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Research Article

ON SOME INEQUALITIES FOR EXPONENTIALLY WEIGHTED FRACTIONAL HARDY OPERATORS WITH ∆-INTEGRAL CALCULUS

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Abstract: *Dynamic equations, inequalities, and operators are the indispensable cornerstones of harmonic analysis and time-scale calculus. Undoubtedly, one of the most important of these operators and inequalities is the Hardy operator and inequality. Because especially when we say variable exponent Lebesgue space, the first thing that comes to our mind is the Hardy operator. We know that the topics in question have many applications in different scientific fields. In this paper, some inequalities will be proved for variable exponentially weighted Hardy operators with ∆-integral calculus.*

Keywords: *Hardy inequality, Variable exponent, Weight function, Time scale.*

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1. Introduction

The variable exponent Lebesgue space was first revealed by [1]. However, this space is based on the paper of [2] together with applications to modeling electrorheological fluids [3]. Many operators and inequalities have been studied in the variable exponent Lebesgue space. One of the most important of these is the Hardy operator, fractional Hardy operator, and inequality. As an example of a few studies, Different approaches have been introduced to the Hardy operator, fractional Hardy operator, and inequalities [4, 5]. In addition, a new dimension was added to the Hardy-type inequalities in the weighted and variable exponent Lebesgue space [6]. Mathematicians and scientists working in different scientific fields have expanded the workspace of operators in harmonic analysis. For example, they have proved the conditions for the boundedness and compactness, etc. of operators and fractional inequalities in variable exponent Lebesgue spaces and they used it in physics, mechanics, electrorheological fluids, optics, economics, etc. [7, 8, 9, 10, 11, 12, 13]. Besides, it has been known that the fractional Hardy operator in a variable exponent Lebesgue space does not satisfy arbitrary non-negative measurable functions; but provides for non-negative monotonic functions. Moreover, the sharp constant of the fractional Hardy-type operator was obtained for non-negative functions [13]. After a while, the monotony was replaced by a weaker condition [14]. However, a relationship has also been established between harmonic analysis in general, the variable exponential Lebesgue space in particular, and other spaces of different types. Time scales, Morrey spaces, and Sobolev spaces can be examples of spaces.

Recently, the calculation of time scales has also attracted authors' attention. Generally, by integrating time scales with the subjects within the field of harmonic analysis, they have revealed the magnificent relationships between them. For example, the dynamic integral inequalities and fractional integral operators have been studied on time scales with variable exponent Lebesgue spaces by many authors [15-28].

We aim of this paper is to obtain some inequalities for exponentially weighted fractional Hardy-type operator and the weighted dual of the classical fractional Hardy operator acting exponentially weighted with Δ-integral in time scale calculus.

2. Auxiallary Statements and Preliminaries

The emergence of the theory of time scales was introduced to the literature in 1988 by Stefan Hilger [29]. Although it later experienced a period of stagnation, its popularity increased especially after the 2000s. Recently, many authors have studied certain dynamic inequalities on time scales, operators, and concepts that fall within the field of harmonic analysis. We see that it still maintains its popularity today. Let $[x, y]$ be a facultative closed interval on $\mathbb{T}$ (time scale). We refer to the references [29-32] for more details. $[x, y]_{\mathbb{T}}$ is denoted by $[x, y] \cap \mathbb{T}$.

Definition 2.1. [31] The functions $\sigma, \rho: \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$, $\rho(t) = sup\{s \in \mathbb{T}: s < t\}$ for $t \in \mathbb{T}$. $\sigma(t)$ is defined as forward jump operator. The function $\rho(t)$ is defined as backward jump operator. If $\sigma(t) > t$, then t is defined as right-scattered. If $\sigma(t) = t$, then t is called as right-dense. If $\rho(t) < t$, then t is defined as left-scattered. If $\rho(t) = t$, then t is called as left-dense.

Definition 2.2. [31] Let functions $\mu, \vartheta: \mathbb{T} \to \mathbb{R}^+$ such that $\mu(t) = \sigma(t) - t,\ \vartheta(t) = t - \rho(t)$. The functions $\mu(t)$ and $\vartheta(t)$ are called as graininess functions. Let $\mathbb{T}$ be a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$.

$\mathbb{T}^k$ is defined as follows

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho\, sup\, \mathbb{T}, sup\, \mathbb{T}], & if \quad sup\, \mathbb{T} < \infty \\ \mathbb{T}, & if \quad sup\, \mathbb{T} = \infty \end{cases}$$

and

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus [inf\, \mathbb{T}, \sigma(inf\, \mathbb{T})], & |inf\, \mathbb{T}| < \infty \\ \mathbb{T}, & inf\, \mathbb{T} = -\infty. \end{cases}$$

Let $g: \mathbb{T} \to \mathbb{R}$ be a mapping and let t be defined as right-dense. We can write the following.

i) Let g be $\Delta-$differentiable at $t \in \mathbb{T}^k (t \neq min\mathbb{T})$, then g is continuous at point t.

ii) Let g be left continuous at t and let t be defined as right-scattered, then g is $\Delta-$differentiable at point t,

$$g^{\Delta}(t) = \frac{g^{\sigma}(t) - g(t)}{\mu(t)}$$

iii) Let g be $\Delta-$differentiable at t and $\lim_{s \to t} \frac{g(t)-g(s)}{t-s}$, then

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

iv) Let g be $\Delta-$differentiable at t, then $g^{\sigma}(t) = g(t) + \mu(t) g^{\Delta}(t)$.

Remark 2.3. [31] If $\mathbb{T} = \mathbb{R}$, then $g^{\Delta}(t) = g'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $h^{\Delta}(t) = \Delta h(t)$.

Definition 2.4. [30] If $G: \mathbb{T} \to \mathbb{R}$ is defined as a $\Delta-$antiderivative of $g: \mathbb{T} \to \mathbb{R}$, then $G^{\Delta} = g(t)$ holds for all $t, s \in \mathbb{T}$ and Δ-integral of g is called as the by

$$\int_{s}^{t} g(\tau)\Delta\tau = G(t) - G(s).$$

The following statements were proved by [14]

Theorem 2.5. [14] Let f be a measurable function on $(0,\infty)$ and satisfies for some $C > 0$. Let $x > 0$, $0 < q < 1$, and $\beta < 1 - \frac{1}{q}$. The inequality

$$f(x) \leq \frac{C}{x}\left(\int_{0}^{x} f^{q}(t)t^{q-1}dt\right)^{1/q}, \tag{1}$$

then

$$\left\|x^{\beta}(Hf)(x)\right\|_{L_q(0,\infty)} \leq K\left\|y^{\beta}f(y)\right\|_{L_q(0,\infty)} \tag{2}$$

where

$$K = C^{1-q}q^{1-\frac{1}{q}}\left(1 - \beta - \frac{1}{q}\right)^{-\frac{1}{q}} \tag{3}$$

Moreover, the constant K is sharp.

Let ω denote a weight non-negative function. The space of $L_{p,\omega}(0,\infty)$ for $0 < p < \infty$ is defined as follows:

$$\|f\|_{L_{p,\omega}(0,\infty)} = \left(\int_{0}^{\infty} |f(x)|^{p}\omega(x)dx\right)^{1/p}.$$

The fractional Hardy operator is defined as follows:

$$(H_{\omega}f)(x) = \frac{1}{K(x)}\frac{1}{x^{1-\propto}}\int_{0}^{x} f(y)\omega(y)dy, \quad x > 0,$$

where $0 \leq \propto < 1$, $0 < K(x) = \int_0^x \omega(y)dy < \infty$ for all $y > 0$.

Lemma 2.6. [14] Let $0 < q < 1$, $k_1 > 0$, $B > 0$, and let ω be a non-negative weight function, then the following inequality can be written.

$$\omega(s) \leq k_1\omega(t) \quad \text{for} \quad 0 < t < s < \infty \tag{4}$$

Let f be a measurable function for almost all $0 < s < \infty$, then

$$f(s) \leq B\left(\int_{0}^{s} \omega(t)t^{q-1}dt\right)^{-1/q}\left(\int_{0}^{s} f^{q}(t)\omega(t)t^{q-1}dt\right)^{1/q} \tag{5}$$

for all $x > 0$

$$(H_{\omega}f)(x) \leq \frac{k_2}{x\omega(x)^{1/q}}\left(\int_{0}^{x} f^{q}(t)\omega(t)t^{q-1}dt\right)^{1/q} \tag{6}$$

where $k_2 = q^{1/q}B^{1-q}c_1^{\frac{z}{q}-1}$.

Remark 2.7. [14] If $\omega = 1$, then inequality (5) turns into inequality (1) with $C = Bp^{1/p}$ and $k_1 = 1$, consequently $k_2 = p^{1/p}B^{1-p}$.

Remark 2.8. [14] If f is non-increasing mapping, then (5) holds for $B = 1$.

Theorem 2.9. [14] Let $k_1 > 0, 0 < q < 1,\ B > 0,\ \omega$ be a positive weight function, and $\beta < 1 - \frac{1}{q}$. Let f be a Lebesgue non-negative measurable function, then

$$\left\| x^{\beta}(H_{\omega}f)(x) \right\|_{L_{q,\omega}(0,\infty)} \leq N \left\| y^{\beta} f(y) \right\|_{L_{q,\omega}(0,\infty)} \tag{7}$$

where

$$N = B^{1-q} c_1^{\frac{2}{q}-1} \left(1 - \beta - \frac{1}{q}\right)^{-\frac{1}{q}} \tag{8}$$

Let H_{ω}^{*} be the dual of the operator H_{ω} in $L_2(0,\infty)$. Then for any $f, g \in L_2(0,\infty)$

$$\int_0^{\infty} \left(\frac{1}{D(x)} \int_0^x f(s)\omega(s)ds \right) g(x)dx = \int_0^{\infty} \left(\int_y^{\infty} \frac{g(x)}{D(x)} dx \right) f(s)\omega(s)ds$$

$$= \int_0^{\infty} \omega(s)(H^{*}g)(x)f(s)ds = \int_0^{\infty} \omega(y) \left(\int_y^{\infty} \frac{g(x)}{D(x)} dx \right) f(s)ds.$$

Hence the equality $(H_{\omega}f, g)_{L_2(0,\infty)} = (f, H_{\omega}^{*}g)_{L_2(0,\infty)}$ is satisfied for the operator H_{ω}^{*} defined by

$$(H_{\omega}^{*}f)(x) = \omega(x) \int_x^{\infty} \frac{g(y)}{D(s)} ds, \quad x > 0.$$

Lemma 2.10. Let ω be a non-negative weight function for $x > 0$, and let $\int_0^x \omega(t)dt < \infty$ be satisfied. Let f be a non-negative measurable function for $0 < x < \infty$. Let $0 < q < 1, B > 0$,

$$\int_x^{\infty} f^q(t)\omega(t)t^{q-1}dt < \infty,$$

and

$$f(x) \leq \frac{B}{x} \left(\int_x^{\infty} f^q(t)\omega(t)t^{q-1}dt \right)^{1/q} \omega(x)^{\frac{1}{1-q}} \left(\int_0^x \omega(t)dt \right)^{\frac{1}{1-q}}, \tag{9}$$

then for $n > 0$

$$(H_{\omega}^{*}f)(x) \leq k_3 \omega(x) \left(\int_x^{\infty} f^q(t)\omega(t)t^{q-1}dt \right)^{1/q} \tag{10}$$

where $k_3 = qB^{1-q}$.

Proof. By (9) it follows that

$$x^{1-q} f(x)^{1-q} \leq B^{1-q} \left(\int_x^{\infty} f^q(t)\omega(t)t^{q-1}dt \right)^{\frac{1}{q}-1} \omega(x) \int_0^x \omega(t)dt.$$

Hence

$$\frac{f(x)}{K(x)} \leq B^{1-q} \omega(x) f(x)^q x^{q-1} \left(\int_x^{\infty} f^q(t)\omega(t)t^{q-1}dt \right)^{\frac{1}{q}-1}$$

$$= qB^{1-q}(-1)\left[\left(\int_x^\infty f^q(t)\omega(t)t^{q-1}dt\right)^{1/q}\right].$$

Integrating over (n,∞), then we get

$$\int_n^\infty \frac{f(x)}{D(x)}dx$$

$$\leq qB^{1-q}\lim_{x\to\infty}\left(\left(\int_x^\infty f^q(t)\omega(t)t^{q-1}dt\right)^q - \left(\int_x^\infty f^q(t)\omega(t)t^{q-1}dt\right)^{1/q}\right)$$

$$\leq qB^{1-q}\left(\int_x^\infty f^q(t)\omega(t)t^{q-1}dt\right)^{1/q}$$

Hence

$$(H_\omega^* f)(x) = \omega(x)\int_x^\infty \frac{f(x)}{D(x)}dx \leq qB^{1-q}\omega(x)\left(\int_x^\infty f^q(t)\omega(t)t^{q-1}dt\right)^{1/q}.$$

If $\omega(x) = 1$ in (9) and (10), then we have the following corollary.

Corollary 2.11. [14] Let f be a non-negative Lebesgue measurable function for $0 < x < \infty$, and $\int_x^\infty f^q(t)t^{q-1}dt < \infty$. Let $B > 0$ and $0 < q < 1$, then the inequality

$$f(x) \leq \frac{B}{x^{q'}}\left(\int_x^\infty f^q(t)t^{q-1}dt\right)^{1/q} \tag{11}$$

is satisfied, then for $x > 0$

$$(H^* f)(x) \leq k_3\left(\int_x^\infty f^q(t)t^{q-1}dt\right)^{1/q} \tag{12}$$

where $k_3 = qB^{1-q}$ and q' is the conjugate exponent of q.

Remark 2.12. Inequalites (11), (12) respectively are analogues of inequality (1) and inequality (2) in [8], for the dual of the classical Hardy operator.

Theorem 2.13. [14] Let f be a non-negative Lebesgue measurable function. Let $-\frac{1}{q} < \beta < 1 - \frac{1}{q}$, $0 < q < 1$, $B > 0, x > 0$, then

$$\left\|\gamma^\beta (H^* f)(\gamma)\right\|_{L_q(0,\infty)} \leq k_4\left\|t^{\beta+1}f(t)\right\|_{L_q(0,\infty)} \tag{13}$$

where $k_4 = qB^{1-q}\ (\beta q + 1)^{-\frac{1}{q}}$.

Proof.

$$L_1 = \left\|\gamma^\beta (H^* f)(\gamma)\right\|_{L_q(0,\infty)} = \left[\int_0^\infty \gamma^{\beta q}(H^* f)^q(\gamma)d\gamma\right]^{1/q} = \left[\int_0^\infty \gamma^{\beta q}\left(\int_\gamma^\infty \frac{f(t)}{t}dt\right)^q d\gamma\right]^{1/q}.$$

Then it follows that

$$L_1 \leq \left[\int_0^\infty \gamma^{\beta q}k_3^q\left(\int_\gamma^\infty f^q(t)t^{q-1}dt\right)^q d\gamma\right]^{1/q} = k_3\left[\int_\gamma^\infty f^q(t)t^{q-1}\left(\int_0^t \gamma^{\beta q}d\gamma\right)dt\right]^{1/q}$$

$$= qB^{1-q} \ (\beta q + 1)^{-\frac{1}{q}} \left\| t^{\beta+1} f(t) \right\|_{L_q(0,\infty)}.$$

Let φ be a measurable positive function in R^m. Suppose that p is a measurable positive function on φ. Assume that $0 < p^- \le p(x) \le p^+ < \infty$, $p^- = ess \inf_{x\in\varphi} p(x)$, $p^+ = ess \sup_{x\in\varphi} p(x)$ and ω is a weight function on φ.

Definition 2.14. Let $L_{p(s),\omega}(\varphi)$ we define as all measurable functions on φ such that

$$I_{p,\omega}(f) = \int_{\varphi} (|f(s)\omega(s)|)^{p(s)} ds < \infty \tag{14}$$

Note that the expression

$$\|f\|_{L_{p(.),\omega}(\varphi)} = \inf\left\{\lambda > 0;\ \int_{\varphi} \left(\frac{|f(s)|\omega(s)}{\lambda}\right)^{p(s)} ds \le 1\right\} \tag{15}$$

denotes on $L_{p(s),\omega}(\varphi)$.

Corollary 2.15. [10] Let $n(s) = \frac{u(s)v(s)}{v(s)-u(s)}$, and let $0 < u^- \le u(s) \le v(s) \le v^+ < \infty$. Assume that ω_1, ω_2 are weight functions in φ satisfying the condition:

$$\left\|\frac{\omega_1}{\omega_2}\right\|_{L_{n(.)}(\varphi)} < \infty.$$

Then the inequality

$$\|f\|_{L_{u(.),\omega_1}(\varphi)} \le \left(A_1 + B_1 + \|\chi\varphi_2\|_{L_\infty(\varphi)}\right)^{\frac{1}{u^-}} \left\|\frac{\omega_1}{\omega_2}\right\|_{L_{n(.)}(\varphi)} \|f\|_{L_{v(.),\omega_2}(\varphi)} \tag{16}$$

holds for every $f \in L_{v(s),\omega_2}(\varphi)$, where

$$\varphi_1 = \{s \in \varphi : u(s) < v(s)\}, \quad \varphi_2 = \{y \in \varphi : u(s) = v(s)\},$$

$$B_1 = \sup_{s\in\varphi_1} \frac{u(s)}{v(s)}, \quad A_1 = \sup_{s\in\varphi_1} \frac{v(s)-u(s)}{v(s)}.$$

Lemma 2.16. [11] Let $t \in \varphi_2 \subset R^m$. If $0 < p^- \le p(x) \le q(t) \le q^+ < \infty$, for all $x \in \varphi_1 \subset R^m$, and if $p \in M(\varphi_1)$, then the inequality

$$\left\| \|f\|_{L_{p(.)}(\varphi_1)} \right\|_{L_{q(.)}(\varphi_2)} \le M_{p,q} \left\| \|f\|_{L_{q(.)}(\varphi_2)} \right\|_{L_{p(.)}(\varphi_1)} \tag{17}$$

holds, where

$$M_{p,q} = \left(\|\chi\Lambda_1\|_\infty + \|\chi\Lambda_2\|_\infty + \frac{p^+}{q^-} + \frac{p^-}{q^+}\right)(\|\chi\Lambda_1\|_\infty + \|\chi\Lambda_2\|_\infty) \tag{18}$$

$$q^- = ess \inf_{\varphi_2} q(x),\ \ q^- = ess \sup_{\varphi_2} q(x)$$

$$\Lambda_1 = \{(x,t) \in \varphi_1 \times \varphi_2; p(x) = q(x)\}, \qquad \Lambda_2 = \varphi_1 \times \varphi_2 \backslash \Lambda_1.$$

If $M(\varphi_1)$ is define as continuous functions on φ_1 and if $f: \varphi_1 \times \varphi_2 \to R$ is define as measurable function, then

$$\left\| \|f\|_{L_{q(.)}(\varphi_2)} \right\|_{L_{p(.)}(\varphi_1)} < \infty.$$

Theorem 2.17. [10] Let f be a non-negative and non-increasing function, and let $0 < p^- \leq p(x) \leq q(t) \leq q^+ < 1$, for $x \in (0,\infty)$, and equality $n(x) = \frac{p^- p(x)}{p(x) - p^-}$ be satisfied. Suppose that φ_1 and φ_2 are weight non-negative functions. The inequality

$$\|Hf\|_{L_{q(x),\omega_2}(0,\infty)} \leq p^{-\frac{1}{p^-}} M_{p,q} b_p \left\| \frac{y^{\frac{1}{p^+}} \left\| \frac{\omega_2}{x} \right\|_{L_{q(.)}(y,\infty)}}{\omega_1} \right\|_{L_{n(.)}(0,\infty)} \|f\|_{L_{p(.),\omega_1}(0,\infty)} \qquad (19)$$

holds for $f \in L_{p(x),\omega_1}(0,\infty)$, where

$$M_{p,q} = \left(\|\chi\Lambda_1\|_{L_\infty(0,\infty)} + \|\chi\Lambda_2\|_{L_\infty(0,\infty)} + p^- \left(\frac{1}{q^-} - \frac{1}{q^+} \right) \right) \left(\|\chi \mathrm{T}_1\|_{L_\infty(0,\infty)} + \|\chi \mathrm{T}_2\|_{L_\infty(0,\infty)} \right),$$

$\mathrm{T}_1 = \{x \in (0,\infty) : p(x) = p^-\}$, $\mathrm{T}_2 = (0,\infty) \backslash \mathrm{T}_1$, and $b_p = \left(1 - \frac{p^+ - p^-}{p^+} + \|\chi \mathrm{T}_1\|_{L_\infty(0,\infty)}\right)^{\frac{1}{p^-}}$.

3. Results and Discussion

The variable exponentially fractional Hardy-type operator with Δ −integral in time scale calculus is defined by as follows

$$(H_{\propto\omega^p} f)(x) = \frac{1}{K(x)} \frac{1}{x^{1-\propto}} \int_0^x f(s)\omega^{p(s)}(s)\Delta s, \quad x > 0,$$

where $0 \leq \propto < 1$, $0 < p(s) < 1$, $0 < K(x) = \int_0^x \omega^{p(s)}(s)\Delta s < \infty$ for all $s > 0$.

Theorem 3.1. Let f is Δ −integrable non-negative Lebesgue measurable function satisfying inequality (5) with p replaced by p^-. ω is a positive weight function. Let $0 \leq \propto < 1$, $0 < p^- \leq p(x) \leq q(x) \leq q^+ < 1$, $\beta < 1 - \frac{1}{p^-}$, $n(x) = \frac{p^- p(x)}{p(x) - p^-}$. Assume that ω_1 and ω_2 are weight positive functions. The inequality

$$\|H_{\propto\omega^{p(t)}} f\|_{L_{q(x),\omega_2}(0,\infty)}$$

$$\leq k_2 M_{p,q} b_p \left\| \frac{\omega^{\frac{p(t)}{p^-}} t^{\frac{1}{p^-}} \left\| \frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right\|_{L_{q(.)}(t,\infty)}}{\omega_1^{p(y)}} \right\|_{L_{n(.)}(0,\infty)} . \|f\|_{L_{p(.),\omega_1^{p(t)}}(0,\infty)} \qquad (20)$$

is valid for $f \in L_{p(x),\omega_1}(0,\infty)$, where $k_2 = p^{-\frac{1}{p^-}} k_1^{\frac{2}{p^-}-1} B^{1-p^-}$.

Proof. Applying Lemma 2.6, we obtain

$$\|H_{\propto\omega^{p(y)}} f\|_{L_{q(x),\omega_2^{p(y)}}(0,\infty)} = \left\| \omega_2^{p(y)} H_{\propto\omega^{p(y)}} f \right\|_{L_{q(x)}(0,\infty)}$$

$$\leq \left\| \frac{k_2 \omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \left(\frac{1}{x^{1-\propto}} \int_0^x f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\Delta t \right)^{1/p^-} \right\|_{L_{q(x)}(0,\infty)}$$

$$= k_2 \left\| \frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \left(\frac{1}{x^{1-\propto}} \int_0^x f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\Delta t \right)^{1/p^-} \right\|_{L_{q(x)}(0,\infty)}.$$

Let $J_1 = \left\| \frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \left(\frac{1}{x^{1-\propto}} \int_0^x f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\Delta t \right)^{1/p^-} \right\|_{L_{q(x)}(0,\infty)}$

$$J_1 = \left\| \left(\frac{1}{x^{1-\propto}} \int_0^\infty \left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(0.x)}(t) \left[\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right]^{p^-} t^{p^- -1}\Delta t \right)^{1/p^-} \right\|_{L_{q(x)}(0,\infty)}$$

$$= \left\| \left(\frac{1}{x^{1-\propto}} \int_0^\infty \left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(0.x)}(t) \left[\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right]^{p^-} t^{p^- -1}\Delta t \right)^{1/p^-} \right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)}$$

$$= \left\| \left\| \frac{1}{x^{1-\propto}} \left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(0.x)}(t) \left[\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right]^{p^-} t^{p^- -1} \right\|_{L_1(0,\infty)} \right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)}^{\frac{1}{p^-}}.$$

Next applying Lemma 2.16, we obtain

$$J_1 \leq M_{p,q} \left(\frac{1}{x^{1-\propto}} \int_0^\infty \left\| \left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(0.x)}(t) \left[\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right]^{p^-} t^{p^- -1} \right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)} \Delta t \right)^{1/p^-}$$

$$= M_{p,q} \left(\frac{1}{x^{1-\propto}} \int_0^\infty f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1} \left\| \chi_{(0.x)}(t) \left[\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right]^{p^-} \right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)} \Delta t \right)^{1/p^-}$$

$$= M_{p,q} \left(\frac{1}{x^{1-\propto}} \int_0^\infty f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1} \left\| \left[\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)} \right]^{p^-} \right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)} \Delta t \right)^{1/p^-}$$

$$= M_{p,q}\left\|\frac{1}{x^{1-\propto}}f(t)\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^--1}{p^-}}\left\|\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)}\right\|_{L_q(t,\infty)}\right\|_{L_{p^-}(0,\infty)}.$$

Let $J_2 = \left\|f(t)\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^--1}{p^-}}\left\|\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)}\right\|_{L_q(t,\infty)}\right\|_{L_{p^-}(0,\infty)}$,

then applying Corollary 2.15, we obtain

$$J_2 \leq b_p\left\|\frac{\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^--1}{p^-}}\left\|\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)}\right\|_{L_q(t,\infty)}}{\omega_1^{p(t)}}\right\|_{L_{n(.)}(0,\infty)}\frac{1}{x^{1-\propto}}\|f\|_{L_{p(.)\omega_1^{p(t)}}(0,\infty)},$$

Hence

$$\|H_{\omega^p}f\|_{L_{q(x),\omega_2}(0,\infty)}$$

$$\leq k_2M_{p,q}b_p\left\|\frac{\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^--1}{p^-}}\left\|\frac{\omega_2^{p(t)}(x)}{x\omega^{\frac{p(t)}{p^-}}(x)}\right\|_{L_q(t,\infty)}}{\omega_1^{p(t)}}\right\|_{L_{n(.)}(0,\infty)}\frac{1}{x^{1-\propto}}\|f\|_{L_{p(.)\omega_1^{p(t)}}(0,\infty)}.$$

Theorem 3.2. Let f be $\Delta-$integrable a non-negative Lebesgue measurable function satisfying inequality (9) with p replaced by p^-. ω is a positive weight function. Let $0 \leq \propto < 1$, $0 < p^- \leq p(x) \leq q(x) \leq q^+ < 1$, $\beta < 1 - \frac{1}{p^-}$, $n(x) = \frac{p^-p(x)}{p(x)-p^-}$, $x \in (0,\infty)$. Assume that ω_1 and ω_2 are positive weight functions. The inequality

$$\left\|H^*_{\propto\omega^{p(t)}}f\right\|_{L_{q(x),\omega_2^p}(0,\infty)}$$

$$\leq k_3M_{p,q}b_p\left\|\frac{\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^--1}{p^-}}\left\|\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right\|_{L_q(0,t)}}{\omega_1^{p(t)}}\right\|_{L_{n(.)}(0,\infty)}\|f\|_{L_{p(.)\omega_1^p}(t)(0,\infty)} \quad (21)$$

is valid for $f \in L_{p(x),\omega_1}(0,\infty)$, where $k_3 = p^-A^{1-p^-}$.

Proof. Applying Lemma 2.6, we obtain

$$\left\|H^*_{\propto\omega^{p(t)}}f\right\|_{L_{q(x),\omega_2^p}(t)(0,\infty)} = \left\|\omega_2^{p(t)}H^*_{\omega^{p(t)}}f\right\|_{L_{q(x)}(0,\infty)}$$

$$\leq k_3 \left\| \omega_2^{p(t)}(x)\omega^{p(t)}(x)\left(\frac{1}{x^{1-\propto}}\int_x^{\infty} f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\Delta t\right)^{1/p^-}\right\|_{L_{q(x)}(0,\infty)}$$

Let $V_1 = \left\| \omega_2^{p(t)}(x)\omega^{p(t)}(x)\left(\frac{1}{x^{1-\propto}}\int_x^{\infty} f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\Delta t\right)^{1/p^-}\right\|_{L_{q(x)}(0,\infty)}$,

$$V_1 = \left\| \left(\frac{1}{x^{1-\propto}}\int_0^{\infty}\left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(x,\infty)}(t)\left[\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right]^{p^-} t^{p^- -1}\Delta t\right)^{1/p^-}\right\|_{L_{q(x)}(0,\infty)}$$

$$= \left\| \frac{1}{x^{1-\propto}}\int_0^{\infty}\left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(x,\infty)}(t)\left[\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right]^{p^-} t^{p^- -1}\Delta t\right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)}^{1/p^-}$$

$$= \left\| \left\| \frac{1}{x^{1-\propto}}\left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(x,\infty)}(t)\left[\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right]^{p^-} t^{p^- -1}\right\|_{L_1(0,\infty)}\right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)}^{1/p^-}$$

Applying Lemma 2.16, we obtain

$$V_1 \leq M_{p,q}\left(\frac{1}{x^{1-\propto}}\int_0^{\infty}\left\|\left[f^{p^-}(t)\omega^{p(t)}(t)\right]\chi_{(x,\infty)}(t)\left[\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right]^{p^-} t^{p^- -1}\right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)}\Delta t\right)^{1/p^-}$$

$$= M_{p,q}\left(\frac{1}{x^{1-\propto}}\int_0^{\infty} f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\left\|\chi_{(x,\infty)}(t)\left[\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right]^{p^-}\right\|_{L_{\frac{q(.)}{p^-}}(0,\infty)}\Delta t\right)^{1/p^-}$$

$$= M_{p,q}\left(\frac{1}{x^{1-\propto}}\int_0^{\infty} f^{p^-}(t)\omega^{p(t)}(t)t^{p^- -1}\left\|\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right\|_{L_q(0,t)}^{p^-}\Delta t\right)^{1/p^-}$$

$$= M_{p,q}\left\|\frac{1}{x^{1-\propto}}f(t)\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^- -1}{p^-}}\left\|\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right\|_{L_q(0,t)}\right\|_{L_{p^-}(0,\infty)}.$$

By applying Corollary 2.15, we obtain

$$\left\|\frac{1}{x^{1-\propto}}f(t)\omega^{\frac{p(t)}{p^-}}(t)t^{\frac{p^- -1}{p^-}}\left\|\omega_2^{p(t)}(x)\omega^{p(t)}(x)\right\|_{L_q(0,t)}\right\|_{L_{p^-}(0,\infty)}$$

$$\leq b_p \left\| \frac{\omega^{\frac{p(t)}{p^-}}(t) t^{\frac{p^- -1}{p^-}} \left\| \omega_2^{p(t)}(x)\omega^{p(t)}(x) \right\|_{L_q(0,t)}}{\omega_1^{p(t)}} \right\|_{L_{n(.)}(0,\infty)} \frac{1}{x^{1-\propto}} \|f\|_{L_{p(.)\omega_1^{p(t)}}(0,\infty)}.$$

Hence

$$\left\| H^*_{\omega^{p(t)}} f \right\|_{L_{q(x),\omega_2^p}(0,\infty)}$$

$$\leq k_3 M_{p,q} b_p \left\| \frac{\omega^{\frac{p(t)}{p^-}}(t) t^{\frac{p^- -1}{p^-}} \left\| \omega_2^{p(t)}(x)\omega^{p(t)}(x) \right\|_{L_q(0,t)}}{\omega_1^{p(t)}} \right\|_{L_{n(.)}(0,\infty)} \frac{1}{x^{1-\propto}} \|f\|_{L_{p(.)\omega_1^{p(t)}}(0,\infty)}.$$

Remark 3.3. In Theorem 3.1 and Theorem 3.2, if we get $\mathbb{T} = \mathbb{R}$, $\propto = 1$ and $p(t) = 0$, then we obtain continuous weighted inequalities as mentioned in [15].

4. Conclusion

In general, operators and variable exponent types of inequalities have become one of the important cornerstones of harmonic analysis. Their boundedness, compactness, etc. have caused them to become the focus of attention of mathematicians. Numerous studies have been conducted in this field, especially since the beginning of this century. Likewise, the issue of time scales has become popular, especially in the last years, although not so much in the past. Holistic studies covering time scales and harmonic analysis have become more popular recently. Inspired by these studies, we took a new present to variable exponentially fractional Hardy operator by reconciling time scales and harmonic analysis.

Ethical Statement

The author declares that this document does not require ethics committee approval or any special permission.

Conflict of Interest

The author declares no conflict of interest.

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