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Research Article

Van der Pol Oscillator, its Control Theoretical Analysis Using Dissipative Canonical Equations and its Lyapunov Function

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ABSTRACT

In this research paper, beginning with the Lagrangian and generalized velocity proportional (Rayleigh) dissipation function of a physical/engineering system, the Lagrange-dissipative model ({L,D}-model briefly) of the system is initially developed. Upon satisfying the prerequisite condition for a Legendre transform, the Hamiltonian function can be obtained. With the Hamiltonian function and the generalized velocity proportional (Rayleigh) dissipation function, dissipative canonical equations can be derived. Using these dissipative canonical equations as the state-space equations of the system allows for the investigation of observability, controllability, and stability properties. In addition to the equilibrium (or critical or fixed) points of the system, stability properties can also be verified through a Lyapunov function as a residual energy function (REF). Since the proposed method is valid for both linear and nonlinear systems, it has been applied to the Van der Pol oscillator/equation.

Keywords: *Observability, Controllability and stability of Van der Pol oscillator/equation, Lyapunov function for stability of Van der Pol oscillator/equation.*

Van der Pol Osilatörü, Yitimli Kanonik Denklemler Kullanarak Kontrol Teorik Analizi ve Lyapunov Fonksiyonu

ÖZ

Bu araştırma makalesinde, bir fiziksel/mühendislik sisteminin Lagrangian ve genelleştirilmiş hız orantılı (Rayleigh) yitim fonksiyonu ile başlayarak, öncelikle bu sistemin Lagrange-dissipatif modeli (kısaca {L,D}-modeli) oluşturulmuştur. Legendre dönüşümü için gerekli koşulunun sağlanmasıyla Hamilton fonksiyonu elde edilebilir. Hamilton fonksiyonu ve genelleştirilmiş hız orantılı (Rayleigh) yitim fonksiyonuyla, yitimli kanonik denklemler elde edilebilir. Yitimli kanonik denklemlerin sistemin durum uzayı denklemleri olarak kullanılması; gözlemlenebilirlik, kontrol edilebilirlik ve kararlılık özelliklerinin araştırılması için kullanılır. Sistemin denge (veya kritik veya sabit) noktalarının yanı sıra, sistemin kararlılık özellikleri, artık enerji fonksiyonu (REF) olarak bir Lyapunov fonksiyonu aracılığıyla da doğrulanabilir. Önerilen yöntem doğrusal ve doğrusal olmayan sistemler için de geçerli olduğundan, yöntem Van der Pol osilatörüne/denklemine uygulanmıştır.

Anahtar Kelimeler : *Van der Pol denklemi, Gözlemlenebilirlik, Denetlenebilirlik, Kararlılık, Van der Pol denklemi kararlılığı için Lyapunov fonksiyonu*.

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I. INTRODUCTION

A non-conservative oscillator with a nonlinear damping was examined by Balthasar van der Pol in 1920, where the related equation and oscillation are called as Van der Pol equation and Van der Pol oscillation respectively.

Dynamical systems described by Van der Pol equations occur in various fields of physical and engineering sciences. These cover also biological and geological/geophysical sciences.

Some of the first articles on Van der Pol equation/oscillator are [1-2-3-4]. Also, different books available on nonlinear dynamics and chaos include Van der Pol equation/oscillator like [5-6-7-8]. The equation/oscillator has different applications in different sciences. The equation was also used even in seismology to model the two plates in a geological fault as in [9].

Physical and mathematical foundations of Lagrangians and Hamiltonians are contained in [10-11-12], where [12] is a very rare source including conditions for a Legendre transform. Modelling a physical or an engineering dynamical system by means of a Lagrangian L, a generalized velocity proportional (Rayleigh) dissipation function D and a Hamiltonian H depending on tensorial variables in contravariant and covariant forms was shown in [13], which also includes the dissipative canonical equations. In different tensorial forms, the extended Hamiltonians to obtain canonical equations in case of dissipative systems directly are given in [14] which contains also higher order {L,D}-models. Therefore, higher order Lagrangians, dissipation functions and nonconservative Hamiltonians are introduced in this study. The method to obtain a Lyapunovfunction as residual energy function in a systematic way was introduced in the reference [15].

In general, observability and controllability of nonlinear systems are given in [16]. Mathematically, Lie derivatives for observability and Lie brackets for controllability for linear and nonlinear systems are explained in [17] in detail. And the approaches convert to the usual linear case of observability and controllability matrices if the system is a linear one.

Physical system analysis related to observability, controllability and stability using its equations of generalized motion and canonical equations both in dissipative forms are studied in [18]. In the article[19], Duffing oscillator/equation was analyzed related the concepts given using equations of generalized motion including dissipation which can also be applied here.

This time, dissipative canonical equations will rather be used in this study. But observability and controllability analysis of Van der Pol oscillator/equation in general lacks in the literature. And this analyze is performed first time using classical mechanical approach including dissipation with statespace method as far as known by us. Moreover, Lyapunov's direct (or second) method is applied as residual energy function, and this is not considered for Van der Pol oscillator/equation in any other literature before which provides more precise results from the point of physics.

Lagrange-dissipative model of the Van der Pol oscillator/equation was derived here firstly. Provided that the condition for a Legendre transform is fulfilled, the Hamiltonian of the system and the dissipative canonical equations can be obtained which differs than the usual approaches with no dissipation. In this context, the method proposed is explained in detail. As far as we know, Van der Pol oscillator is analyzed for the first time by means of observability, controllability using the method and stability using Lyapunov function which is valid for linear and nonlinear cases. Lyapunov function as a residual energy function obtained as sum of Hamiltonian and negative form of dissipative energy will be used for the stability analysis. Finally, the conclusions which were established was stated.

II.OBTAINING THE {L,D}-MODEL AND THE EQUATIONS OF GENERALIZED MOTION

The Lagrangian in its most general form is

$$L(\dot{q},q,t) = T(\dot{q},t) - U(\dot{q},q,t) \tag{1}$$

$T(\dot{q},t)$ is the kinetic and $U(\dot{q},q,t)$is the potential energy parts of the Lagrangian. And the generalized velocity proportional (Rayleigh) dissipation function has the following form:

$$\frac{\partial D}{\partial \dot{q}_k} = R_k \dot{q}_k \Rightarrow D(\dot{q}_k) = \frac{R_k}{2} \dot{q}_k^2 \ ; R_k = const.|_{\dot{q}_k} \tag{2}$$

where k presents the degree of freedom. The extended Euler-Lagrange differential equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = F_k \ ; k = 1,\cdots,\mathrm{f} \tag{3}$$

The external generalized force F_k may be included in the generalized velocity proportional (Rayleigh) dissipation function $D(\dot{q}_k)$ in form $-\dot{q}_k F_k$ as negative loss or in the Lagrangian as negative potential in form of $-q_k F_k$ to obtain the equations of generalized motion using extended Euler-Lagrange differential equation.

The second-degree differential equation of the generalized motion obtained through extended Euler-Lagrange differential equation has the most general form given below:

$$M_k \ddot{q}_k + R_k \dot{q}_k + K_k q_k + C_k = F_k \Rightarrow \ddot{q}_k + \frac{R_k}{M_k}\dot{q}_k + \frac{K_k}{M_k} q_k + \frac{C_k}{M_k} = \frac{F_k}{M_k} \ ; \ M_k \neq 0 \tag{4}$$

where M_k is the generalized mass, R_k is the generalized resistive and K_k is the generalized capacitive elements respectively, C_k is the constant when available.

Equations of generalized motion can be written in state space form to analyze the generalized motion related to observability, controllability, and stability. But we here, will prefer to analyze the system related the concepts using state space form of dissipative canonical equations since the Hamiltonian is needed to construct a Lyapunov function in form of a residual energy function.

III.LEGENDRE TRANSFORM, HAMILTONIAN H AND OBTAINING DISSIPATIVE CANONICAL EQUATIONS

Prerequisite that the condition for a Legendre transform given as

$$\left|\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_k}\right| = \left|\frac{\partial p_r}{\partial \dot{q}_k}\right| \neq 0 \ ; r,k = 1,\cdots,\mathrm{f} \tag{5}$$

İs fulfilled, one can obtain the Hamiltonian *H* related to the Lagrangian *L* given as

$$H(p_k, q_k, t) = \sum_{k=1}^{f} p_k \dot{q}_k - L(\dot{q}_k, q_k, t) \tag{6}$$

The canonical equations are:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \;, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \tag{7}$$

The dissipative canonical equations are in what follows:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \;, \quad \dot{q}_k = \frac{\partial H}{\partial p_k} \;; \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} - \frac{\partial D}{\partial \dot{q}_k} \tag{8}$$

Prerequisite that the external generalized force(s) are not included in any potential or dissipation function in any form, the dissipative canonical equations in general can be written in matrix form as below:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} q_k \\ p_k \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \frac{\partial H}{\partial p_k} \\ -\frac{\partial H}{\partial q_k} - \frac{\partial D}{\partial \dot{q}_k} \end{bmatrix}}_{f(q)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} F_k}_{g(q)u} \;; \quad \underbrace{y = q_k}_{h(q)} \tag{9}$$

IV. OBSERVABILITY USING THE DISSIPATIVE CANONICAL EQUATIONS

For a physical system, observability is proven using Lie derivatives of $\boldsymbol{h}$ with respect to $\boldsymbol{f}$

$$L_f^0 \boldsymbol{h} = \boldsymbol{h}(\boldsymbol{q}) \;; \; L_f^1 \boldsymbol{h} = \nabla \boldsymbol{h}(\boldsymbol{q}).\boldsymbol{f}(\boldsymbol{q}) \tag{10}$$

where the first Lie derivative is as follows:

$$L_f^1 \boldsymbol{h} = \underbrace{\begin{bmatrix} \frac{\partial h}{\partial p_k} & \frac{\partial h}{\partial q_k} \end{bmatrix}}_{\nabla \boldsymbol{h}(\boldsymbol{q})} \underbrace{\begin{bmatrix} f_1(p_k, q_k) \\ f_2(p_k, q_k) \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{q})} \tag{11}$$

The related vector of all Lie derivatives $\boldsymbol{G}$ and the observability matrix $\boldsymbol{O}$ that is the multiplication of the matrix **O** with the operator ∇ have the forms given below:

$$\boldsymbol{G} = \begin{bmatrix} L_f^0 \boldsymbol{h} \\ L_f^1 \boldsymbol{h} \end{bmatrix} \Rightarrow \boldsymbol{O} = \nabla \boldsymbol{G} = \begin{bmatrix} \frac{\partial L_f^0 \boldsymbol{h}}{\partial p_k} & \frac{\partial L_f^0 \boldsymbol{h}}{\partial q_k} \\ \frac{\partial L_f^1 \boldsymbol{h}}{\partial p_k} & \frac{\partial L_f^1 \boldsymbol{h}}{\partial q_k} \end{bmatrix} \tag{12}$$

For such a physical system to be observable (locally, if the system is a nonlinear one) at a point q_0, the rank of the observability matrix $\boldsymbol{O}$ must be $n = 2$.

V. CONTROLLABILITY USING THE DISSIPATIVE CANONICAL EQUATIONS

With the Lie bracket given below

$$[\boldsymbol{f},\boldsymbol{g}] = \begin{bmatrix} \frac{\partial g}{\partial p_k} & \frac{\partial g}{\partial q_k} \end{bmatrix} \boldsymbol{f} - \begin{bmatrix} \frac{\partial f}{\partial p_k} & \frac{\partial f}{\partial q_k} \end{bmatrix} \boldsymbol{g} \tag{13}$$

controllability of a physical system modelled by the Hamiltonian *H* together with dissipation function *D*) can be proven, if the controllability matrix $\boldsymbol{Q} = [\boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}]]$has the rank $n = 2$.

VI. EQUILIBRIUM POINTS AND THE STABILITY USING THE EIGENVALUES AND THE LYAPUNOV FUNCTION IN FORM OF A RESIDUAL ENERGY FUNCTION

Equilibrium points for dissipative Hamiltonian systems are obtained using the term below:

$$(\dot{q}_k; \dot{p}_k) = \left(\frac{\partial H}{\partial p_k}; -\frac{\partial H}{\partial q_k} - \frac{\partial D}{\partial \dot{q}_k}\right) = (0\,;0) \tag{14}$$

The Hamiltonian leads to a system of 2f differential equations of order one compared to the f differential equations of order two obtained through extended Euler-Lagrange differential equations. Its state space form is given in the following:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} q_k \\ p_k \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & \frac{1}{M_k} \\ -\frac{\partial H}{\partial q_k}\frac{1}{q_k} & -\frac{\partial D}{\partial \dot{q}_k}\frac{1}{p_k} \end{bmatrix}}_{[J]} \underbrace{\begin{bmatrix} q_k \\ p_k \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} F_k}_{g(q)u}; M_k, q_k, p_k \neq 0 \quad ; \quad \underbrace{y = q_k}_{h(q)} \tag{15}$$

where $x = [q_k, p_k]^T$ is the vector of the canonical variables, *T* stands for transpose and $[J]$ is the Jacobian of the matrix $[f(q)]$. This is valid for linear and also, for nonlinear systems if the nonlinear terms are linearized in the matrix $[f(q)]$ and written in form of a Jacobian matrix. And the eigenvalues of characteristic equation of the Jacobian matrix are given by the equation

$$c_k(\lambda) = |\lambda[I] - [J]| = \begin{vmatrix} \lambda & -\frac{1}{M_k} \\ \frac{\partial H}{\partial q_k}\frac{1}{q_k} & \lambda + \frac{\partial D}{\partial \dot{q}_k}\frac{1}{p_k} \end{vmatrix} = \lambda^2 + \lambda\frac{\partial D}{\partial \dot{q}_k}\frac{1}{p_k} + \frac{1}{M_k}\frac{\partial H}{\partial q_k}\frac{1}{q_k} = 0 \tag{16}$$

The system is (asymptotically) stable if

$$\left.\begin{aligned} \lambda_1.\lambda_2 &= \frac{1}{M_k}\frac{\partial H}{\partial q_k}\frac{1}{q_k} > 0 \Rightarrow \frac{\partial H}{\partial q_k} > 0 \\ \lambda_1 + \lambda_2 &= -\frac{\partial D}{\partial \dot{q}_k}\frac{1}{p_k} < 0 \Rightarrow -\frac{\partial D}{\partial \dot{q}_k} < 0 \end{aligned}\right\} \Rightarrow \lambda_1 < 0,\ \lambda_2 < 0; \lambda_1 \neq \lambda_2 \tag{17}$$

and marginally stable in case of a lossless system, when the roots are in complex conjugate form:

$$\frac{\partial D}{\partial \dot{q}_k} = 0\,; \frac{1}{M_k}\frac{\partial H}{\partial q_k}\frac{1}{q_k} > 0 \Rightarrow \frac{\partial H}{\partial q_k} > 0 \Rightarrow \lambda_{1,2} = \pm j\sqrt{\frac{1}{M_k}\frac{\partial H}{\partial q_k}\frac{1}{q_k}} \quad (18)$$

When the Hamiltonian $H(p_k, q_k)$ together with the generalized velocity proportional (Rayleigh) dissipation function $D(\dot{q}_k)$ is known, then the system can be analyzed for observability, controllability and stability.

If the Jacobian matrix $[J]$covers nonlinear terms through which the eigenvalues can not be found, the Lyapunov function in form of residual energy function can be used for stability.

Since the REF is defined as

$$\mathrm{H} = H - \int \left[\sum_{k=1}^{f} \frac{\partial D(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_f)}{\partial \dot{q}_k}\dot{q}_k\right] dt \quad ; \quad 0 < \mathrm{H} < \infty \; \forall\, t \in \mathbb{R}_0^+ \quad (19)$$

and the first-time derivative of the REF, is total power, which reads

$$\frac{d\mathrm{H}}{dt} = -\sum_{k=1}^{f} \frac{\partial D(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_f)}{\partial \dot{q}_k}\dot{q}_k \quad ; \quad 0 < \mathrm{H} < \infty \; \forall\, t \in \mathbb{R}_0^+ \quad (20)$$

VII. OBSERVABILITY, CONTROLLABILITY AND STABILITY ANALYSIS OF VAN DER POL OSCILLATOR USING THE DISSIPATIVE CANONICAL EQUATIONS

A. OBTAINING the {L,D}-MODEL and the EQUATIONS of GENERALIZED MOTION of VAN DER POL OSCILLATOR

In its most general form, the Van der Pol oscillator with a nonlinear damping is a non-conservative dynamical system and the dynamics of the system is governed by the differential equation:

$$\ddot{q} + \mu(q^2 - 1)\dot{q} + q = F \quad (21)$$

which is an important special case of the *Liénard equation*, where the variable *q* is the time dependent generalized coordinate, *F* is the external generalized force and the scalar μ is the parameter indicating nonlinearity and also the strength of the damping. Accordingly, if $\mu = 0$, that is no damping, then the case converts to harmonic oscillator which is always conservative. Other cases are:

$$\mu > 0 \Rightarrow \begin{cases} q^2 - 1 > 0 \Rightarrow |q| > 1: \textit{damping is positive} \\ q^2 - 1 < 0 \Rightarrow |q| < 1: \textit{damping is negative} \end{cases}$$
$$\mu < 0 \Rightarrow \begin{cases} q^2 - 1 > 0 \Rightarrow |q| > 1: \textit{damping is negative} \\ q^2 - 1 < 0 \Rightarrow |q| < 1: \textit{damping is positive} \end{cases} \quad (22)$$

Here, we will be interested of positive damping factor μ which will deliver positive or negative damping depending on $|q| > 1$ or $|q| < 1$, where $\mu = 0$ means the non-dissipative case.

In our case, through comparing one obtains

$$\left.\begin{array}{l}\ddot{q}+\mu(q^2-1)\dot{q}+q=F\\ \ddot{q}+\frac{R}{M}\dot{q}+\frac{K}{M}q+\frac{C}{M}=\frac{F}{M}\end{array}\right\}\begin{cases}\frac{F}{M}=F\Rightarrow M=1\,;\frac{R}{M}=R=\mu(q^2-1)\\ \frac{C}{M}=0\Rightarrow C=0\,;\frac{K}{M}=1\Rightarrow K=M=1\end{cases} \tag{23}$$

The generalized velocity proportional (Rayleigh) dissipation function in this case is:

$$\frac{\partial D(\dot{q},q)}{\partial \dot{q}}=\mu(q^2-1)\dot{q}\Rightarrow D(\dot{q},q)=\frac{\mu(q^2-1)}{2}\dot{q}^2 \tag{24}$$

The related $\{L,D\}$-model with an autonomous Lagrangian L and the related momentum is

$$\{L,D\}=\begin{cases}L=\underbrace{\frac{\dot{q}^2}{2}}_{T(\dot{q})}-\underbrace{\frac{q^2}{2}}_{U(q)}\Rightarrow p=\dot{q}\\ D(q,\dot{q})=\frac{\mu(q^2-1)}{2}\dot{q}^2\end{cases} \tag{25}$$

Here, the generalized velocity proportional (Rayleigh) dissipation function D does not only depend on generalized velocity $\dot{q}$ in form of $D=D(\dot{q})$ but also generalized coordinate in form of $D=D(\dot{q},q)$.

B. LEGENDRE TRANSFORM, HAMILTONIAN H, the EXTENDED HAMILTONIAN H OBTAINING DISSIPATIVE CANONICAL EQUATIONS of VAN DER POL OSCILLATOR and THEIR MATRIX EQUATION FORM

The condition for a Legendre transform is already fulfilled since:

$$\left|\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_k}\right|=\left|\frac{\partial p_r}{\partial \dot{q}_k}\right|=1\neq 0\ \ ;\ r,k=1 \tag{26}$$

Thus, the related Hamiltonian reads:

$$H(p,q)=\underbrace{\frac{p^2}{2}}_{T(p)}+\underbrace{\frac{q^2}{2}}_{U(q)} \tag{27}$$

The extended Hamiltonian cannot be used here to obtain the dissipative canonical equations directly as the generalized velocity proportional (Rayleigh) dissipation function has the form of $D=D(\dot{q},q)$ and not $D=D(\dot{q})$. And the dissipative canonical equations for this system are

$$\begin{aligned}\dot{q}&=\frac{\partial H}{\partial p}=p\\ \dot{p}&=-\frac{\partial H}{\partial q}-\frac{\partial D}{\partial \dot{q}}=-q-\mu\dot{q}(q^2-1)=-q-\mu p(q^2-1)\end{aligned} \tag{28}$$

The dissipative canonical equations in state space form for Van der Pol oscillator can be rewritten in form of a matrix equation including generalized external force as follows:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} p \\ -q - \mu p(q^2 - 1) \end{bmatrix}}_{f(q)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} F}_{g(q)u} \; ; \; \underbrace{y = q}_{h(q)} \tag{29}$$

C. OBSERVABILITY USING the DISSIPATIVE CANONICAL EQUATIONS of the VAN DER POL OSCILLATOR

The Lie derivatives of $\boldsymbol{h}$ with respect to $\boldsymbol{f}$ are

$$L_f^0 \boldsymbol{h} = \boldsymbol{h}(\boldsymbol{q}) = q \; ; \; \nabla \boldsymbol{h}(\boldsymbol{q}) = [0 \quad 1] \tag{30}$$

Accordingly;

$$L_f^1 \boldsymbol{h} = [0 \quad 1]\begin{bmatrix} p \\ -q - \mu p(q^2 - 1) \end{bmatrix} = -q - \mu p(q^2 - 1) \tag{31}$$

The vector of all Lie derivatives and its multiplication with ∇ are for the case is

$$\boldsymbol{G} = [L_f^0 \boldsymbol{h} \quad L_f^1 \boldsymbol{h}] \Rightarrow \nabla \boldsymbol{G} = \begin{bmatrix} 0 & 1 \\ -\mu(q^2 - 1) & -1 - 2\mu pq \end{bmatrix} \tag{32}$$

And the determinant of which has rank two, when

$$\mu(q^2 - 1) \neq 0 \Rightarrow q \neq \pm 1 \tag{33}$$

i.e. the system is (locally) observable everywhere except the points $|q| = 1$.

D. CONTROLLABDLDTY USING THE DISSIPATIVE CANONICAL EQUATIONS of the VAN DER POL OSCILLATOR

Calculating the Lie brackets yields here:

$$[\boldsymbol{f}, \boldsymbol{g}] = [0 \quad 0]\begin{bmatrix} p \\ -q - \mu p(q^2 - 1) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -\mu(q^2 - 1) & -1 - 2\mu pq \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 + 2\mu pq \end{bmatrix} \tag{34}$$

Hence the controllability matrix $\boldsymbol{Q}$ is in the form

$$\boldsymbol{Q} = [\boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}]] = \begin{bmatrix} 0 & 0 \\ 1 & 1 + 2\mu pq \end{bmatrix} \tag{35}$$

$\boldsymbol{Q}$ has rank zero, implying that, this system is not controllable.

E. EQUILIBRIUM POINTS and the STABILITY USING the EIGENVALUES of the VAN DER POL OSCILLATOR and its LYAPUNOV FUNCTION in FORM of a RESIDUAL ENERGY FUNCTION

Equilibria condition for such a dissipative Hamiltonian system in general is

$$(\dot{q}; \dot{p}) = \left(\frac{\partial H}{\partial p}; -\frac{\partial H}{\partial q} - \frac{\partial D}{\partial \dot{q}}\right) = (0; 0) \tag{36}$$

Therefore, equilibrium points are:

$$\begin{aligned} &\dot{q} = 0 \Rightarrow p = 0 \\ &\dot{p} = -q - \mu p(q^2 - 1) = 0 \Rightarrow p = -\frac{q}{\mu(q^2 - 1)} = 0 \Rightarrow q = 0 \end{aligned} \quad (37)$$

The state space equation related the dissipative canonical equations of Van der Pol equation can be written as:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -\mu(q^2 - 1) \end{bmatrix}}_{[J]} \underbrace{\begin{bmatrix} q \\ p \end{bmatrix}}_{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F \quad ; \quad \underbrace{y = q}_{h(q)} \quad (38)$$

Using Jacobian matrix $[J]$ of the system and its eigenvalues, stability of the equilibrium points can be understood better. With the characteristic equation, results can be obtained as follows:

$$|\lambda[\mathrm{I}] - [J]| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + \mu(q^2 - 1) \end{vmatrix} = \lambda^2 + \lambda\mu(q^2 - 1) + 1 = 0 \quad (39)$$

The (asymptotic) stability is proven when

$$\left.\begin{aligned} \lambda_1\lambda_2 = 1 \\ \lambda_1 + \lambda_2 = -\mu(q^2 - 1) < 0 \Rightarrow |q| > 1 \end{aligned}\right\} \Rightarrow \lambda_1 < 0 \; ; \; \lambda_2 < 0 \quad (40)$$

The characteristic equation for the equilibrium point $q = 0$ is

$$\lambda^2 + \lambda\mu(q^2 - 1) + 1\big|_{q=0} = \lambda^2 - \lambda\mu + 1 = 0 \Rightarrow \begin{Bmatrix} \lambda_1\lambda_2 = 1 \\ \lambda_1 + \lambda_2 = \mu \end{Bmatrix} \Rightarrow \lambda_1 > 0 \, ; \, \lambda_2 > 0 \quad (41)$$

Thus, the equilibrium point $q = 0$ is unstable.
Lyapunov function in form of a residual energy function, that must be positive definite, is

$$\mathrm{H} = \frac{p^2}{2} + \frac{q^2}{2} - \int \mu(q^2 - 1)\dot{q}^2 \; dt \quad (42)$$

with the first-time derivative

$$\dot{\mathrm{H}}(p_k, q^k) = -\mu(q^2 - 1)\dot{q}^2 \quad (43)$$

which must be in negative semidefinite form to ensure stability and fulfill the following condition:

$$-\mu(q^2 - 1)\dot{q}^2 \leq 0 \quad (44)$$

where equality means marginal while less means asymptotic stability. Accordingly, one obtains the result satisfying the Eq.(44)

$$\dot{q} \in \mathbb{R} \Rightarrow \dot{q}^2 \geq 0 \;\; ; \; q^2 - 1 \geq 0 \Rightarrow |q| \geq 1 \quad (45)$$

And as such the regions with border lines $|q| \geq 1$ in phase space are stable regions while the region $|q| < 1$ is unstable.

VIII. CONCLUSIONS

It was demonstrated that for a Van der Pol oscillator/equation as a nonlinear system, when {L,D}-model and thus Hamiltonian are known, then the system can be analyzed by means of observability, controllability and stability using the approach which was performed for the first time in order to Van der pol oscillator/equation in general here. Moreover, stability analysis can also be performed using Lyapunov function as residual energy function. As can be seen that this kind of Lyapunov function is constructed using Hamiltonian and dissipative function together for linear and also nonlinear systems. This kind of analysis method is valid for every linear and nonlinear system where kinetic, potential energies and (generalized velocity proportional) Rayleigh dissipation functions are involved. It can also be applied specially to coupled engineering/physical systems, where different physical quantities are available, with no need to convert the physical quantities to the other.

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