International Electronic Journal of Algebra
Volume 36 (2024) 89-100
DOI: 10.24330/ieja.1446322

ALMOST-REDUCTIVE AND ALMOST-ALGEBRAIC LEIBNIZ ALGEBRA

David A. Towers

Received: 8 September 2023; Revised: 6 January 2024; Accepted: 7 January 2024
Communicated by Burcu Üngör

Abstract. This paper examines whether the concept of an almost-algebraic Lie algebra developed by Auslander and Brezin in [J. Algebra, 8(1968), 295-313] can be introduced for Leibniz algebras. Two possible analogues are considered: almost-reductive and almost-algebraic Leibniz algebras. For Lie algebras these two concepts are the same, but that is not the case for Leibniz algebras, the class of almost-algebraic Leibniz algebras strictly containing that of the almost-reductive ones. Various properties of these two classes of algebras are obtained, together with some relationships between ϕ-free, elementary, E-algebras and A-algebras.

Mathematics Subject Classification (2020): 17B05, 17B20, 17B30, 17B50
Keywords: Leibniz algebra, symmetric Leibniz algebra, Frattini ideal, ϕ-free, elementary, E-algebra, A-algebra, almost-algebraic, almost-reductive

1. Introduction

An algebra L over a field F is called a *Leibniz algebra* if for every $x, y, z \in L$, we have

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

In other words, the right multiplication operator $R_x : L \to L : y \mapsto [y, x]$ is a derivation of L. As a result, such algebras are sometimes called *right* Leibniz algebras and there is a corresponding notion of *left* Leibniz algebras which satisfy

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

Clearly the opposite of a right (left) Leibniz algebra is a left (right) Leibniz algebra, so, in most situations, it does not matter which definition we use. A *symmetric* Leibniz algebra L is one which is both a right and left Leibniz algebra and in which $[[x, y], [x, y]] = 0$ for all $x, y \in L$. This last identity is only needed in characteristic two, as it follows from the right and left Leibniz identities otherwise (see [8, Lemma 1]). Symmetric Leibniz algebras L are flexible, power associative and have $x^3 = 0$

for all $x \in L$ (see [6, Proposition 2.37]) and so, in a sense, are not far removed from Lie algebras.

Put $I = \langle \{x^2 : x \in L\} \rangle$. Then I is an ideal of L and L/I is a Lie algebra called the *liesation* of L. We define the following series:

$$L^1 = L, L^{k+1} = [L^k, L] \text{ and } L^{(1)} = L, L^{(k+1)} = [L^{(k)}, L^{(k)}] \text{ for all } k = 2, 3, \ldots$$

Then L is *nilpotent* (resp. *solvable*) if $L^n = 0$ (resp. $L^{(n)} = 0$) for some $n \in \mathbb{N}$. The *nilradical*, $N(L)$, (resp. *radical*, $\Gamma(L)$) is the largest nilpotent (resp. solvable) ideal of L.

Throughout, L will denote a (right) Leibniz algebra over a field F of characteristic zero unless otherwise specified. The Frattini ideal of L, $\phi(L)$, is the largest ideal of L contained in all maximal subalgebras of L. The Leibniz algebra L is called *ϕ-free* if $\phi(L) = 0$ and *elementary* if $\phi(B) = 0$ for every subalgebra B of L. Leibniz algebras all of whose nilpotent subalgebras are abelian are called *A-algebras*; Leibniz algebras L such that $\phi(B) \leq \phi(L)$ for all subalgebras B of L are called *E-algebras*. The *abelian socle*, $Asoc(L)$, of a Leibniz algebra L is the sum of its minimal abelian ideals.

A linear Lie algebra $L \leq \text{gl}(V)$ is *almost-algebraic* if L contains the nilpotent and semisimple Jordan components of its elements; an abstract Lie algebra L is then called almost-algebraic if $\text{ad}L \leq \text{gl}(L)$ is almost-algebraic. Here we are exploring whether an analogous concept to this last one can be developed for Leibniz algebras and then to determine properties and inter-relationships between these five classes of algebras analogous to those obtained by Towers and Varea in [15].

In Section 2, the concepts of an almost-reductive Leibniz algebra and of an almost-algebraic Leibniz algebra are introduced and various basic properties of them are produced. Descriptions of symmetric Leibniz algebras which are almost-algebraic, and of those with an almost-reductive radical are obtained. In addition, some analogues of the results in [15] are found for symmetric Leibniz algebras. In Section 3, the inner derivation algebra, $R(L)$, of L is defined. This is a Lie algebra and some properties of almost-reductive and almost-algebraic Leibniz algebras L are related to corresponding properties of $R(L)$. We also introduce the concept of an L-split element of a Leibniz algebra L and show that L is almost-algebraic if every element of L is L-split. It is also shown that if every element of a subalgebra B of L is L-split, then the idealiser of B in L is almost-algebraic. In the final section we determine some consequences of these results for symmetric Leibniz A-algebras.

2. Definitions and preliminary results

Definition 2.1. We call L **almost-reductive** if $L = N(L)\dot{+}\Sigma$, where Σ is a Lie algebra and $N(L)$ is a completely reducible Σ-bimodule.

Lemma 2.2. *Let L be an almost-reductive Leibniz algebra. Then $\Sigma = C \oplus S$, where S is a semisimple Lie algebra, C is an abelian Lie algebra and $R_c|_{N(L)}$ is semisimple for all $c \in C$.*

Proof. If A is an irreducible Σ-bimodule of L, we have that $[\Sigma, A] = 0$ or $[\sigma, a] = -[a, \sigma]$ for all $a \in A, \sigma \in \Sigma$, by [4, Lemma 1.9]. It follows that A is an irreducible right Σ-module of L and the result now follows from [7, Theorem 11, p.47]. □

Definition 2.3. We call L **almost-algebraic** if L/I is an almost-algebraic Lie algebra.

Theorem 2.4. *Let L be an almost-reductive Leibniz algebra. Then L is almost-algebraic.*

Proof. Let $L = N(L)\dot{+}\Sigma$ be an almost-reductive Leibniz algebra where $\Sigma = C \oplus S$, and let N/I be the nilradical of L/I. Then $N(L) \subseteq N$ and $N = N(L) + N \cap \Sigma$. Since $N \cap \Sigma$ is a solvable ideal of Σ, $N \cap \Sigma \subseteq C$. Let $C = N \cap \Sigma \oplus D$. Then $L/I = N/I\dot{+}(\Sigma' + I)/I$ where $\Sigma' = D \oplus S$ is reductive. Hence L/I is an almost-algebraic Lie algebra. □

The converse of the above result is false, as the following example shows.

Example 2.5. Let L be the four-dimensional solvable cyclic Leibniz algebra with basis a, a^2, a^3, a^4 and $[a^4, a] = a^4$. Then $I = L^2$ and L/I is trivially almost-algebraic. But $N(L) = I$ and L is not completely reducible: for example, there is no ideal A of L such that $I = A \oplus (Fa^3 + Fa^4)$.

In fact, it even fails for symmetric Leibniz algebras, as shown in the next example.

Example 2.6. Let L be the three-dimensional symmetric Leibniz algebra with basis e_1, e_2, e_3 and non-zero products $[e_1, e_2] = e_1$, $[e_2, e_1] = -e_1$, $e_2^2 = e_3$. Then $I = Fe_3$ and $L/I \cong Fe_1 + Fe_2$, which has nilradical Fe_1 and is clearly almost-algebraic. However, $N(L) = Fe_1 + Fe_3$ and L does not split over this ideal, so L is not almost-reductive.

Theorem 2.7. *Let L be a Leibniz algebra.*

(i) *If L is ϕ-free, then L is almost-reductive (and so, almost-algebraic).*

(ii) *Let L be almost-reductive. Then L is ϕ-free if and only if its nilradical is abelian.*

Proof. (i) Let L be ϕ-free. By [5, Theorem 2.4 and Corollary 2.9], we have that $L = N(L)\dot{+}V$ where V is a Lie subalgebra of L acting completely reducibly on $N(L)$ and $N(L) = \text{Asoc } L$. It follows that L is almost-reductive.

(ii) Suppose that L is almost-reductive and that $N(L)$ is abelian. Then $N(L) = \text{Asoc } L$, so L is ϕ-free by the same argument as in [11]. The converse follows from [5, Theorem 2.4 and Corollary 2.9]. □

Corollary 2.8. *Let L be an almost-reductive Leibniz algebra. Then L is ϕ-free if and only if $L = \Lambda\dot{+}I$, where $\Lambda = \Sigma\dot{+}A$ is a ϕ-free almost-algebraic Lie algebra with nilradical A, $N(L) = A \oplus I$, $[I, \Sigma] = I$ and I is a completely reducible Σ-bimodule.*

Proof. First, let L be ϕ-free. Then $L = (A_1 \oplus \ldots \oplus A_n)\dot{+}\Sigma$ where Σ is as described in Lemma 2.2 and A_i is an abelian irreducible Σ-bimodule for each $1 \leq i \leq n$. Suppose that $[a, \sigma] = -[\sigma, a]$ for all $a \in A_1 \oplus \ldots \oplus A_r$, $\sigma \in \Sigma$, but that there is an $a_i \in A_i$ and a $\sigma_i \in \Sigma$ such that $[a_i, \sigma_i] \neq -[\sigma_i, a_i]$ for each $r+1 \leq i \leq n$. Put $\Lambda = A_1 \oplus \ldots \oplus A_r\dot{+}\Sigma$. Then $[A_i, \Sigma]$ is a Σ-bimodule, and so $[A_i, \Sigma] = A_i$ or 0 for each i. Now $[\Sigma, A_i] = 0$ for $r+1 \leq i \leq n$, by [4, Lemma 1.9], so $[A_i, \Sigma] = A_i$ for $r+1 \leq i \leq n$, by the choice of r. If $a \in A_i$ and $\sigma \in \Sigma$, then $[a, \sigma] = [a, \sigma + [\sigma, a]] \in I$ for $r+1 \leq i \leq n$. It follows that $A_i = [A_i, \Sigma] \subseteq I$ for each $r+1 \leq i \leq n$. Clearly, $A_{r+1} \oplus \ldots \oplus A_n \subseteq I$ since Λ is a Lie algebra.

Now let $L = \Lambda\dot{+}I$, where $\Lambda = \Sigma\dot{+}A$ is a ϕ-free almost-algebraic Lie algebra with nilradical A, $N(L) = A \oplus I$, $[I, \Sigma] = I$ and I is a completely reducible Σ-bimodule. Then L is ϕ-free since its nilradical is abelian. □

The following result is a generalisation of [2, Theorem 6 and its Corollary].

Corollary 2.9. *Every ϕ-free Leibniz algebra in which $I \subseteq Z(L)$ is a Lie algebra; in particular, every ϕ-free symmetric Leibniz algebra is a Lie algebra.*

Proof. Let L be a ϕ-free Leibniz algebra in which $I \subseteq Z(L)$. It follows from Corollary 2.8 that $I = 0$ and so L is a Lie algebra. □

Definition 2.10. The **right centre** of a Leibniz algebra is the set $Z_r(L) = \{z \in L \mid [x, z] = 0 \text{ for all } x \in L\}$.

For any (right) Leibniz algebra L, $Z_r(L)$ is an abelian ideal of L (see [6, Proposition 2.9]). A special case of the above result is the following.

Proposition 2.11. (*cf.* [2, Theorem 6 and its Corollary]) *If* $L/Z_r(L)$ *is semisimple and* $\dim Z_r(L) = 1$*, then* L *is a Lie algebra.*

Proof. By Levi's Theorem for Leibniz algebras, $L = Z_r(L)\dot{+}S$ where S is a semisimple Lie algebra. Let $Z_r(L) = Fz$ and let $s_1, s_2 \in S$. Then $[z, s_i] = \lambda_i z$ for $i = 1, 2$ and $[z, [s_1, s_2]] = [[z, s_1], s_2] - [[z, s_2], s_1] = \lambda_1\lambda_2 z - \lambda_2\lambda_1 z = 0$. Thus $[Z_r, L] = [Z_r, S] = [Z_r, S^2] = 0$, whence $Z_r(L) = Z(L)$. It is now clear that L is a Lie algebra. Note that L is ϕ-free and $I = 0$, so this is a special case of Corollary 2.9. □

Lemma 2.12. *Let* B *be a subalgebra of a Leibniz algebra* L*. If* B *is almost-algebraic, then so is the Lie algebra* $(B + I)/I$*.*

Proof. Let J be the Leibniz kernel of B. Then B/J is almost-algebraic. But now

$$\frac{B+I}{I} \cong \frac{B}{B \cap I} \cong \frac{B/J}{(B \cap I)/J},$$

and $(B \cap I)/J$ is abelian and so is almost-algebraic. The result then follows from [1, Lemma 4.1]. □

Theorem 2.13. *Let* L *be a Leibniz algebra with radical* Γ*. Then,*

(i) *if* Γ *is almost-algebraic, then so is* L*; and*
(ii) *if* L *is almost-reductive, then so is* Γ*.*

Proof. (i) If Γ is the radical of L, Γ/I is the radical of L/I. Let Γ be almost-algebraic. Then Γ/I is almost-algebraic by Lemma 2.12. It follows from [1, Corollary 3.1] that L/I and hence L is almost-algebraic.

(ii) This is clear from Lemma 2.2. □

For Lie algebras the converse is true. However, it appears that this may not be the case even for symmetric Leibniz algebras, though examples are not easy to construct. The best that we can achieve at the moment is given by the following two results.

Theorem 2.14. *Let* L *be an almost-algebraic symmetric Leibniz algebra with radical* Γ *and nilradical* N*. Then* $L = N + \Sigma$*, where* $N \cap \Sigma = I$ *and* $\Sigma = \Gamma \cap \Sigma \oplus S$ *with* $(\Gamma \cap \Sigma)^3 = 0$*,* S *semisimple and* R_{c+I} *acting semisimply on* N/I *for all* $c \in \Gamma \cap \Sigma$*.*

Proof. We have that $L = \Gamma\dot{+}S$, where S is a semisimple Lie algebra, by Levi's Theorem for Leibniz algebras. Also, L/I is almost-reductive. Thus, $L/I = N/I\dot{+}\Sigma/I$ where Σ is a subalgebra of L, $(\Gamma \cap \Sigma)/I$ is abelian, $\Sigma/I = (\Gamma \cap \Sigma)/I \oplus (S\dot{+}I)/I$ with

S a semisimple Lie algebra and R_{c+I} acting semisimply on N/I for all $c \in \Gamma \cap \Sigma$. Hence $L = N + \Sigma$ where $N \cap \Sigma = I$ and $(\Gamma \cap \Sigma)^2 \subseteq I$, so $(\Gamma \cap \Sigma)^3 = 0$. Moreover,

$$[\Gamma \cap \Sigma, S] = [\Gamma \cap \Sigma, S^2] \subseteq [[\Gamma \cap \Sigma, S], S] \subseteq [I, S] = 0 \text{ and}$$
$$[S, \Gamma \cap \Sigma] = [S^2, \Gamma \cap \Sigma] \subseteq [S, [S, \Gamma \cap \Sigma]] + [[S, \Gamma \cap \Sigma], S]$$
$$\subseteq [S, I] + [I, S] = 0.$$

□

Theorem 2.15. *Let L be a symmetric Leibniz algebra with an almost-reductive radical Γ. Then L is as described in Theorem 2.14 above and $\Gamma = N \dot{+} C$ where C is an abelian subalgebra and $R_c \mid_N$ is semisimple for all $c \in C$.*

Proof. Suppose that Γ is almost-reductive, so that $\Gamma = N \dot{+} C$ where C is an abelian subalgebra and $R_c \mid_N$ is semisimple for all $c \in C$. Moreover, Γ is almost-algebraic by Theorem 2.4 and hence so is L by Theorem 2.13(i). The result follows. □

Note that C in the above result is just a maximal torus of Γ and $\Gamma = N \rtimes C$.

Proposition 2.16. *Let L be a Leibniz algebra.*

(i) *If L is almost-algebraic and J is an almost-algebraic ideal of L, then L/J is almost-algebraic.*
(ii) *If L is almost-reductive and J is an ideal of L with $J \subseteq \phi(L)$, then L/J is almost-reductive.*

Proof. (i) It follows from Lemma 2.12 and [1, Lemma 4.1] that $(J+I)/I$ and hence $L/(J+I) \cong (L/I)/((J+I)/I)$ is an almost-algebraic Lie algebra. But $(J+I)/J$ is the Leibniz kernel of L/J and $(L/J)/((J+I)/J) \cong L/(J+I)$. Thus L/J is almost-algebraic.
(ii) We have that $N(L/J) = N(L)/J$ as in [14, Lemma 2.3]. Let $N(L) = A_1 \oplus \ldots \oplus A_n$ where A_i is an irreducible Σ-bimodule of L. Then $A_i \cap J = 0$ or A_i for each $i = 1, \ldots, n$. Let $A_1 \cap J = \ldots = A_r \cap J = 0$, $J = A_{r+1} \oplus \ldots \oplus A_n$, so $L/J \cong (A_1 \oplus \ldots \oplus A_r) \dot{+} \Sigma$, which is almost-reductive. □

Corollary 2.17. *Let L be an almost-reductive Leibniz algebra. Then $\phi(L) = N^2$, where N is the nilradical of L.*

Proof. First, $N^2 = \phi(N) \subseteq \phi(L)$, as in [11, Theorem 6.5]. Hence $N(L/N^2) = N/N^2$, by [14, Lemma 2.3], giving that $N(L/N^2)$ is abelian. Moreover, L/N^2 is almost-reductive by Proposition 2.16 (ii) and so L/N^2 is ϕ-free by Theorem 2.7 (ii). It follows that $\phi(L) \subseteq N^2$. □

Note that the above Corollary is false if 'almost-reductive' is replaced by 'almost-algebraic', as the following example shows.

Example 2.18. Let L be as in Example 2.5. Then the only maximal subalgebras are I and $F(a-a^2)+F(a^2-a^3)+F(a^3-a^4)$ (see [9, proof of Proposition 6.1]). Hence $\phi(L)=F(a^2-a^3)+F(a^3-a^4)\neq 0=N^2$.

Once again, it is not even true if L is a symmetric Leibniz algebra.

Example 2.19. Let L be as in Example 2.6. Then $\phi(L)=Fe_3\neq 0=N^2$.

Proposition 2.20. *Let L be an almost-reductive symmetric Leibniz algebra. If every almost-algebraic subalgebra of L is ϕ-free, then L is an elementary Lie algebra.*

Proof. Since $\phi(L)=0$, we have that L is a Lie algebra by Corollary 2.9. The result now follows from [15, Proposition 2.3]. □

Proposition 2.21. *Let L be an almost-reductive symmetric Leibniz algebra. If every almost-algebraic subalgebra of L/I is ϕ-free, then $\phi(L)=N^2=I$ and L is an E-algebra.*

Proof. We have that L/I is almost-algebraic by Theorem 2.4, so $N^2=\phi(L)\subseteq I$ by Corollary 2.17. Now, if M is a maximal subalgebra of L with $Z(L)\not\subseteq M$, $L=M+Z(L)$ which gives that $L^2\subseteq M$. Hence $Z(L)\cap L^2\subseteq\phi(L)$. Thus

$$N^2\subseteq I\subseteq Z(L)\cap L^2\subseteq\phi(L)=N^2.$$

Let B be a subalgebra of L. Then $I_{L/I}((B+I)/I)$ is almost-algebraic by [1, Theorem 2.3] and so is ϕ-free by assumption. It follows that $(B+I)/I$ is ϕ-free by [11, Lemma 4.1] whence $\phi(B)\subseteq I=\phi(L)$ and L is an E-algebra. □

3. The inner derivation algebra of a Leibniz algebra

Definition 3.1. The **inner derivation algebra** of L is the set

$$R(L)=\{R_x\mid x\in L\}.$$

Note that $R(L)$ is a Lie algebra under bracket product. For every subset U of L, we will write $R_U=\{R_x\mid x\in U\}$. It is easy to check that $R_{[y,x]}=[R_x,R_y]$. To simplify notation, put $[y,_n x]=R_x^n(y)$. Then a simple induction proof yields the following.

Lemma 3.2. $R_{[y,_n x]}=(-1)^n[R_y,_{n-1}R_x]$.

Definition 3.3. For any algebra A, the **opposite algebra**, A°, has the same underlying vector space and the opposite multiplication, $(x, y) \mapsto x \star y = yx$, where juxtaposition denotes the multiplication in A.

The following is easy to check (see, for example, [6, Proposition 2.26] or [3, page 42]).

Proposition 3.4. *For any (right) Leibniz algebra L, the map $\theta : L \to R(L)^\circ : x \mapsto R_x$ is a homomorphism with kernel $Z_r(L)$, so the Lie algebra $L/Z_r(L)$ is isomorphic to $R(L)^\circ$.*

The following two lemmas are easy to see.

Lemma 3.5. *For any Lie algebra L,*

(i) *U is a subalgebra of L if and only if U° is a subalgebra of L°;*
(ii) *U is an ideal of L if and only if U° is an ideal of L°;*
(iii) *U is solvable if and only if U° is solvable;*
(iv) *U is nilpotent if and only if U° is nilpotent.*

Lemma 3.6. *For every Leibniz algebra L, we have:*

(i) *If U is a subalgebra of L, then R_U is a subalgebra of $R(L)$.*
(ii) *Every subalgebra of $R(L)$ is of the form R_U where U is a subalgebra of L.*
(iii) *If U is an ideal of L, then R_U is an ideal of $R(L)$.*
(iv) *Every ideal of $R(L)$ is of the form R_U where U is an ideal of L.*

Lemma 3.7. *Let L be a Leibniz algebra. Then*

(i) *Γ is the radical of $L \Leftrightarrow R_\Gamma$ is the radical of $R(L)$ (This is also given in* [3, page 44]*);*
(ii) *if $Z_r(L) \subseteq \phi(L)$, then N is the nilradical of $L \Leftrightarrow R_N$ is the nilradical of $R(L)$.*

Proof. (i) Clearly, $\Gamma(L/Z_r(L)) = \Gamma/Z_r(L)$ and so $\Gamma/Z_r(L) \cong \Gamma(R(L)^\circ) = \Gamma(R(L))$. Moreover, $\theta\,|_\Gamma$ is a homomorphism from Γ onto R_Γ whence the result.
(ii) Clearly, $Z_r(L) \subseteq N$ and so $N/Z_r(L) \subseteq N(L/Z_r(L)) = K/Z_r(L)$, say. But K is nilpotent, by [4, Theorem 5.5], so $N/Z_r(L) = N(L/Z_r(L))$. The proof now follows in similar manner to (i). □

Proposition 3.8. *If the Leibniz algebra L is almost-algebraic, then so is the Lie algebra $R(L)$.*

Proof. Let L be almost-algebraic. Then L/I is almost-algebraic. Now $L/Z_r(L) \cong (L/I)/(Z_r(L)/I)$ and $Z_r(L)/I$ is abelian and so is almost-algebraic. It follows that $L/Z_r(L)$ is almost-algebraic, by [1, Lemma 4.1]. The result follows from Proposition 3.4. □

Definition 3.9. If B is a subalgebra of L, the **idealiser of** B **in** L, $I_L(B) = \{x \in L \mid [x,b],[b,x] \in B \text{ for all } b \in B\}$.

Corollary 3.10. *Let B be a subalgebra of an almost-algebraic Leibniz algebra L. Then the idealiser, $I_{R(L)}(R_B)$, of R_B in $R(L)$ is an almost-algebraic Lie algebra.*

Proof. This follows from Proposition 3.8 and [1, Theorem 2.3]. □

Definition 3.11. The element $x \in L$ is called ***L*-split** if there exist elements $s, n \in L$ such that $R_x = R_s + R_n$ is the decomposition of R_x into its semisimple and nilpotent parts.

Proposition 3.12. *If every element of the Leibniz algebra L is L-split, then L is almost-algebraic.*

Proof. Let $x \in L$. Then $R_{x+I} = R_{s+I} + R_{n+I}$ if $R_x = R_s + R_n$ and R_{s+I}, R_{n+I} are the semisimple and nilpotent parts of R_{x+I}, so the result follows from [1, Theorem 2]. □

The following result is now proved as in [1, Theorem 2.3].

Proposition 3.13. *Let B be a subalgebra of a Leibniz algebra L in which every element is L-split. Then the idealiser, $I_L(B)$, of B in L is almost-algebraic.*

Proof. Let $J = I_L(B)$. Since $R_L(B)$ leaves B invariant, so does its algebraic hull. In particular, if $x \in J$, both the semisimple and nilpotent parts of $R_L(x)$ leave B invariant. Hence, every element of J is J-split and so by Proposition 3.12, J is almost-algebraic. □

4. Leibniz A-algebras

Proposition 4.1. *Let L be a Lie A-algebra and let K be an ideal of L with $K \subseteq Z(L)$. If L/K is almost-algebraic, then so is L.*

Proof. Let L/K be almost-algebraic and let R be the radical of L. Then R/K is almost-algebraic by [1, Corollary 3.1] and so $\phi(R/K) = 0$ by [15, Lemma 2.1 (ii)]. Hence $\phi(R) \subseteq K \subseteq Z(R)$ by [11, Corollary 4.4]. It follows that $\phi(R) \subseteq Z(R) \cap R^2 = 0$, by [12, Theorem 3.3] since R is an A-algebra. Thus R is almost-algebraic, by [15, Proposition 2.1] whence so is L by [1, Corollary 3.1] again. □

Corollary 4.2. *Let L be a Leibniz A-algebra and let K be an ideal of L with $K \subseteq Z(L)$. If L/K is almost-algebraic, then so is L.*

Proof. Let L/K be almost-algebraic. Then

$$\frac{L/I}{(I+K)/I} \cong \frac{L/K}{(I+K)/K},$$

which is almost-algebraic by Proposition 2.16. Moreover, $(I+K)/I \subseteq Z(L/I)$ and L/I is a Lie A-algebra by [12, Lemma 2], so L/I is almost-algebraic by Proposition 4.1. Hence L is almost-algebraic. □

Lemma 4.3. *Let L be an almost-reductive symmetric Leibniz A-algebra. Then L is a Lie algebra.*

Proof. Since L is almost-reductive, $\phi(L) = N^2$ by Corollary 2.17 and $N^2 = 0$ since L is an A-algebra. Hence L is a ϕ-free symmetric Leibniz algebra and so is a Lie algebra by Corollary 2.9. □

Lemma 4.4. *If L is a symmetric Leibniz A-algebra, then L/I is a Lie A-algebra.*

Proof. If K/I is a nilpotent subalgebra of L/I, $K^r \subseteq I$ for some $r > 0$, whence $K^{r+1} = 0$. It follows that K is nilpotent and thus abelian. □

Theorem 4.5. *Let L be a symmetric Leibniz A-algebra. Then L is an almost-reductive algebra if and only if it is an elementary Lie algebra.*

Proof. ($\Rightarrow$) Let L be an almost-reductive symmetric Leibniz A-algebra. Then L is a Lie algebra by Lemma 4.3. It now follows that it is elementary by [15, Theorem 2.4].
($\Leftarrow$) The converse follows from [15, Theorem 2.4]. □

Corollary 4.6. *Let L be a symmetric Leibniz A-algebra with radical Γ. If Γ is ϕ-free, then L is an elementary Lie algebra.*

Proof. Assume that Γ is ϕ-free. Then Γ is almost-reductive by Theorem 2.7 (i). It follows that L is as described in Theorem 2.15. Moreover, $(\Gamma \cap \Sigma)^2 = 0$ since L is an A-algebra and if $\sigma \in \Gamma \cap \Sigma$, $\sigma = n + c$ for some $n \in N$, $c \in C$. Hence $[n', \sigma] = [n', c]$ for all $n' \in N$ and so $R_\sigma \mid_N$ is semisimple. It follows that L is almost-reductive and hence that L is an elementary Lie algebra by Theorem 4.5. □

Corollary 4.7. *Let L be an almost-reductive symmetric Leibniz A-algebra. Then L splits over each of its ideals.*

Proof. This follows from Lemma 4.3 and [15, Corollary 2.6]. □

Proposition 4.8. *Let L be a Leibniz algebra over any field. Then L is an E-algebra if and only if $L/\phi(L)$ is elementary.*

Proof. The proof is the same as for the Lie case in [10, Proposition 2]. □

Proposition 4.9. *Let L be a symmetric Leibniz A-algebra. Then L is an E-algebra.*

Proof. Let L be a Leibniz A-algebra. Then $L/\phi(L)$ is an A-algebra by [13, Lemma 2]. But $L/\phi(L)$ is ϕ-free and so is almost-reductive by Corollary 2.17 (i). Hence $L/\phi(L)$ is elementary by Theorem 4.5 and so L is an E-algebra by Proposition 4.8. □

Acknowledgement. The author would like to thank the referees for their careful reading of the paper.

Declarations

Disclosure statement: The author reports that there are no competing interests to declare.

References

[1] L. Auslander and J. Brezin, *Almost algebraic Lie algebras*, J. Algebra, 8 (1968), 295-313.

[2] Sh. A. Ayupov and B. A. Omirov, *On Leibniz algebras*, Algebra and operator theory, Tashkent (1997), Kluwer Academic Publishers, (1998), 1-12.

[3] Sh. A. Ayupov, B. Omirov and I. Rakhimov, Leibniz Algebras-Structure and Classification, CRC Press, Boca Raton, 2020.

[4] D. W. Barnes, *Some theorems on Leibniz algebras*, Comm. Algebra, 39(7) (2011), 2463-2472.

[5] C. Batten, L. Bosko-Dunbar, A. Hedges, J. T. Hird, K. Stagg and E. Stitzinger, *A Frattini theory for Leibniz algebras*, Comm. Algebra, 41(4) (2013), 1547-1557.

[6] J. Feldvoss, *Leibniz algebras as non-associative algebras*, Nonassociative mathematics and its applications, Contemp. Math., 721 (2019), 115-149.

[7] N. Jacobson, Lie Algebras, Interscience Tracts in Pure and Applied Mathematics, Interscience, New York-London, 1962.

[8] M. Jibladze and T. Pirashvili, *Lie theory for symmetric Leibniz algebras*, J. Homotopy Relat. Struct., 15(1) (2020), 167-183.

[9] S. Siciliano and D. A. Towers, *On the subalgebra lattice of a Leibniz algebra*, Comm. Algebra, 50(1) (2022), 255-267.

[10] E. L. Stitzinger, *Frattini subalgebras of a class of solvable Lie algebras*, Pacific J. Math., 34 (1970), 177-182.

[11] D. A. Towers, *A Frattini theory for algebras*, Proc. London Math. Soc. (3), 27 (1973), 440-462.

[12] D. A. Towers, *Solvable Lie A-algebras*, J. Algebra, 340 (2011), 1-12.

[13] D. A. Towers, *Leibniz A-algebras*, Commun. Math., 28(2) (2020), 103-121.

[14] D. A. Towers, *On the nilradical of a Leibniz algebra*, Comm. Algebra, 49(10) (2021), 4345-4347.

[15] D. A. Towers and V. R. Varea, *Further results on elementary Lie algebras and Lie A-algebras*, Comm. Algebra, 41(4) (2013), 1432-1441.

David A. Towers
Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF, England
e-mail: d.towers@lancaster.ac.uk