.

The corresponding (copula) density has a rather complicated form, so we omit it. In order to understand its shape behavior, we present its plot in Figure 2. From this figure, it is clear that the contour curves become steeper for larger values of b, i.e., the density increases faster for larger values of b. For $b>1$, the density reaches its maximum at $(1,1)$, and its minimum at $(0,1)$ and $(1,0)$. For $b=1$, the peaks are at $(0,1)$ and $(1,0)$, and the minimum is at $(0,0)$.

3. Dependence

This section is devoted to the overall dependence properties of the GJ copula as described in Equation (2.2), always assuming that $b\geq 1$ and $a\in(0,1/b]$ as established in Proposition 2.4.

3.1. Quadrant dependence

The subsequent proposition is about the quadrant dependence of the GJ copula.

Proposition 3.1. *The GJ copula is not quadrant dependent (negative or positive) in a uniform way, i.e., for all the possible values of the parameters.*

Proof. We recall that $C(x,y)$ is said to be positive quadrant dependent if, for any $(x,y)\in[0,1]^2$, $C(x,y)\geq xy$. On the other hand, $C(x,y)$ is said to be negative quadrant dependent if, for any $(x,y)\in[0,1]^2$, $C(x,y)\leq xy$. See [22] and [18] for more information. With this in mind, let us consider the following parameter values: $a=0.5$ and $b=2$. Then numerical calculations with respect to x and y give

$$C\left(\frac{1}{2},\frac{1}{2}\right)=0.3>0.25=\frac{1}{2}\times\frac{1}{2}$$

and

$$C\left(\frac{1}{10},\frac{1}{10}\right)=0.005<0.01=\frac{1}{10}\times\frac{1}{10}.$$

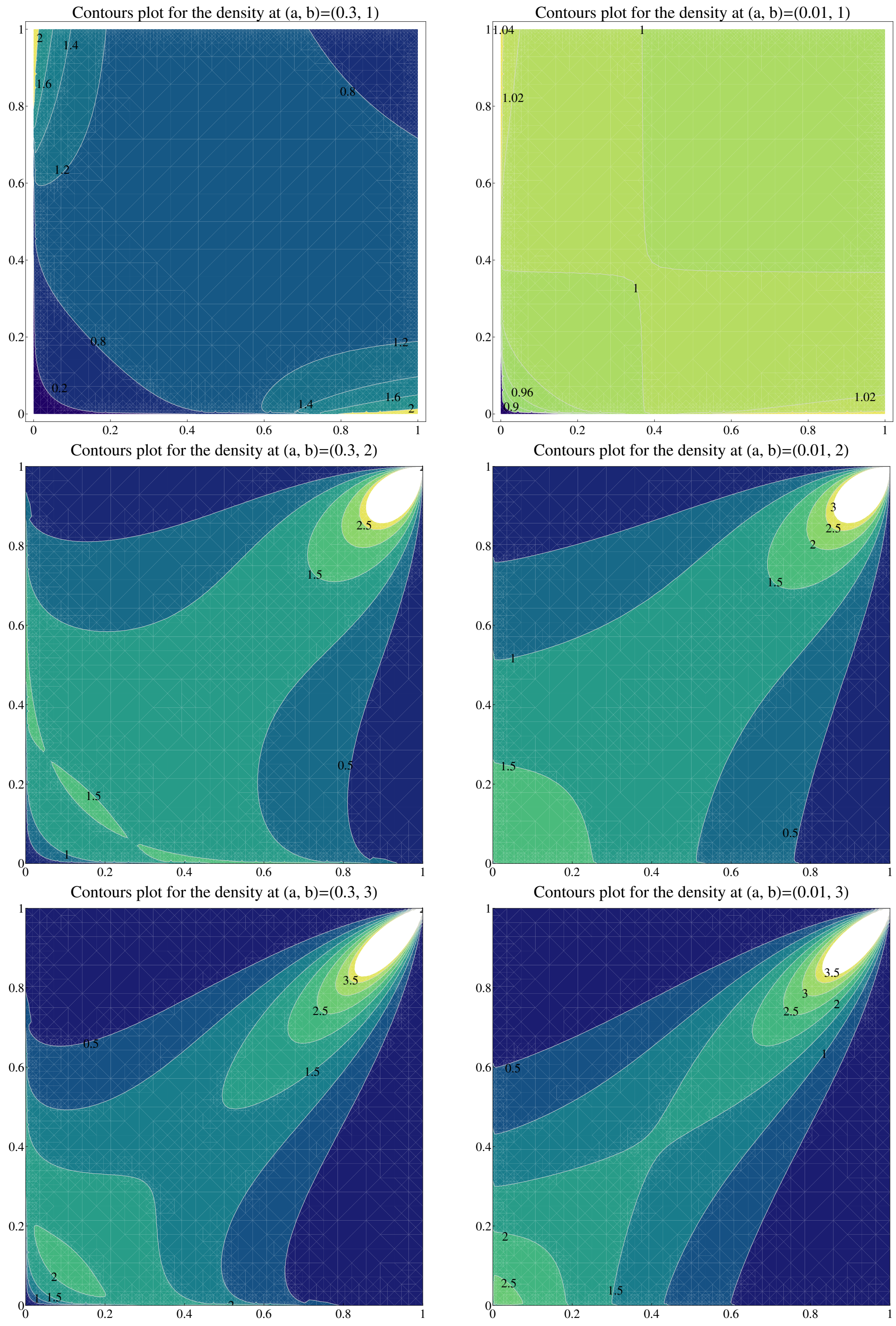


Figure 2. Contour plots for the density at different values of the parameters.

Thus, for some fixed parameter values, the GJ copula is not necessarily quadrant dependent. This concludes the proof. □

However, to complete Proposition 3.1, we should mention that for some restricted values of the parameters, the GJ copula can be positive quadrant dependent (like the Joe copula) or negative quadrant dependent (like the Gumbel-Barnett copula). In addition, the GJ copula and its density are generally not totally positive of order two (TP_2). (see [18]).

3.2. Stochastic order

Some stochastic order properties of the GJ copula are described in the proposition below.

Proposition 3.2. *The GJ copula is*

- *negatively ordered in a for fixed b, i.e., for any $0 \leq a_1 \leq a_2 \leq 1/b$ and $(x,y) \in [0,1]^2$, we have $C(x,y;a_1,b) \geq C(x,y;a_2,b)$, where $C(x,y;a,b)$ denotes the GJ copula with parameters a and b.*
- *positively ordered in b for fixed a, i.e., for any $0 \leq a \leq 1/b_2$, $1 \leq b_1 \leq b_2$, and $(x,y) \in [0,1]^2$, we have $C(x,y;a,b_1) \leq C(x,y;a,b_2)$.*

Proof. We employ a step-by-step proof.

- For $0 \leq a_1 \leq a_2 \leq 1/b$ and $b \geq 1$, since $-\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right] \leq 0$, we have

$$-a_1\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right] \geq -a_2\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right],$$

which implies that

$$\left[1-(1-x)^b\right]\left[1-(1-y)^b\right]\exp\left\{-a_1\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\} \geq$$
$$\left[1-(1-x)^b\right]\left[1-(1-y)^b\right]\exp\left\{-a_2\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\}.$$

Therefore, we have

$$1-\left\{1-\left[1-(1-x)^b\right]\left[1-(1-y)^b\right]\exp\left\{-a_1\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\}\right\}^{1/b} \geq$$
$$1-\left\{1-\left[1-(1-x)^b\right]\left[1-(1-y)^b\right]\exp\left\{-a_2\log\left[1-(1-x)^b\right]\log\left[1-(1-y)^b\right]\right\}\right\}^{1/b},$$

i.e., $C(x,y;a_1,b) \geq C(x,y;a_2,b)$.

- For $0 \leq a \leq 1/b_2$ and $1 \leq b_1 \leq b_2$, we have $1-(1-x)^{b_1} \leq 1-(1-x)^{b_2}$, so

$$\log\left[1-(1-x)^{b_1}\right] \leq \log\left[1-(1-x)^{b_2}\right],$$

thus

$$-a\log\left[1-(1-x)^{b_1}\right]\log\left[1-(1-y)^{b_1}\right] \leq -a\log\left[1-(1-x)^{b_2}\right]\log\left[1-(1-y)^{b_2}\right],$$

which implies that

$$\exp\left\{-a\log\left[1-(1-x)^{b_1}\right]\log\left[1-(1-y)^{b_1}\right]\right\} \leq$$
$$\exp\left\{-a\log\left[1-(1-x)^{b_2}\right]\log\left[1-(1-y)^{b_2}\right]\right\}.$$

Furthermore, we have $1-(1-x)^{b_1} \leq 1-(1-x)^{b_2}$ and $1-(1-y)^{b_1} \leq 1-(1-y)^{b_2}$, and, as a result,

$$\left[1-(1-x)^{b_1}\right]\left[1-(1-y)^{b_1}\right]\exp\left\{-a\log\left[1-(1-x)^{b_1}\right]\log\left[1-(1-y)^{b_1}\right]\right\} \leq$$
$$\left[1-(1-x)^{b_2}\right]\left[1-(1-y)^{b_2}\right]\exp\left\{-a\log\left[1-(1-x)^{b_2}\right]\log\left[1-(1-y)^{b_2}\right]\right\}.$$

This implies that

$$1-\left[1-(1-x)^{b_1}\right]\left[1-(1-y)^{b_1}\right]\exp\left\{-a\log\left[1-(1-x)^{b_1}\right]\log\left[1-(1-y)^{b_1}\right]\right\}\geq$$
$$1-\left[1-(1-x)^{b_2}\right]\left[1-(1-y)^{b_2}\right]\exp\left\{-a\log\left[1-(1-x)^{b_2}\right]\log\left[1-(1-y)^{b_2}\right]\right\},$$

and, equivalently,

$$1-\left\{1-\left[1-(1-x)^{b_1}\right]\left[1-(1-y)^{b_1}\right]\exp\left\{-a\log\left[1-(1-x)^{b_1}\right]\log\left[1-(1-y)^{b_1}\right]\right\}\right\}^{1/b_1}\leq$$
$$1-\left\{1-\left[1-(1-x)^{b_2}\right]\left[1-(1-y)^{b_2}\right]\exp\left\{-a\log\left[1-(1-x)^{b_2}\right]\log\left[1-(1-y)^{b_2}\right]\right\}\right\}^{1/b_2},$$

i.e., $C(x,y;a,b_1)\leq C(x,y;a,b_2)$.

This ends the proof □

This result allows us to understand how the sub-models derived from the GJ copula compare with each other in terms of a and b. We can refer to [22] for the details on this aspect.

3.3. Spearman's rho and Kendall's tau correlation coefficients

Based on the classical integral definitions (see [22] and [18]), Spearman's rho and Kendall's tau correlation coefficients associated with the GJ copula are given by

$$\rho_C = 12\iint_{[0,1]^2} C(x,y)dxdy - 3$$

and

$$\tau_C = 4\iint_{[0,1]^2} C(x,y)dC(x,y) - 1,$$

respectively. Unfortunately, due to the complexity of the GJ copula, both integrals lack closed-form expressions. However, they can be evaluated numerically for various parameter choices a and b, as will be shown later in Table 1.

Despite the inability to obtain exact forms of Spearman's rho and Kendall's tau coefficients, we were fortunate enough to derive Blomqvist's β correlation coefficient, which depends on the copula only through C(1/2,1/2). However, this coefficient often gives results close to those of Spearman's rho and Kendall's tau (see [22]).

3.4. Blomqvists correlation coefficient

Based on the definition in [22], the medial correlation coefficient or Blomqvists β of the GJ copula is given by

$$\begin{aligned}\beta_C &= 4C\left(\frac{1}{2},\frac{1}{2}\right)-1\\ &= 3-4\left\{1-\left[1-\left(\frac{1}{2}\right)^b\right]^2\exp\left[-a\left\{\log\left[1-\left(\frac{1}{2}\right)^b\right]\right\}^2\right]\right\}^{1/b}.\end{aligned}$$

Table 1 lists the values of Spearman's rho, Kendall's tau and Blomqvists β for different choices of the parameters a and b. Figure 3 illustrates Table 1 by presenting plots of these correlation coefficients.

Table 1. Spearman's rho, Kendall's tau and Blomqvists β of the GJ copula at some choices of the parameters.

b	a	ρ_C	τ_C	β_C	b	a	ρ_C	τ_C	β_C
1	1	-0.5239	-0.3613	-0.3815	2	0.5	0.3938	0.2689	0.2862
	0.9	-0.4848	-0.3330	-0.3511		0.4	0.4137	0.2843	0.2994
	0.8	-0.4437	-0.3035	-0.3191		0.3	0.4345	0.3004	0.3129
	0.7	-0.4002	-0.2727	-0.2856		0.2	0.4565	0.3175	0.3265
	0.6	-0.3542	-0.2404	-0.2504		0.1	0.4796	0.3356	0.3403
	0.5	-0.3053	-0.2064	-0.2136	3	0.33	0.6642	0.4846	0.5180
	0.4	-0.2531	-0.1704	-0.1748		0.23	0.6744	0.4940	0.5227
	0.3	-0.1972	-0.1323	-0.1342		0.13	0.6850	0.5038	0.5274
	0.2	-0.1369	-0.0916	-0.0916	4	0.25	0.7832	0.5974	0.6359
	0.1	-0.0715	-0.0477	-0.0469		0.15	0.7890	0.6036	0.6377

Based on this computational and graphical work, we observe that the copula allows correlations approximately between -0.52 at $(a, b) = (1, 1)$ and 1 at $b \to \infty$. Thus, unlike the Gumbel-Barnett and Joe copulas, this extension can accommodate datasets with either positive or negative correlations. In some sense, the GJ copula combines the correlation powers of the Gumbel-Barnett and Joe copulas.

3.5. Tail dependence

The tail dependence parameters are examined in the next result. We refer to [22] for the relevant definitions and interpretations.

Proposition 3.3. *The upper and lower tail dependence parameters of the GJ copula are determined as*

$$\lambda_U = 2 - 2^{1/b}, \qquad \lambda_L = 0,$$

respectively.

Proof.

- According to [22], by using the generator function, the upper tail dependence parameter λ_U is given by

$$\lambda_U = 2 - \lim_{t\to 0} \frac{1-\varphi^{-1}(2t)}{1-\varphi^{-1}(t)} = 2 - \lim_{t\to 0} \frac{\left\{1-\exp\left[\frac{1}{a}(1-e^{2t})\right]\right\}^{1/b}}{\left\{1-\exp\left[\frac{1}{a}(1-e^{t})\right]\right\}^{1/b}}$$

$$= 2 - \left\{\lim_{t\to 0} \frac{1-\exp\left[\frac{1}{a}(1-e^{2t})\right]}{1-\exp\left[\frac{1}{a}(1-e^{t})\right]}\right\}^{1/b} = 2 - \left\{\lim_{t\to 0} \frac{\frac{1}{a}(1-e^{2t})}{\frac{1}{a}(1-e^{t})}\right\}^{1/b}$$

$$= 2 - \left\{\lim_{t\to 0} \frac{2t}{t}\right\}^{1/b} = 2 - 2^{1/b}.$$

- From [22], by using the generator function, the lower tail dependence parameter λ_L is given by

$$\lambda_L = \lim_{t\to +\infty} \frac{\varphi^{-1}(2t)}{\varphi^{-1}(t)} = \lim_{t\to\infty} \frac{1-\left\{1-\exp\left[\frac{1}{a}(1-e^{2t})\right]\right\}^{1/b}}{1-\left\{1-\exp\left[\frac{1}{a}(1-e^{t})\right]\right\}^{1/b}}$$

$$= \lim_{t\to +\infty} \frac{\frac{1}{b}\exp\left[\frac{1}{a}(1-e^{2t})\right]}{\frac{1}{b}\exp\left[\frac{1}{a}(1-e^{t})\right]} = \lim_{t\to +\infty} \exp\left[\frac{1}{a}e^{t}(1-e^{t})\right] = 0.$$

The desired results are obtained. □

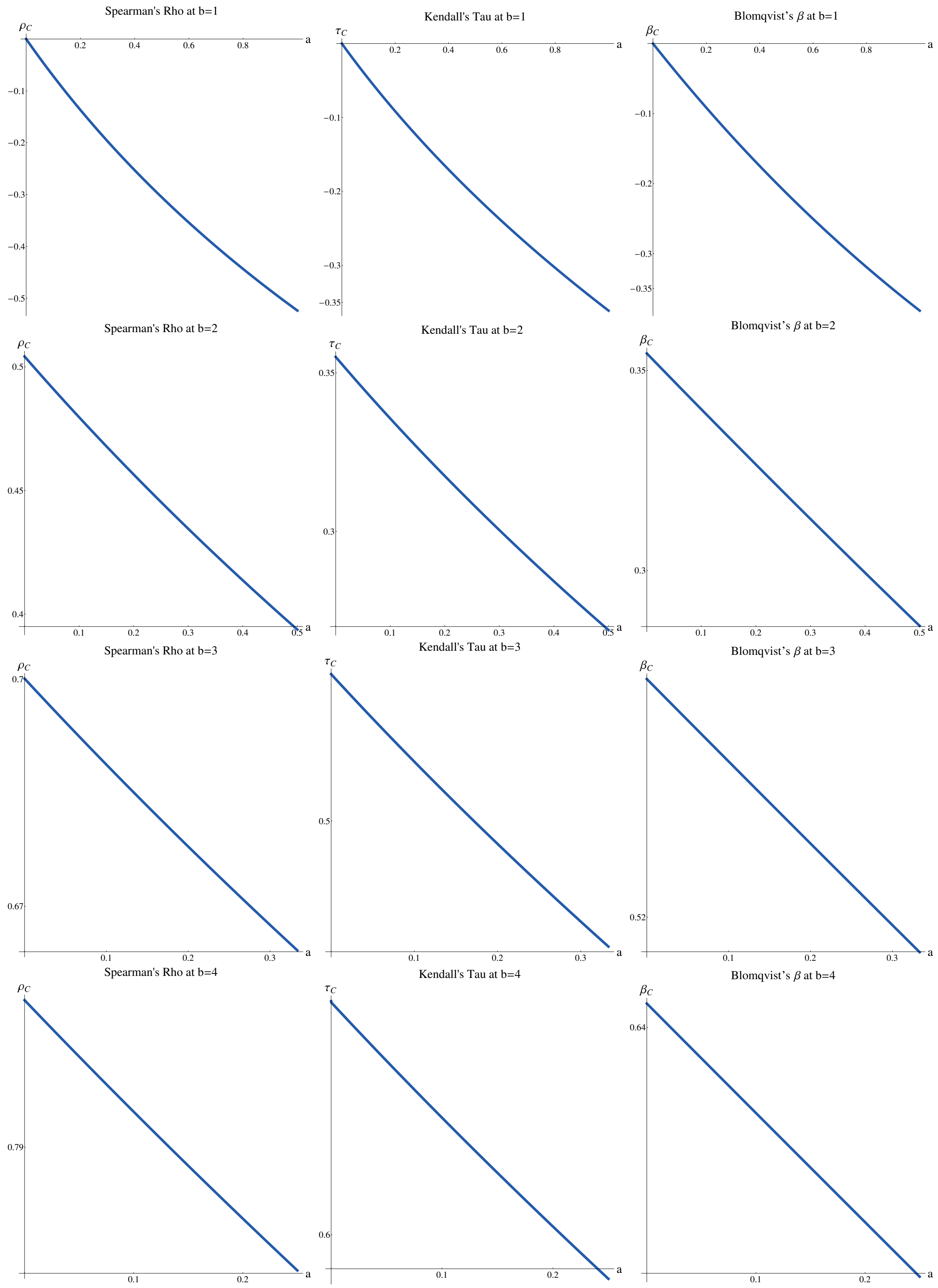


Figure 3. Spearman's rho, Kendall's tau and Blomqvists β for the GJ copula at different values of the parameters.

From this result, we see that the tail dependence parameters are independent of a, or indirectly dependent since we must have $b \in [1, 1/a)$. In particular, the GJ copula has no lower tail dependence, while its upper tail dependence depends only on the value of b.

4. Simulation study

In this section, we carry out a brief simulation study based on the GJ copula for various choices of the parameters. For a fixed sample size of 30, 100, or 200, ten thousand samples of data are generated from the GJ copula with uniform margins. The maximum likelihood estimates (MLEs) are then obtained for the two parameters a and b. The unbiasedness and efficiency of these estimates are examined by calculating the bias and the mean square error (MSE). We recall that the bias is the average of the difference between the actual and the estimated values of the parameter for all the runs, and the MSE is the average squared difference between the estimated and the actual values for all runs. The estimation results are presented in Table 2. The results for the one-parameter copula case, where $a = 1/b$, are shown in the last section of the table.

Table 2. Simulation results of the GJ copula.

Actual values (a, b)	Sample size	Bias(a)	MSE(a)	Bias(b)	MSE(b)
$(0.2, 1)$	30	0.144	0.097	0.078	0.023
	100	0.052	0.026	0.032	0.004
	200	0.029	0.012	0.019	0.002
$(0.13, 1.3)$	30	0.120	0.077	0.065	0.069
	100	0.048	0.029	0.022	0.020
	200	0.024	0.014	0.011	0.010
$(0.1, 2)$	30	0.112	0.054	0.088	0.160
	100	0.063	0.029	0.034	0.045
	200	0.040	0.019	0.025	0.024
Actual value $a = 1/b$	Sample size		Bias(b)		MSE(b)
0.13	30		0.019		0.025
	100		0.006		0.007
	200		0.009		0.003
0.2	30		0.049		0.102
	100		0.013		0.027
	200		0.007		0.013
0.25	30		0.062		0.184
	100		0.012		0.026
	200		0.009		0.023

This table shows that the two estimates are consistent, as the bias and the MSE decrease with increasing sample size in all cases. In general, both estimates tend to overestimate the parameter value (i.e., positive bias). For the case $a = 1/b$, we observe that smaller values of the bias and the MSE are associated with smaller values of b.

5. Application to data

In this section, we examine the behavior of the GJ copula as a comprehensive model for handling real datasets. It is worth noting that two-parameter copulas are used not only as models, but also as model selection tools. When analyzing the data, the two-parameter copula is usually reduced to one of its one-parameter sub-families if it gives the best fit, i.e., analyzing the data with a two-parameter copula is equivalent to making a comparison between all of its one-parameter sub-families, saving us the trouble of fitting each sub-family individually. In the following application, we fit a real dataset with the two-parameter GJ copula and its special case where $a = 1/b$. We will also use some well-known copulas for comparison, namely the Plackett, Clayton and Gumbel-Hougaard copulas, as shown in Table 3.

Table 3. Some well-known copulas

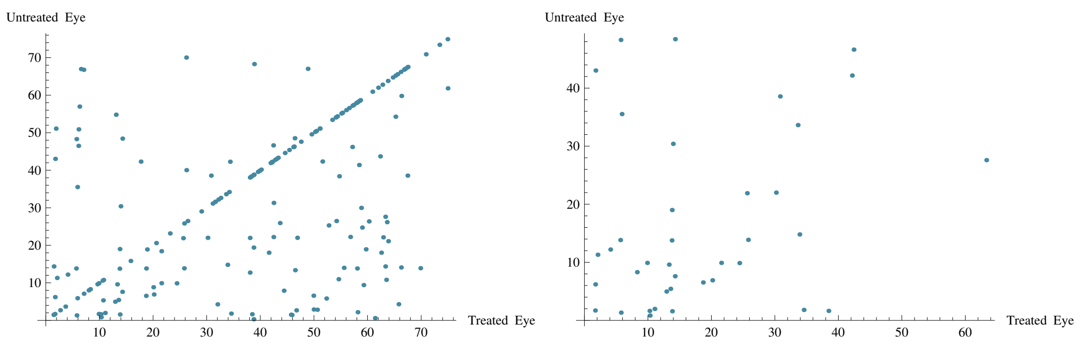
Copula	Formula of $C(x,y)$ for $(x,y) \in [0,1]^2$		Parameter(s)
Plackett	$\begin{cases} \frac{1+(\alpha-1)(x+y)-\sqrt{\{1+(\alpha-1)(x+y)\}^2-4\alpha(\alpha-1)xy}}{2(\alpha-1)} \\ xy \end{cases}$	$;\alpha \neq 1$ $;\alpha = 1$	$a \geq 0$
Clayton	$\left(x^{-a}+y^{-a}-1\right)^{-\frac{1}{a}}$		$a \geq 0$
Gumbel-Hougaard	$e^{-([-\log x]^a+[-\log y]^a)^{\frac{1}{a}}}$		$a \geq 1$

5.1. Diabetic Retinopathy Data

This dataset, originally discussed by [14], consists of 197 patients with diabetic retinopathy. Each patient was randomly assigned to receive laser treatment in one eye. For each eye, the time to blindness was recorded in months. The aim of the study was to determine the effect of laser treatment on delaying blindness in patients with diabetic retinopathy. The data were censored for various reasons, such as death or dropout, leaving only 38 complete records. They are available in the R package "SurvCorr".

Many researchers looked at diabetic retinopathy data, some of which was based on copulas. For example, Ghosh[12] fitted these data using the Plackett copula with and without a covariate indicating the type of diabetes (juvenile or adult). The author proposed a goodness-of-fit test and used it to show that the Plackett copula was an appropriate choice. Jones et al.[19] considered the power generalized Weibull distribution as a marginal distribution with the two-parameter BB9 copula, which has the Gumbel-Hougaard copula as a sub-family. As in [12], these authors discussed two models, one with and one without the covariate. Coelho-Barros et al.[8] analyzed these data with the FGM and Gumbel-Barnett copulas, using the Weibull distribution for the marginal distributions. According to their results, the Gumbel-Barnett copula fits better. Franco et al.[10] introduced a model based on the Marshall-Olkin copula, called the GBD family, and used it to fit the diabetic retinopathy data. The EM algorithm was used to estimate its parameters. After analyzing the data, Alqallaf and Kundu[2] considered a bivariate inverse generalized exponential (BIGE) distribution and compared it with two alternative models. They showed that the BIGE distribution gave a better fit.

For the 197 records, the mean time to blindness in the treated eye is 38.9 compared to 32.3 in the untreated eye. The sample Spearman's rho correlation coefficient is 0.464, while Kendall's tau correlation coefficient is 0.385. For the uncensored data, the mean time to blindness in the treated eye for the 38 patients is 38.7 compared to 32.6 in the untreated eye. The sample Spearman's rho and Kendall's tau correlation coefficients are 0.255 and 0.209 respectively. Scatter plots for the full dataset and the 38 uncensored observations are shown in Figure 4.



(a) The whole dataset **(b)** Uncensored observations

Figure 4. Scatter plots for the diabetic retinopathy data.

5.2. Method and results

In this study, we consider the 38 uncensored records with the GJ copula and its special case $a = 1/b$ using the Weibull distributions for both marginal distributions, where their cumulative distribution functions are given by $F_i(x) = 1 - e^{-(x/\lambda_i)^{\alpha_i}}$, $x > 0$, and $\alpha_i, \lambda_i > 0$, with $i = 1, 2$. The p-value for the Anderson Darling goodness-of-fit test for the time to blindness of the treated eye is 0.903 and that of the untreated eye is 0.678. Thus, for both marginals, there is no evidence against the hypotheses that the data arise from a Weibull distribution. For the sake of comparison, we also analyzed the data using the Plackett, Clayton, and Gumbel-Hougaard copulas, as described in Table 3. In Table 4, we obtain the MLEs for the parameters along with the Akaike information criterion (AIC) and Bayesian information criterion (BIC).

Table 4. Estimation results for the uncensored diabetic retinopathy data using the Weibull distribution as the marginal distributions.

Copula	AIC	BIC	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$	$\hat{a}$	$\hat{b}$
GJ	590	600	1.326	20.061	1.036	16.730	0.078	1.390
GJ with $a = 1/b$	590	599	1.256	20.262	0.971	16.765		1.689
Plackett	589	598	1.365	20.471	1.054	16.910	2.301	
Clayton	591	599	1.333	19.996	1.048	17.026	0.213	
Gumbel-Hougaard	589	597	1.339	20.012	1.046	16.700	1.214	

Although the best fits to the censored data are the Plackett and Gumbel-Hougaard copulas, the other three copulas, i.e., the GJ, GJ with $a = 1/b$ and Clayton copulas, give similar fits as the differences between their AICs and the minimum AIC are less than 5. For more details, see [4].

When we consider the censoring in the data, the Weibull distribution is no longer appropriate to fit the marginals, as the p-values of the Anderson Darling test for both marginal values are close to zero. We therefore fit all the data using the GJ copula with empirical marginal distributions. The Plackett, Clayton, and Gumbel-Hougaard copulas are once again used to compare the results. Table 5 shows these results. The best models are the Gumbel-Hougaard and GJ copulas (the GJ copula, which, in this case, is almost reduced to the Joe copula $a \to 0$).

Table 5. Estimation results for the entire diabetic retinopathy data using empirical margins.

Copula	AIC	BIC	$\hat{a}$	$\hat{b}$
GJ	595	601	5×10^{-8}	1.356
GJ with $a = 1/b$	616	619		1.630
Plackett	616	619	3.401	
Clayton	636	640	0.422	
Gumbel-Hougaard	594	597	1.255	

6. Conclusion

In this research, we have introduced a new two-parameter Archimedean copula that includes two famous one-parameter Archimedean copulas, the Gumbel-Barnett and the Joe copulas. Each of these copulas has its own special dependence structure. This suggests that the GJ copula is able to capture different types of dependence in the datasets. With this in mind, this new copula was thoroughly investigated. Among the results, we found that it covers a correlation range approximately of $(-0.5, 1)$ and has an upper tail dependence. A brief simulation was performed to examine the estimates of the parameters involved, which were found to be unbiased and consistent. Finally, the GJ copula was used to fit a real dataset. The results showed that the GJ copula provided a reasonable fit to the data. However, the main advantage of modelling with two-parameter copulas is that they often return the most appropriate model from their one-parameter sub-family, which in our case is the Joe copula and the Gumbel-Barnett copula. This was evident in the discussion of the diabetic retinopathy data, as the GJ copula reduced to the Joe copula.

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Author contributions. C.C.: conceived the idea, developed the theory, performed the calculations, revised and edited, supervised the results of this work. A.W.: performed the numerical simulations, designed the figures, performed the application calculations. All authors have read and approved the published version of the manuscript.

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Data availability. The data used in this article is available in the R package "SurvCorr".

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