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On Some *k*- Oresme Polynomials with Negative Indices

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Abstract
In this study, *k*- Oresme polynomials with negative indices, which are the generalization of Oresme polynomials, were examined and defined. By examining the algebraic properties of recently defined polynomial sequences, some important identities were given. The matrices of negative indices *k*- Oresme polynomials was defined. Some sum formulas were given according to this definition.

Keywords: Matrices, Recurrences, Special sequences and polynomials
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1. Introduction

In [1], Alwyn Francis Horadam defined the well-known number sequence called Horadam numbers denoted by second order linear recurrence relation. The author examined the principle properties of an arbitrary generalized integer sequence and studied particular cases of this sequence [1]-[3]. The sequence studied by Horadam is re-examined by various authors and several applications of this sequence are included in [4]-[7].
For nonzero integers p and q, Horadam sequence is given by the recurrence relation

$$w_{n+2} = pw_{n+1} - qw_n, n \geq 0, \tag{1.1}$$

where $w_n = w_n(w_0, w_1; p, q)$ is the general term. Nicole Oresme, one of the scientists in the 14th century, investigated the sum of the sequences of rational numbers and the properties of this sum [8]. Later in 1974, this author expanded and defined a new integer sequence denoted by $\{\mathbf{O}_n\}$ and this defined sequence is known in the literature as the Oresme sequence [9]. Different sequences are obtained by customizing the coefficients p, q in the Horadam sequence, which has been studied by many authors. The Oresme sequences we are working with here is the version of the coefficients p, q obtained by taking special numbers.The recurrence relation of this sequence is as follows.

$$\mathbf{O}_n = \mathbf{O}_{n-1} - \frac{1}{4}\mathbf{O}_{n-2}; \mathbf{O}_0 = 0, \mathbf{O}_1 = \frac{1}{2}. \tag{1.2}$$

Horadam examined these numbers in more detail and obtained both linear and non-linear relations involving these numbers and gave the generating functions for them. Cook [6] generalized the these numbers as k- Oresme numbers denoted by $\mathbf{O}_n^{(k)}$ and defined by, for $k>2$,

$$\mathbf{O}_n^{(k)}=\mathbf{O}_{n-1}^{(k)}-\frac{1}{k^2}\mathbf{O}_{n-2}^{(k)}, \tag{1.3}$$

in here the initial conditions are $\mathbf{O}_0^{(k)}=0$ and $\mathbf{O}_1^{(k)}=\frac{1}{k}$.
It can be noticed that these numbers are reduced to standard Oresme numbers by taking $k=2$. In [6], for $k^2-4>0$, the closed formula of k- Oresme numbers is given by

$$\mathbf{O}_n^{(k)}=\frac{\alpha^n-\beta^n}{\sqrt{k^2-4}}. \tag{1.4}$$

In the last equation $\alpha=\frac{k+\sqrt{k^2-4}}{2k}$ and $\beta=\frac{k-\sqrt{k^2-4}}{2k}$. Some identities and sum formulas for this number sequence are studied in [6], [10]. Moreover, see [6], [10]-[13] for recent studies. In [14], Halici et al. generalized the k- Oresme numbers as k- Oresme polynomials denoted by $\mathbf{O}_n^{(k)}(x)$. The recurrence relation of nth k- Oresme polynomials is as follows.

$$\mathbf{O}_{n+2}^{(k)}(x)=\mathbf{O}_{n+1}^{(k)}(x)-\frac{1}{k^2x^2}\mathbf{O}_n^{(k)}(x),\mathbf{O}_0^{(k)}(x)=0,\mathbf{O}_1^{(k)}(x)=\frac{1}{kx}, \tag{1.5}$$

where $x\in\mathbb{R}$ and $n\in\mathbb{N}$. Taking $k=1$ and $x=1$ in (1.5) respectively, one can get Oresme polynomials and k- Oresme numbers. In [12], k- Oresme numbers are extended to negative indices and gave the following recurrence relation

$$\mathbf{O}_{-n}^{(k)}=k^2\left(\mathbf{O}_{-n+1}^{(k)}-\mathbf{O}_{-n+2}^{(k)}\right), \tag{1.6}$$

where $\mathbf{O}_{-1}^{(k)}=-k$ and $\mathbf{O}_0^{(k)}=0$ are the initial conditions. The nth term of this sequence is defined by

$$\mathbf{O}_{-n}^{(k)}=-k^{2n}\frac{(\alpha^n-\beta^n)}{\sqrt{k^2-4}}. \tag{1.7}$$

The values α and β are as in the equation (1.4).
Also, the authors in [15] worked on k- Oresme polynomials and derivatives. Some results obtained about these polynomials are given below.

$$i)\sum_{i=1}^{n}\mathbf{O}_i^{(k)}(x)=k^2x^2\left(\frac{1}{kx}-\mathbf{O}_{n+2}(x)\right). \tag{1.8}$$

$$ii)\sum_{i=1}^{n}(-1)^i\mathbf{O}_i^{(k)}(x)=\frac{k^2x^2}{2k^2x^2+1}\left(\frac{1}{kx}+(-1)^{n+1}\left(\mathbf{O}_{n+2}^{(k)}(x)-2\mathbf{O}_{n+1}^{(k)}(x)\right)\right). \tag{1.9}$$

$$iii)\sum_{i=1}^{n}\mathbf{O}_{2i+1}^{(k)}(x)=\frac{k^2x^2}{2k^2x^2+1}\left(\frac{k^2x^2}{kx+1}+\frac{k^2x^2}{k^2x^2+1}\mathbf{O}_{2n+1}^{(k)}(x)-k^2x^2\mathbf{O}_{2n+2}^{(k)}(x)\right). \tag{1.10}$$

$$iv)\sum_{i=1}^{n}\mathbf{O}_{2i}^{(k)}(x)=\frac{k^2x^2}{2k^2x^2+1}\left(kx-\left(k^2x^2+1\right)\mathbf{O}_{2n+2}^{(k)}(x)+\mathbf{O}_{2n+1}^{(k)}(x)\right). \tag{1.11}$$

In [16], Soykan studied a different generalization of Oresme sequences.
In this study, we examined the corresponding generation matrix for the polynomial sequence we define in this paper. We gave some combinatorial equations for this new sequence studied with the help of basic matrix calculations. Also, we derived new identities by using the concepts of trace and determinant of a matrix. We also calculated sum formulas for the elements of this sequence.

2. Main Results

Definition 2.1. *For $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, k- Oresme polynomial with negative indices is denoted by $\mathbf{O}_{-n}^{(k)}(x)$ and defined by the recurrence relation*

$$\mathbf{O}_{-n}^{(k)}(x) = (kx)^2\left(\mathbf{O}_{-n+1}^{(k)}(x) - \mathbf{O}_{-n+2}^{(k)}(x)\right), \tag{2.1}$$

with initial conditions $\mathbf{O}_{-1}^{(k)}(x) = -kx$ and $\mathbf{O}_{0}^{(k)}(x) = 0$.

Some terms of this sequence are

$$\left\{\mathbf{O}_{-n}^{(k)}(x)\right\}_{n\geq 0} = \left\{0, -kx, -(kx)^3, (kx)^3-(kx)^5, \ldots\right\}.$$

In the case of $k = 2$ and $x = 1$, the recurrence relation (2.1) is reduced to the equation (1.6). If the equation (2.1) is solved, the roots of this equation are

$$\alpha = \frac{kx+\sqrt{(kx)^2-4}}{2kx} \text{ and } \beta = \frac{kx-\sqrt{(kx)^2-4}}{2kx}, \tag{2.2}$$

respectively.

Corollary 2.2. *The Binet formula for the sequence $\left\{O_{-n}^{(k)}(x)\right\}_{n\geq 0}$ is*

$$\mathbf{O}_{-n}^{(k)}(x) = -(kx)^{2n}\frac{(\alpha^n-\beta^n)}{\sqrt{(kx)^2-4}}. \tag{2.3}$$

Proof. For the k- Oresme polynomials with negative indices, let us substitute the closed formula for the k- Oresme numbers with negative indices in equation (1.7).

$$\mathbf{O}_{-n}^{(k)}(x) = \frac{1}{\sqrt{(kx)^2-4}}\left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right),$$

$$\mathbf{O}_{-n}^{(k)}(x) = -\frac{1}{\sqrt{(kx)^2-4}}\left(\frac{\alpha^n-\beta^n}{(\alpha\beta)^n}\right),$$

which implies

$$\mathbf{O}_{-n}^{(k)}(x) = -(kx)^{2n}\frac{1}{\sqrt{(kx)^2-4}}\left(\left(\frac{kx+\sqrt{(kx)^2-4}}{2kx}\right)^n - \left(\frac{k-\sqrt{(kx)^2-4}}{2kx}\right)^n\right).$$

By some elementary operations, the following equation is obtained

$$\mathbf{O}_{-n}^{(k)}(x) = -(kx)^{2n}\frac{(\alpha^n-\beta^n)}{\sqrt{(kx)^2-4}}.$$

This proves the corollary. □

Using the terms of the sequence $\left\{\mathbf{O}_{-n}^{(k)}(x)\right\}_{n\geq 0}$, the generating matrix corresponds to these polynomials with negative indices is defined as

$$\mathbb{O} = \frac{1}{kx}\begin{bmatrix} (kx)^2\mathbf{O}_{0}^{(k)}(x) & -\mathbf{O}_{-1}^{(k)}(x) \\ (kx)^2\mathbf{O}_{-1}^{(k)}(x) & -\mathbf{O}_{-2}^{(k)}(x) \end{bmatrix}. \tag{2.4}$$

In the following Theorems some fundamental identities for the polynomials mentioned above are deduced by using the matrices $\mathbb{O}$.

Theorem 2.3. *For the matrix $\mathbb{O}$, the following equation is true.*

$$\mathbb{O}^n = \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix}. \tag{2.5}$$

Proof. To prove by induction observe that for $n = 1$, then the equation (2.5) is true. Using the fact that $\mathbb{O}^{n+1} = \mathbb{O}^n\mathbb{O}$, we have

$$\mathbb{O}^{n+1} = \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n}(x) & kx\mathbf{O}^{(k)}_{-n+1}(x) - kx\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n-1}(x) & kx\mathbf{O}^{(k)}_{-n}(x) - kx\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix}$$

and when the necessary procedures and arrangements are made

$$\mathbb{O}^{n+1} = \begin{bmatrix} kx\mathbf{O}^{(k)}_{-(n+1)+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+1)}(x) \\ kx\mathbf{O}^{(k)}_{-(n+1)}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+1)-1}(x) \end{bmatrix}$$

is obtained. Thus, the proof is completed. □

In the following theorem, we give the generating function of $\left\{\mathbf{O}^{(k)}_{-n}(x)\right\}_{n\geq 0}$.

Theorem 2.4. *The generating function for these polynomials is derived below:*

$$\sum_{i=1}^{\infty}\mathbf{O}^{(k)}_{-n}(x)z^i = -\frac{-kxz}{1-z(kx)^2+z^2(kx)^2}, \tag{2.6}$$

where $x \in \mathbb{R}$.

Proof. Using the definition of generating number function and some elementary operations, we have following equations.

$$f(z) = \mathbf{O}^{(k)}_{0}(x) + z\mathbf{O}^{(k)}_{-1}(x) + z^2\mathbf{O}^{(k)}_{-2}(x) + z^3\mathbf{O}^{(k)}_{-3}(x)\cdots.$$

$$-z(kx)^2 f(z) = -z(kx)^2\mathbf{O}^{(k)}_{0}(x) - z^2(kx)^2\mathbf{O}^{(k)}_{-1}(x) - z^3(kx)^2\mathbf{O}^{(k)}_{-2}(x) - z^4(kx)^2\mathbf{O}^{(k)}_{-3}(x)\cdots.$$

$$z^2(kx)^2 f(z) = z^2(kx)^2\mathbf{O}^{(k)}_{0}(x) + z^3(kx)^2\mathbf{O}^{(k)}_{-1}(x) + z^4(kx)^2\mathbf{O}^{(k)}_{-2}(x) + z^5(kx)^2\mathbf{O}^{(k)}_{-3}(x)\cdots.$$

From this, the following equation is obtained:

$$f(z) - z(kx)^2 f(z) - z^2(kx)^2 f(z) = \mathbf{O}^{(k)}_{0}(x) + z\left(\mathbf{O}^{(k)}_{-1}(x) - (kx)^2\mathbf{O}^{(k)}_{0}(x)\right) + z^2\left(\mathbf{O}^{(k)}_{-2}(x) - (kx)^2\mathbf{O}^{(k)}_{-1}(x) + (kx)^2\mathbf{O}^{(k)}_{0}(x)\right)\cdots.$$

By using the relation (2.1), it is obviously seen that

$$f(z) - z(kx)^2 f(z) + z^2(kx)^2 f(z) = -kxz.$$

Which implies

$$f(z) = \frac{-kxz}{1-z(kx)^2+z^2(kx)^2}.$$

This completes the proof. □

The well-known Catalan and Cassini identities for the sequence $\left\{\mathbf{O}^{(k)}_{-n}(x)\right\}_{n\geq 0}$ are given in the following two Theorems.

Theorem 2.5. *For $n \geq 0$, we have*

$$\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-n-1}(x) - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 = -(kx)^{2n}. \tag{2.7}$$

Proof. By using the matrix $\mathbb{O}$ given in the equation (2.5) and the fact that $(det\,(\mathbb{O}))^n = det\,(\mathbf{O}^n)$, we can write

$$det \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix} = det \begin{bmatrix} 0 & 1 \\ (kx)^2 & (kx)^2 \end{bmatrix}^n .$$

Hence, we have

$$(det\,(\mathbb{O}))^n = -\mathbf{O}^{(k)}_{n+1}(x)\mathbf{O}^{(k)}_{n-1}(x) + \left(\mathbf{O}^{(k)}_{n}(x)\right)^2 = -(kx)^{2n}.$$

Thus, the desired result is obtained. ☐

We have given an important identity provided by the elements of this polynomial sequence in the Theorem below.

Theorem 2.6. *For $n \geq r$, the following equality is true.*

$$\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x) - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 = -(kx)^{2n-2r}\left(\mathbf{O}^{(k)}_{-r}(x)\right)^2 . \tag{2.8}$$

Proof. By substituting the equation (2.5) into the left-hand side of the equation (2.8), we get

$$LHS = \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+r+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n+r}(x) \\ kx\mathbf{O}^{(k)}_{-n+r}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n+r-1}(x) \end{bmatrix} \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n-r+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-r}(x) \\ kx\mathbf{O}^{(k)}_{-n-r}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-r-1}(x) \end{bmatrix} - \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix}^2 .$$

By the matrix operation, the LHS equals to

$$LHS = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

where

$$\begin{aligned}
A &= (kx)^2\mathbf{O}^{(k)}_{-n+r+1}(x)\mathbf{O}^{(k)}_{-n-r+1}(x) - \mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x),\\
B &= -\mathbf{O}^{(k)}_{-n+r+1}(x)\mathbf{O}^{(k)}_{-n-r}(x) + \tfrac{1}{(kx)^2}\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r-1}(x),\\
C &= (kx)^2\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r+1}(x) - \mathbf{O}^{(k)}_{-n+r-1}(x)\mathbf{O}^{(k)}_{-n-r}(x),\\
D &= -\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x) + \tfrac{1}{(kx)^2}\mathbf{O}^{(k)}_{-n+r-1}(x)\mathbf{O}^{(k)}_{-n-r-1}(x),\\
A' &= (kx)^2\left(\mathbf{O}^{(k)}_{-n+1}(x)\right)^2 - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2,\\
B' &= \mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-n}(x) + \tfrac{1}{(kx)^2}\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-n-1}(x),\\
C' &= (kx)^2\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-n}(x) - \mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-n-1}(x)
\end{aligned}$$

and

$$D' = -\left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 + \tfrac{1}{(kx)^2}\left(\mathbf{O}^{(k)}_{-n-1}(x)\right)^2 .$$

Hence, we obtain

$$\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x) - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 = -(kx)^{2n-2r}\left(\mathbf{O}^{(k)}_{-r}(x)\right)^2 ,$$

which proves the theorem. ☐

In the case of $r = 1$, one can get the Cassini identity from the equation (2.8).
In the below, we give an important identity for these polynomials we are considering with negative indices is given.

Theorem 2.7. *For $n, m \in \mathbb{Z}^+$, we have*

$$\mathbf{O}^{(k)}_{-(n+m)}(x) = kx\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x) - \frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x)\mathbf{O}^{(k)}_{-m}(x). \tag{2.9}$$

Proof. By using (2.5), we can get

$$\mathbb{O}^{n+m}=\begin{bmatrix} kx\mathbf{O}^{(k)}_{-(n+m)+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+m)}(x) \\ kx\mathbf{O}^{(k)}_{-(n+m)}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+m)-1}(x) \end{bmatrix}.$$

Since $\mathbb{O}^{n+m}=\mathbb{O}^{n}\mathbb{O}^{m}$, equating the corresponding elements of the matrices we have

$$kx\mathbf{O}^{(k)}_{-(n+m)}(x)=(kx)^2\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x)-\mathbf{O}^{(k)}_{-n-1}(x)\mathbf{O}^{(k)}_{-m}(x).$$

Hence,

$$\mathbf{O}^{(k)}_{-(n+m)}(x)=kx\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x)-\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x)\mathbf{O}^{(k)}_{-m}(x).$$

□

The well-known an important identity for these polynomials with negatives indices is deduced in the following Theorem.

Theorem 2.8. *For the positive integers m,n, the following is satisfied.*

$$\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-m}(x)-\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x)=-(kx)^{2m}\mathbf{O}^{(k)}_{-(n-m)}(x). \tag{2.10}$$

Proof. Using the closed formula, we can write $\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-m}(x)-\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x)$ as,

$$LHS=\frac{1}{(kx)^2-4}\left[\left(-(kx)^{2n-2}\left(\alpha^{n-1}-\beta^{n-1}\right)-(kx)^{2m}\left(\alpha^{m}-\beta^{m}\right)\right)-\left((kx)^{2n}\left(\alpha^{n}-\beta^{n}\right)-(kx)^{2m-2}\left(\alpha^{m-1}-\beta^{m-1}\right)\right)\right],$$

$$LHS=\frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(-\alpha^{n-1}\beta^{m}-\beta^{n-1}\alpha^{m}+\alpha^{n}\beta^{m-1}+\beta^{n}\alpha^{m-1}\right)\right],$$

$$LHS=\frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(\alpha^{n}\beta^{m}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)-\alpha^{m}\beta^{n}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right)\right],$$

$$LHS=\frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(\alpha^{n}\beta^{m}-\alpha^{m}\beta^{n}\right)\frac{\alpha-\beta}{\alpha\beta}\right],$$

where α and β are the roots of equation (2.1). By substituting $\alpha-\beta=\frac{\sqrt{(kx)^2-4}}{kx}$ and $\alpha\beta=\frac{1}{(kx)^2}$ into the last equation, we obtain

$$\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-m}(x)-\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x)=\frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(\alpha^{n}\beta^{m}-\alpha^{m}\beta^{n}\right)\frac{\alpha-\beta}{\alpha\beta}\right].$$

Making necessary arrangements, we get

$$LHS=-(kx)^{2m}\mathbf{O}^{(k)}_{-(n-m)}(x)$$

which completes the proof. □

Now, we have given some sum formulas of this polynomials with negative indices in the Theorem below.

Theorem 2.9. *For $n\geq 1$, we have the followings.*

$$i)\sum_{i=1}^{n}\mathbf{O}^{(k)}_{-i}(x)=-kx\left(1-kx\mathbf{O}_{-n+1}(x)\right). \tag{2.11}$$

$$ii)\sum_{i=1}^{n}(-1)^{i}\mathbf{O}_{-i}^{(k)}(x)=\frac{1}{2(kx)^{2}+1}\left(kx+(-1)^{n}\left((kx)^{2}\mathbf{O}_{-n}^{(k)}(x)+\mathbf{O}_{-n-1}^{(k)}(x)\right)\right). \quad (2.12)$$

$$iii)\sum_{i=1}^{n}\mathbf{O}_{-(2i+1)}^{(k)}(x)=\frac{1}{2}\left(\frac{\mathbf{O}_{-2n}^{(k)}(x)\left(-2(kx)^{3}-kx\right)}{2(kx)^{2}+1}\left(-kx+(kx)^{2}(2\mathbf{O}_{-2n-1}^{(k)}(x)+1)+(kx)^{4}-1\right)\right). \quad (2.13)$$

$$iv)\sum_{i=1}^{n}\mathbf{O}_{-(2i)}^{(k)}(x)=\frac{(kx)^{2}}{2(kx)^{2}+1}\left(kx-\left((kx)^{2}+1\right)\mathbf{O}_{-2n-2}^{(k)}(x)+\mathbf{O}_{-2n-1}^{(k)}(x)\right). \quad (2.14)$$

Proof. i) This equation,

$$\sum_{i=1}^{n}\mathbf{O}_{-i}^{(k)}(x)=-kx(1-kx\mathbf{O}_{0}(x))$$

is true for $n=1$. Let us assume that equality is true for $n\leq m$. Then, we get

$$LHS=-kx\left(1-kx\mathbf{O}_{-n+1}^{(k)}(x)\right)+\left(\mathbf{O}_{-n-1}^{(k)}(x)\right),$$

$$LHS=-kx\left(1-kx\mathbf{O}_{-n+1}^{(k)}(x)\right)+(kx)^{2}\left(kx\mathbf{O}_{-n}^{(k)}(x)-\mathbf{O}_{-n+1}^{(k)}(x)\right)$$

and

$$LHS=-kx\left(1-kx\mathbf{O}_{-n}^{(k)}(x)\right).$$

ii) The proof can be done similarly by using induction method.
iii) By observing that

$$\sum_{i=0}^{n}\mathbf{O}_{-2i-1}^{(k)}(x)=\frac{1}{2}\left(\sum_{i=0}^{2n+1}\mathbf{O}_{-i}^{(k)}(x)-\sum_{i=0}^{2n+1}(-1)^{i}\mathbf{O}_{-i}^{(k)}(x)\right),$$

and using *i* and *ii*, the proof is clear.
iv) Similarly, by observing that

$$\sum_{i=0}^{n}\mathbf{O}_{-2i}^{(k)}(x)=\frac{1}{2}\left(\sum_{i=0}^{2n}\mathbf{O}_{-i}^{(k)}(x)-\sum_{i=0}^{2n}(-1)^{i}\mathbf{O}_{-i}^{(k)}(x)\right),$$

the desired equality can be shown. □

In 2004, Laughlin calculated powers of an arbitrary second order matrix A by using the trace and determinant of this matrix. In [4],[5], Halici and Akyuz deduced and gave some combinatorial identities involving Horadam sequence. The help of these studies, we give some important and proper identity for the polynomials we examined with negative indices in the rest of the section. *nth* power of an arbitrary matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is given by the following formula:

$$A^{n}=z_{n}A-z_{n-1}DI_{2},$$

where

$$z_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}=\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{2^{n-1}}\binom{n}{2m+1}T^{n-2m-1}(T^{2}-4D)$$

and α, β are the roots of the characteristic equation of Horadam sequence. Notice that, T and D denotes the trace and determinant of the matrix A respectively.
The matrix A^n is given by Laughlin as

$$A^n = \begin{bmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_n & y_n - ay_{n-1} \end{bmatrix},$$

where

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i} D^i.$$

Theorem 2.10. *For $n \geq 1$, we have*

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(\frac{n\left(2-(kx)^2\right) - 2i\left(1-(kx)^2\right)}{n-i} \right) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (kx)^{-i} \left(\sqrt{(kx)^2 - 4} \right). \tag{2.15}$$

Proof. Applying (2.15) to generating matrix $\mathbb{O}$, we can write

$$\mathbb{O}^n = \begin{bmatrix} y_n - (kx)^2 y_{n-1} & y_{n-1} \\ (kx)^2 y_n & y_n \end{bmatrix}.$$

For $k > 2$, notice that trace and determinant of $\mathbb{O}$ are calculated as $T = (kx)^2$ and $D = -(kx)^2$. Hence, we write y_n as

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i}. \tag{2.16}$$

Using the fact that $\lambda_1^n + \lambda_2^n = 2y_n - (kx)^2 y_{n-1}$, we obtain the left-hand side as

$$LHS = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{n-i} \left(\frac{\sqrt{(kx)^2-4}}{2kx} \right)^i - \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{n-i} \left(-\frac{\sqrt{(kx)^2-4}}{2kx} \right)^i,$$

$$LHS = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^n} \frac{1}{(kx)^i} \left[\left(\sqrt{(kx)^2-4} \right)^i - \left(-\sqrt{(kx)^2-4} \right)^i \right],$$

$$LHS = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (kx)^{-i} \left(\sqrt{(kx)^2-4} \right)^i.$$

Furthermore, by using equation (2.16), we can write right-hand side as

$$RHS = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} - (kx)^2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} (kx)^{2n-2i}$$

which equals to

$$RHS = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} - (kx)^2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i}{n-2i} \frac{n-2i}{n-i} (kx)^{2n-2i}.$$

Since $\frac{n-2i}{n-i} = \frac{n-2\lfloor \frac{n}{2} \rfloor}{n-\lfloor \frac{n}{2} \rfloor} = 0$, we get the desired result as:

$$RHS = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(2 - (kx)^2 \frac{n-2i}{n-i} \right),$$

$$RHS = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(\frac{n(2-(kx)^2) - 2i(1-(kx)^2)}{n-i} \right).$$

Equating the left and right hand sides, we get

$$RHS = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(\frac{n(2-(kx)^2) - 2i(1-(kx)^2)}{n-i} \right) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (kx)^{-i} \left(\sqrt{(kx)^2-4} \right)^i.$$

□

3. Conclusion

In this study, we define the corresponding generation matrix for the polynomial sequence we define in this work. We obtained some combinatorial equations for this new sequence studied with the help of basic matrix calculations. Moreover, we gave new identities by using the concepts of trace and determinant of a matrix. We also derived sum formulas for the elements of this sequence.

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