GU J Sci 30(4): 474-487 (2017)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Proximity Coincidence Points for a Pair of Maps with Three Auxiliary Functions in Partially Ordered Metric Spaces

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Article Info

Received: 02/11/2016
Accepted: 10/03/2017

Keywords

proximity coincidence point,
proximally increasing map,
best proximity point,
partially ordered set.

Abstract

In this paper, we establish proximity coincidence point results using three auxiliary functions, which need not be continuous, in partially ordered metric spaces for a pair of maps. We also discuss several corollaries and give illustrative examples in support of our results. The results presented in this paper generalize the results of Wangkeeree and Sisarat [17].

1. INTRODUCTION

The famous Banach's contraction principle is an important tool to assert the uniqueness of fixed point for selfmaps in complete metric spaces. When a map from a metric space into itself has no fixed points, it could be interesting to study the existence and uniqueness of some points that minimize the distance between an origin and its corresponding image. That is, it may be speculated to determine an element x for which the error $d(x, Tx)$ is minimal, in the sense x and Tx are in close proximity to each other. This concept gives rise to the best proximity theory.

Let A be a nonempty subset of a metric space (X, d) and $f: A \to X$ is a map. If the fixed point equation $fx = x$ does not possess a solution, then $d(x, fx) > 0$ for all $x \in A$. In such a situation, it is the aim of best proximity theory to find an element $x \in A$ such that $d(x, fx)$ is minimum in some sense. A point $x \in A$ is called best proximity point of $T: A \to B$ if $d(x, Tx) = d(A, B)$ where $d(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\}$. A best proximity point becomes a fixed point if the underlying mapping is a selfmapping. Therefore, it can be concluded that best proximity point theorems generalize fixed point theorems in a natural way.

In recent years, the existence and convergence of best proximity points is an interesting topic of optimization theory which attracted the attention of many authors [1-3, 5-6, 12, 15]. The best proximity point evolves as a generalization of the concept of the best approximation. The authors [7, 9-11, 13-14] and reference therein obtained best proximity point theorems under certain contraction conditions for non-selfmaps.

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2. PRELIMINARIES

We recall the following notations and definitions. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and let A and B be nonempty subsets of X.

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 2.1 [16] *Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property, if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,*

$$\left.\begin{array}{c} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_1).$$

Definition 2.2 *A mapping $T: A \to A$ is said to be increasing if for all $x, y \in A$, $x \preccurlyeq y \Rightarrow Tx \preccurlyeq Ty$.*

Definition 2.3 [8] Let $(X, \preccurlyeq)$ *be a partially ordered set and $F, g: X \to X$ be maps.*
(i) *F is called g-nondecreasing if $gx \preccurlyeq gy$ implies $Fx \preccurlyeq Fy$ for all $x, y \in X$.*
(ii) *F is called g-non-increasing if $gx \preccurlyeq gy$ implies $Fy \preccurlyeq Fx$ for all $x, y \in X$.*

Definition 2.4 [6] *A mapping $T: A \to B$ is said to be proximally increasing (nondecreasing) if for all $u_1, u_2, x_1, x_2 \in A$,*

$$\left.\begin{array}{c} x_1 \preccurlyeq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B \end{array}\right\} \Longrightarrow u_1 \preccurlyeq u_2.$$

Similarly, a mapping $T: A \to B$ is said to be proximally decreasing (non-increasing) if for all $u_1, u_2, x_1, x_2 \in A$,

$$\left.\begin{array}{c} x_1 \preccurlyeq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B \end{array}\right\} \Longrightarrow u_2 \preccurlyeq u_1.$$

Definition 2.5 [17] (g-proximally increasing). *Suppose $(X, \preccurlyeq)$ is a partially ordered set. Let $f: A \to B$ and $g: A \to A$ be maps. A map f is said to be g-proximally increasing if for all $x_1, x_2, y_1, y_2 \in A$,*

$$\left.\begin{array}{c} gy_1 \preccurlyeq gy_2 \\ d(x_1, fy_1) = d(A, B) \\ d(x_2, fy_2) = d(A, B \end{array}\right\} \Longrightarrow x_1 \preccurlyeq x_2.$$

Here we note that if g is an identity map of A, then clearly f is proximally increasing (nondecreasing) and if $A = B$, then f is g increasing (nondecreasing).

Definition 2.6 [17] (Proximity coincidence point). *Let A and B be nonempty subsets of a metric space (X,d). Let $f: A \to B$ be a non-selfmap and $g: A \to A$ be a self map on A. A point $x \in A$ is said to be a proximity coincidence point of f and g if $d(gx, fx) = d(A,B)$.*

In 2015, Wangkeeree and Sisarat [17] proved some proximity coincidence point for non-selfmap and selfmap in partially ordered metric space.

Theorem 2.7 [17] *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let (A,B) be a pair of nonempty subsets of X. Assume that A_0 and B_0 are nonempty subsets of A and B respectively. Let $f : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(i) *f is a g-proximally increasing and (A,B) satisfy the P-property,*
(ii) *$g(A_0)$ is closed and $f(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$,*
(iii) *$\psi(d(fx,fy)) \le \alpha(d(gx,gy)) - \beta(d(gx,gy))$ for all $x,y \in A$ such that $gx \preccurlyeq gy$, where ψ, α, β:[0,∞) → $[0,\infty)$ are such that ψ is continuous and monotone nondecreasing, α is continuous and β is lower semi-continuous, $\psi(t) = 0$ if and only if $t = 0, \alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$,*
(iv) *there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1, fx_0) = d(A,B)$ and $gx_0 \preccurlyeq gx_1$.*

Also, we assume that if any nondecreasing sequence $\{x_n\}$ in gA_0 converges to z, then $x_n \preccurlyeq z$ for all $n \ge 0$.
Then there exists an element $x^* \in A$ such that $d(gx^*, fx^*) = d(A,B)$.

We denote by Ψ the set of all functions $\psi: [0,\infty) \to [0,\infty)$ such that
(i) ψ is nondecreasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$ and
(iii) if $\{t_n\} \subseteq (0,\infty)$ is any bounded sequence such that $\lim_{n\to\infty} \psi(t_n) = 0$, then $\lim_{n\to\infty} t_n = 0$.

We denote by Θ the set of all functions $\varphi: [0,\infty) \to [0,\infty)$ such that
(i) φ is bounded on any bounded interval in $[0,\infty)$ and
(ii) φ is continuous at 0 and $\varphi(0) = 0$.

In Section 3 of this paper, we prove our main results by using three auxiliary functions in which we drop the continuity assumption from the result of Wangkeeree and Sisarat [17], so that our result is more general. In Section 4, we draw some corollaries and provide examples in support of our results.

We state the following lemma, which we use in our main results.

Lemma 2.8. [4] *Suppose that (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) \ge \varepsilon$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon$* and

(i) $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \varepsilon$ $\quad$ (ii) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon$
(iii) $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k}) = \varepsilon$ $\quad$ (iv) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon.$

3. MAIN RESULTS

Theorem 3.1 *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Let (A,B) be a pair of nonempty subsets of X. Assume that A_0 is a nonempty subset of A. Let $f : A \to B$ and $g : A \to A$ satisfy the following conditions***:**

(i) f is a g-proximally increasing and (A,B) satisfy the P-property,
(ii) $g(A_0)$ is closed and $f(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$,
(iii) there exist $\psi \in \Psi$ and φ, $\theta \in \Theta$ with the condition

$$\psi(t) - \overline{\lim}\, \varphi(x_n) + \underline{\lim}\, \theta(x_n) > 0, \tag{1}$$

where $\{x_n\}$ is any sequence in $[0,\infty)$ with $x_n \to t > 0$ and

$$\psi\big(d(fx,fy)\big) \le \varphi\big(d(gx,gy)\big) - \theta(d(gx,gy)) \tag{2}$$

for all $x, y \in A_0$ with $gx \preccurlyeq gy$ and also,

$$\psi(x) \le \varphi(y) \Rightarrow x \le y. \tag{3}$$

Also, suppose that if $\{gx_n\}$ is a nondecreasing sequence in gA_0 such that $gx_n \to gz$, then $g\, x_n \preccurlyeq gz$ for all $n \ge 0$. Furthermore, assume that there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1, fx_0) = d(A,B)$ and $gx_0 \preccurlyeq gx_1$.
Then f and g have proximity coincidence point.

Proof. By our assumpsion, there exist $x_0, x_1 \in A_0$ such that

$$d(gx_1, fx_0) = d(A,B) \text{ and } gx_0 \preccurlyeq gx_1. \tag{4}$$

As $x_1 \in A_0$ so $f(x_1) \subseteq B_0$. Hence there exists $z \in A$ such that $d(z, fx_1) = d(A,B)$. Therefore $z \in A_0$. Since $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $z = gx_2$. Hence

$$d(gx_2, fx_1) = d(A,B) . \tag{5}$$

By g-proximally increasing property of f, from (4) and (5) , we obtain $gx_1 \preccurlyeq gx_2$. On continuing this process, we get a sequence $\{gx_n\}$ in gA_0 such that

$$d(gx_{n+1}, fx_n) = d(A,B) \text{ for all } n \ge 0, \tag{6}$$

satisfying

$$gx_0 \preccurlyeq gx_1 \preccurlyeq \cdots \preccurlyeq gx_n \preccurlyeq gx_{n+1} \preccurlyeq \cdots \ . \tag{7}$$

By the P- property of (A,B), from (4) and (5), we obtain $d(gx_1, gx_2) = d(fx_0, fx_1)$.
On continuing this step, we have,

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \text{ for all } n \ge 0. \tag{8}$$

As $gx_n \preccurlyeq gx_{n+1}$ for all $n \ge 0$, by applyin the inequality (2), we have

$$\begin{aligned}\psi(d(gx_n, gx_{n+1})) = \psi(d(fx_{n-1}, fx_n)) &\le \varphi(d(gx_{n-1}, gx_n)) - \theta(d(gx_{n-1}, gx_n)) \\ &\le \varphi\big(d(gx_{n-1}, gx_n)\big).\end{aligned}$$

This implies, by (3), that $d(gx_n, gx_{n+1}) \le d(gx_{n-1}, gx_n)$ and hence $\{\, d(gx_n, gx_{n+1})\}$ is a decreasing sequence of non-negative real numbers. Therefore there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = r.$$

Since $d(gx_n, gx_{n+1})$ is a decreasing sequence which converges to r , we have $r \leq d(gx_n, gx_{n+1})$ for all $n \geq 0$. From nondecreasing property of ψ, we get

$$\psi(r) \leq \psi(d(gx_n, gx_{n+1})).$$

Suppose $r > 0$. By applying the inequality (2), using (7) and (8), it follows that

$$\psi(r) \leq \psi(d(gx_n, gx_{n+1})) = \psi(d(fx_{n-1}, fx_n)) \leq \varphi\big(d(gx_{n-1}, gx_n)\big) - \theta\big(d(gx_{n-1}, gx_n)\big). \tag{9}$$

On taking the limit supremum in (9), we have

$$\psi(r) - \overline{\lim}\, \varphi(d(gx_{n-1}, gx_n)) + \underline{\lim}\, \theta(d(gx_{n-1}, gx_n)) \leq 0,$$

a contradiction. Hence

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{10}$$

We now show that the sequence $\{gx_n\}$ is Cauchy. Let $gx_n = y_n$. Suppose that $\{y_n\}$ is not a Cauchy sequence. Then by Lemma 2.8, then there exists an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that n_k is the smallest index with $n_k > m_k > k$,

$$d\big(y_{m_k}, y_{n_k}\big) \geq \varepsilon \text{ and } d\big(y_{m_k}, y_{n_k-1}\big) < \varepsilon, \tag{11}$$

satisfying $\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \varepsilon$.

From (11) and by the nondecreasing property of ψ, we obtain $\psi(\varepsilon) \leq \psi(d(y_{m_k}, y_{n_k}))$. Since $y_{m_k} \leq y_{n_k}$ for $k \geq 0$, by applying the inequality (2) and by using (8), we have

$$\psi(\varepsilon) \leq \psi(d\big(y_{m_k}, y_{n_k}\big)) = \psi(d(fy_{m_k-1}, fy_{n_k-1})) \leq \varphi(d(y_{m_k-1}, y_{n_k-1})) - \theta(d(y_{m_k-1}, y_{n_k-1})).$$

On taking the limit supremum as $k \to \infty$ in the above inequality, we obtain

$$\psi(\in) \leq \overline{\lim_{k\to\infty}}\, \varphi(d(y_{m_k}, y_{n_k})) + \underline{\lim_{k\to\infty}}\, \theta(d(y_{m_k}, y_{n_k})) \leq 0,$$

a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. i.e. $\{gx_n\}$ is a Cauchy sequence in $g(A_0)$.

Since $g(A_0)$ is a closed subset of a complete metric space X and hence complete, so that there exists $x^* \in A_0$ such that $gx_n \to gx^* \in g(A_0)$. By the hypothesis of the theorem, we have $g\, x_n \preccurlyeq gx^*$ for all $n \in \mathbb{N}$. Since $x^* \in A_0$, we have $fx^* \in f(A_0) \subseteq B_0$. Therefore there exists a point $z \in A_0$ such that

$$d(z, fx^*) = d(A, B). \tag{12}$$

Since the pair (A,B) satisfy the P-property, from (12) and (6), we have $d(gx_{n+1}, z) = d(fx_n, fx^*)$.
By appling the inequality (2), it follows that

$$\psi(d(gx_{n+1}, z)) = \psi(d(fx_n, fx^*)) \leq \varphi(d(gx_n, gx^*)) - \theta(d(gx_n, gx^*)). \tag{13}$$

On taking the limit as $n \to \infty$ in (13), using the fact that $gx_n \to gx^*$ as $n \to \infty$, by the property (*ii*) of φ and θ and the property of ψ, we optain

$$\lim_{n\to\infty} \psi\big(d(gx_{n+1}, z)\big) = 0$$

Therefore by hypothesis (*iv*), we get $d(gx_{n+1}, z) \to 0$ as $n \to \infty$. i.e. $\lim_{n\to\infty} gx_{n+1} = z$ which implies by the uniqueness of limit, that $z = gx^*$. Hence, we have $d(gx^*, fx^*) = d(A, B)$. Therefore x^* is the proximity coincidence point of f and g.

Theorem 3.2 *In addition to the hypotheses of Theorem 3.1, assume the following*:
Condition H : *Suppose that* g *is one-to-one and for every* $x, y \in A$ *there exists* $u \in A_0$ *such that gu is comparable to gx and gy. Then* f *and* g *have a unique proximity coincidence point.*

Proof. In view of the proof of Theorem 3.1, the set of proximity coincidence points of f and g is nonempty. Suppose that $x, y \in A$ are the two distinct proximity coincidence points of f and g That is,

$$d(gx, fx) = d(A, B) \text{ and } d(gy, fy) = d(A, B). \tag{14}$$

Case (*i*) : *gx* is comparable to *gy*. i.e., either $gx \preccurlyeq gy$ or $gy \preccurlyeq gx$.

We assume, without loss of generality, that $gx \preccurlyeq gy$. Since (A,B) satisfies the P-property, from (14), it follows that

$$d(gx, gy) = d(fx, fy). \tag{15}$$

Since $gx \preccurlyeq gy$, by the inequality (2), we get

$$\psi\big(d(gx, gy)\big) = \psi\big(d(fx, fy)\big) \le \varphi\big(d(gx, gy)\big) - \theta\big(d(gx, gy)\big).$$

Since *x* and *y* are distinct and *g* is one-to-one, it follows that $d(gx, gy) > 0$. Therefore

$$\psi(d(gx, gy)) - \overline{\lim}\,\varphi(d(gx, gy)) + \underline{\lim}\,\theta(d(gx, gy)) \le 0,$$

a contradiction. Hence *gx* = *gy*. This implies that *x* = *y*.

Case (*ii*) : *gx* is not comparable to *gy*.

By assumption, there exists $u \in A_0$ such that *gu* is comparable to *gx* and *gy*. Now, we set $gu_0 = gu$. Suppose that either

$$gu_0 \succcurlyeq gx \text{ or } gu_0 \preccurlyeq gx. \tag{16}$$

We assume, without loss of generality, that

$$gu_0 \preccurlyeq gx. \tag{17}$$

As $u_0 = u \in A$, so $f(A_0) \subseteq B_0$. Hence there exists $z \in A$ such that $d(z, fu_0) = d(A, B)$. Therfore $z \in A_0$. Since $A_0 \subseteq g(A_0)$, there exists $u_1 \in A_0$ such that $z = gu_1$. Hence

$$d(z, fu_0) = d(gu_1, fu_0) = d(A, B). \tag{18}$$

Since f is g-proximally inceasing, from (14), (17) and (18), we obtain

$$gu_1 \preccurlyeq gx.$$

By using the P-property of the pair (A, B), from (14) and (18), we have

$$d(gx, gu_1) = d(fx, fu_0).$$

On continuing this process, we can constract a sequence $\{gu_n\}$ in gA_0 such that

$$d(gx, gu_{n+1}) = d(fx, fu_n) \quad \text{and} \quad gu_n \preccurlyeq gx \text{ for all } n \ge 0. \tag{19}$$

Hence by using (19) and the inequality (2), we have

$$\psi(d(gx,gu_{n+1})) = \psi(d(fx,fu_n)) \leq \varphi(d(gx,gu_n)) - \theta(d(gx,gu_n))$$
.
$$\leq \varphi(d(gx,gu_n). \tag{20}$$

Therefore by condition (3), it follows that $d(gx,gu_{n+1}) \leq d(gx,gu_n)$ so that $\{d(gx,gu_n)\}$ is a decreasing sequence of non-negative real numbers. Hence there exists $t \geq 0$ such that

$$\lim_{n\to\infty} d(gx,gu_{n+1}) = t. \tag{21}$$

Suppose that $t > 0$. Since $\{d(gx,gx_n)\}$ is a decreasing sequence which converges to t, we have $t \leq d(gx,gu_{n+1})$ for all $n \geq 0$. Hence by nondecreasing property of ψ, it follows that

$$\psi(t) \leq \psi(d(gx,gx_{n+1})). \tag{22}$$

Combinig (20), (22) and on taking limit supremum, we get

$\psi(t) \leq \limsup \varphi(d(gx,gu_n) + \limsup\ (-\theta(d(gx,gu_n)))$. i.e.,
$\psi(t) - \limsup \varphi(d(gx,gu_n) + \liminf (\theta(d(gx,gu_n))) \leq 0$,

which is a contradiction. Hence $t = 0$.

Similarly, we can show that $\lim_{n\to\infty} d(gy,gu_n) = 0$. Hence by triangle inequality, we have $d(gx,gy) \leq d(gx,gu_n) + d(gu_n,gy) \to 0$ as $n \to \infty$. Hence $gx = gy$. Since g is one-to-one, we have $x = y$.

4. COROLLARIES AND EXAMPLES

If ψ is the identity mapping and $\theta(t) = 0$ for all $t \in [0,\infty)$ in Theorem 3.1, we have the following.

Corollary 4.1 *Let $(X,d,\preccurlyeq)$ be a partially ordered complete metric space. Let (A,B) be a pair of nonempty subsets of X. Assume that A_0 is a nonempty subset of A. Let $f : A \to B$ and $g : A \to A$ satisfy the following conditions***:**

(*i*) *f is a g-proximally increasing and (A,B) satisfy the P-property,*
(*ii*) *$g(A_0)$ is closed, $f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$,*
(*iii*) *there exists $\varphi \in \Theta$ with the condition*

$$\overline{\lim}\, \varphi(x_n) < t \tag{23}$$

where $\{x_n\}$ is any sequence in $[0,\infty)$ with $x_n \to t > 0$ and

$$d(fx,fy) \leq \varphi\big(d(gx,gy)\big) \tag{24}$$

for all $x, y \in A_0$ with $gx \preccurlyeq gy$.
Also, suppose that if $\{gx_n\}$ is a nondecreasing sequence in gA_0 such that $gx_n \to gz$, then $g\, x_n \preccurlyeq gz$ for all $n \geq 0$. Furthermore, assume that there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1,fx_0) = d(A,B)$ and $gx_0 \preccurlyeq gx_1$.
Then f and g have proximity coincidence point.

If $\psi(t) = \varphi(t)$ for all $t \in [0,\infty)$ in Theorem 3.1, we have the following.

Corollary 4.2 *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Let (A, B) be a pair of nonempty subsets of X. Assume that A_0 is a nonempty subset of A. Let $f : A \rightarrow B$ and $g : A \rightarrow A$ satisfy the following conditions*:

(i) f is a g-proximally increasing and (A,B) satisfy the P-property,
(ii) $g(A_0)$ is closed, $f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$,
(iii) there exists $\psi \in \Psi$ and $\varphi \in \Theta$ with the condition

$$\overline{\lim}\, \theta(x_n) > 0, \tag{24}$$

where $\{x_n\}$ is any sequence in $[0,\infty)$ with $x_n \rightarrow t > 0$ and

$$\psi\big(d(fx, fy)\big) \leq \psi\big(d(gx, gy)\big) - \theta\big(d(gx, gy)\big) \tag{25}$$

for all $x, y \in A_0$ with $gx \preccurlyeq gy$.
Also, suppose that if $\{gx_n\}$ is a nondecreasing sequence in gA_0 such that $gx_n \rightarrow gz$, then $g\, x_n \preccurlyeq gz$ for all $n \geq 0$. Furthermore, assume that there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1, fx_0) = d(A, B)$ and $gx_0 \preccurlyeq gx_1$.
Then f and g have proximity coincidence point.

If ψ and φ are identity mappings and $\theta(t) = (1 - k)t$, where $0 \leq k < 1$ in Theorem 3.1, we have the following.

Corollary 4.3 *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Let (A, B) be a pair of nonempty subsets of X. Assume that A_0 is a nonempty subset of A. Let $f : A \rightarrow B$ and $g : A \rightarrow A$ satisfy the following conditions*:

(i) f is a g-proximally increasing and (A,B) satisfy the P-property,
(ii) $g(A_0)$ is closed, $f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.
Suppose that there exists $k \in [0,1)$ such that for all $x, y \in A_0$ with $gx \preccurlyeq gy$,

$$d(fx, fy) \leq kd(gx, gy), \quad \text{for all } x, y \in A_0 . \tag{26}$$

Also, suppose that if $\{gx_n\}$ is a nondecreasing sequence in gA_0 such that $gx_n \rightarrow gz$, then $g\, x_n \preccurlyeq gz$ for all $n \geq 0$. Furthermore, assume that there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1, fx_0) = d(A, B)$ and $gx_0 \preccurlyeq gx_1$.
Then f and g have proximity coincidence point.

Since, for any nonempty subset A of X, the pair (A, A) satisfies the P-property if $A = B$ in Theorem 3.1, we have the following fixed point result.

Corollary 4.4 *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Let A be a nonempty subset of X. Let $f : A \rightarrow A$ and $g : A \rightarrow A$ satisfy the following conditions*:

(i) f is a g-nondecreasing,
(ii) $g(A)$ is closed and $f(A) \subseteq g(A)$,
(iii) there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ with the condition

$$\psi(t) - \overline{\lim}\, \varphi(x_n) + \underline{\lim}\, \theta(x_n) > 0, \tag{27}$$

where $\{x_n\}$ is any sequence in $[0,\infty)$ with $x_n \rightarrow t > 0$ and

$$\psi\big(d(fx, fy)\big) \leq \varphi\big(d(gx, gy)\big) - \theta(d(gx, gy)) \tag{28}$$

for all $x, y \in A$ *with* $gx \preccurlyeq gy$ *and also,* $\psi(x) \leq \varphi(y) \Rightarrow x \leq y$.

Also, suppose that if $\{gx_n\}$ *is a nondecreasing sequence in gA such that* $gx_n \rightarrow gz$, *then* $g\,x_n \preccurlyeq gz$ *for all* $n \geq 0$. *Furthermore, assume that there exists an element* $x_0 \in A$ *such that* $gx_0 \preccurlyeq fx_0$.
Then f and g have a coincidence point in A.

If ψ is the identity mapping and $\theta(t) = 0$ for all $t \in [0, \infty)$ in Corollary 4.4, we have the following.

Corollary 4.5 *Let* $(X, d, \preccurlyeq)$ *be a partially ordered complete metric space. Let A be a nonempty subset of X. Let* $f : A \rightarrow A$ *and* $g : A \rightarrow A$ *satisfy the following conditions*:

(i) f *is a g-nondecreasing,*
(ii) $g(A)$ *is closed,* $f(A) \subseteq g(A)$,
(iii) *there exists* $\varphi \in \Theta$ *with the condition*

$$\overline{\lim}\, \varphi(x_n) < t \tag{29}$$

where $\{x_n\}$ *is any sequence in* $[0,\infty)$ *with* $x_n \rightarrow t > 0$ *and*

$$d(fx, fy) \leq \varphi\big(d(gx, gy)\big) \tag{30}$$

for all $x, y \in A_0$ *with* $gx \preccurlyeq gy$.
Also, suppose that if $\{gx_n\}$ *is a nondecreasing sequence in gA such that* $gx_n \rightarrow gz$, *then* $g\,x_n \preccurlyeq gz$ *for all* $n \geq 0$. *Furthermore, assume that there exists an element* $x_0 \in A$ *such that* $gx_0 \preccurlyeq fx_0$.
Then f and g have a coincidence point in A.

If $\psi(t) = \varphi(t)$ for all $t \in [0, \infty)$ in Corollary 4.4, we have the following.

Corollary 4.6 *Let* $(X, d, \preccurlyeq)$ *be a partially ordered complete metric space. Let A be a nonempty subset of X. Let* $f : A \rightarrow A$ *and* $g : A \rightarrow A$ *satisfy the following conditions*:

(i) f *is a g-nondecreasing,*
(ii) $g(A)$ *is closed and* $f(A) \subseteq g(A)$,
(iii) *there exist* $\psi \in \Psi$ *and* $\varphi \in \Theta$ *with the condition*

$$\underline{\lim}\, \theta(x_n) > 0, \tag{31}$$

where $\{x_n\}$ *is any sequence in* $[0,\infty)$ *with* $x_n \rightarrow t > 0$ *and*

$$\psi\big(d(fx, fy)\big) \leq \psi\big(d(gx, gy)\big) - \theta(d(gx, gy)) \tag{32}$$

for all $x, y \in A$ *with* $gx \preccurlyeq gy$ *and also,* $\psi(x) \leq \psi(y) \Rightarrow x \leq y$.

Also, suppose that if $\{gx_n\}$ *is a nondecreasing sequence in gA such that* $gx_n \rightarrow gz$, *then* $g\,x_n \preccurlyeq gz$ *for all* $n \geq 0$. *Furthermore, assume that there exists an element* $x_0 \in A$ *such that* $gx_0 \preccurlyeq fx_0$.
Then f and g have a coincidence point in A.

If ψ and φ are identity mappings and $\theta(t) = (1-k)t$, where $0 \leq k < 1$ in Corollary 4.4, we have the following.

Corollary 4.7 *Let* $(X, d, \preccurlyeq)$ *be a partially ordered complete metric space. Let A be a nonempty subset of X. Let* $f : A \rightarrow A$ *and* $g : A \rightarrow A$ *satisfy the following conditions*:

(i) f is a g-nondecreasing,
(ii) $g(A)$ is closed and $f(A) \subseteq g(A)$,
(iii) there exists $k \in [0,1)$ such that for all $x, y \in A$ with $gx \preccurlyeq gy$,

$$d(fx, fy) \leq kd(gx, gy), \tag{33}$$

Also, suppose that if $\{gx_n\}$ is a nondecreasing sequence in gA such that $gx_n \to gz$, then $g\, x_n \preccurlyeq gz$ for all $n \geq 0$. Furthermore, assume that there exists an element $x_0 \in A$ such that $gx_0 \preccurlyeq fx_0$.
Then f and g have a coincidence point in A.

The following example is in support of Theorem 3.1.

Example 4.8 Let $X = [0,3] \times [0,3]$ with $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We define a partial order $\preccurlyeq$ on X by:

$$\preccurlyeq := \{((x_1, x_2),(y_1, y_2)) \in X \times X | x_1 = y_1,\ x_2 = y_2\} \cup \{((x_1, x_2),(y_1, y_2)) \in X \times X | x_1 = y_1 = 0,\ x_2,\ y_2 \in (0,1],\ x_2 \geq y_2\}.$$

Λετ $A = \{(0, x): 0 \leq x \leq 3\}$, $B = \{(1, x): 0 \leq x \leq 3\}$,
$A_0 = \{(0, x): 0 \leq x \leq 1\}$, $B_0 = \{(1, x): 0 \leq x \leq 1\}$.
We define functions $f: A \to B$ and $g: A \to A$ by

$$f(0, x) = \left(1, \frac{x^2}{2+x}\right) \text{ and } g(0, x) = \left(1, \frac{3x}{2+x}\right).$$

Clearly $d(A, B) = 1$, $f(A_0) \subseteq B_0$, $g(A_0)$ is closed and $A_0 \subseteq g(A_0)$. We now show that the pair (A,B) satisfies the P-property. For this purpose, let $(0, x)$, $(0, y) \in A_0$ and $(1, u)$, $(1, v) \in B_0$ such that

$$d\big((0, x), (1, u)\big) = d(A, B) = 1 \text{ and} \tag{34}$$

$$d\big((0, x), (1, u)\big) = d(A, B) = 1. \tag{35}$$

Hence from (34) and (35), we have $x = u$ and $y = v$. This implies that

$$d\big((0, x), (0, y)\big) = d\big((0, u), (0, v)\big) = d\big((1, u), (1, v)\big).$$

Hence the pair (A,B) satisfies the P-property.

Now, we show that f is g-proximally increasing. In this case, let $(0, x)$, $(0, y)$, $(0, u)$ and $(0, v) \in A$ such that

$$\left.\begin{array}{r} g(0, y) \preccurlyeq g(0, v) \\ d\big((0, x), f(0, y)\big) = 1 \\ d\big((0, u), f(0, v)\big) = 1. \end{array}\right\}$$

Since $g(0, y) \preccurlyeq g(0, v)$, it follows that

$$\left(0, \frac{3y}{2+y}\right) \preccurlyeq \left(0, \frac{3v}{2+v}\right) \Leftrightarrow \left(0, \frac{3y}{2+y}\right) \geq \left(0, \frac{3v}{2+v}\right) \Leftrightarrow y \geq v$$

$$\Leftrightarrow 2y^2 + vy^2 \geq 2v^2 + yv^2 \Leftrightarrow \frac{y^2}{2+y} \geq \frac{v^2}{2+v}. \tag{36}$$

From $d\big((0, x), f(0, y)\big) = d\left((0, x), \left(1, \frac{y^2}{2+y}\right)\right) = 1$, we have

$$x = \frac{y^2}{2+y}. \tag{37}$$

From $d\big((0,u), f(0,v)\big) = d\left((0,u), \left(1, \frac{v^2}{2+v}\right)\right) = 1$, we have

$$u = \frac{v^2}{2+v}. \tag{38}$$

By (36), (37) and (38), we obtain $x \geq u \Leftrightarrow (0,x) \preccurlyeq (0,u)$. Hence f is g-proximally increasing.

We choose $x_0 = \left(0, \frac{1}{2}\right)$, $x_1 = \left(0, \frac{2}{29}\right) \in A_0$ such that $d\left(g\left(0, \frac{2}{29}\right), f\left(0, \frac{1}{2}\right)\right) = d(A, B)$ and

$$g\left(0, \tfrac{1}{2}\right) \preccurlyeq g\left(0, \tfrac{2}{29}\right).$$

We define functions $\psi, \varphi, \theta: [0, \infty) \rightarrow: [0, \infty)$ by

$$\psi(t) = \begin{cases} \frac{7}{8}t & if \quad t \in [0,1] \\ \frac{3}{2}t \ if & t > 1 \end{cases}, \qquad \varphi(t) = \begin{cases} \frac{5}{6}t & if \quad t \in [0,1] \\ \frac{1}{2}t \ if & t > 1 \end{cases} \quad \text{and} \quad \theta(t) = \begin{cases} \frac{1}{16}t & if \quad t \in [0,1] \\ \frac{1}{8}t \ if & t > 1. \end{cases}$$

Let $(0,x), (0,y) \in A$ such that $g(0,x) \preccurlyeq g(0,y)$. i.e., necessarily $x, y \in (0,1]$. Hence

$$\psi\left(d\big(f(0,x), f(0,y)\big)\right) = \psi\left(d\left(\left(1, \frac{x^2}{2+x}\right), \left(1, \frac{y^2}{2+y}\right)\right)\right) = \psi\left(\frac{2x^2 + x^2y - 2y^2 - y^2x}{(2+x)(2+y)}\right)$$

$$= \frac{7}{8}\left(\frac{2x^2 + x^2y - 2y^2 - y^2x}{(2+x)(2+y)}\right) = \frac{7}{8}\left(\frac{(2x+2y+xy)(x-y)}{(2+x)(2+y)}\right) \leq \frac{37}{8}\frac{(x-y)}{(2+x)(2+y)}$$

$$= \frac{5}{6}\left(\frac{6(x-y)}{(2+x)(2+y)}\right) - \frac{1}{16}\left(\frac{6(x-y)}{(2+x)(2+y)}\right)$$

$$= \varphi\left(d\big(g(0,x), g(0,y)\big)\right) - \theta\left(d\big(g(0,x), g(0,y)\big)\right).$$

Hence the inequality (2) holds. Therefore the functions ψ, φ, θ, f and g satisfy all the conditions of Theorem 3.1 and (0, 0), (0, 3) are the proximity coincidence points of *f* and *g*.

Here we observe that g(0,2) and g(0,$\frac{5}{6}$) are not comparable, but there is no $u \in A$ such that $g(u)$ is comparable to both g(0,2) and g(0,$\frac{5}{6}$). Therefore condition *H* in Theorem 3.2 fails to hold and *f* and *g* have more than one proximity coincidence point.

Remark 4.9 The functions ψ, φ and θ in Example 4.8 are not continuous, so that Theorem 2.7 is not applicable. Hence our result is more general than the result of Wangkeeree and Sisarat [17] in which continuous control functions are considered.

The following example is in support of Theorem 3.2.

Example 4.10 Let $X = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \cup \{(0,1), (0,2), (1,1), (1,2)\}$, with the Euclidean metric *d.* We define a partial order $\preccurlyeq$ on *X* by

$\preccurlyeq := \{((x_1, x_2), (y_1, y_2)) \in X \times X | x_1 = y_1,\ x_2 = y_2\} \cup \{((x_1, x_2), (y_1, y_2)) \in X \times X | x_1 = y_1 = 0,$ $x_2, y_2 \in [0, \frac{1}{2}],\ x_2 \geq y_2\} \cup \{((0,1),(0,0)), ((0,2),(0,0))$ with $(x_1, x_2) \preccurlyeq (y_1, y_2) \Leftrightarrow x_1 = y_1 = y_2 = 0,$ $x_2 \geq y_2$, where $x_2 \in \{1,2\}\}$.

Let $A=\{(0,x): 0\le x\le \frac{1}{2}\}\cup\{(0,1),(0,2)\}=A_0$ and $B=\{(1,x): 0\le x\le \frac{1}{2}\}\cup\{(1,1),(1,2)\}=B_0$.

We define $f: A\to B$ and $g: A\to A$ by

$f(0,x)=(1,\frac{x^2}{2})$ for all $0\le x\le\frac{1}{2}$, $f(0,1)=f(0,2)=(1,\frac{1}{4})$ and $g(0,x)=(0,2x^2)$ for all $0\le x\le\frac{1}{2}$,

$$g(0,1)=(0,1),\qquad g(0,2)=(0,2).$$

Clearly $d(A,B)=1$, $f(A_0)\subseteq B_0)$ and $A_0\subseteq gA_0$. We choose $x_0=\left(0,\frac{1}{2}\right)$ and $x_1=\left(0,\frac{1}{4}\right)$. Then clearly $d\left(g\left(0,\frac{1}{4}\right),f\left(0,\frac{1}{2}\right)\right)=d(A,B)$ and $g\left(0,\frac{1}{2}\right)\preccurlyeq g\left(0,\frac{1}{4}\right)$.

We now show that the pair (A,B) satisfies P-property. For this purpose, let $(0,x_1),(0,y_1)\in A_0$ and $(1,u_1),(1,v)\in B_0$ such that $d\big((0,x_1),(1,u_1)\big)=d(A,B)=1$ and $d\big((0,y_1),(1,v_1)\big)=d(A,B)=1$. Then $x_1=u_1$ and $y_1=v_1$. Hence $d\big((0,x_1),(0,y_1)\big)=d\big((1,u_1),(1,v_1)\big)$ so that the pair (A,B) satisfies the P-property.

Now, we show that *f* is g-proximally increasing on *A*. In this case, let $(0,x)$, $(0,u)$ and $(0,v)\in A$ such that

$$\left.\begin{array}{r} g(0,y)\preccurlyeq g(0,v)\\ d\big((0,x),f(0,y)\big)=1\\ d\big((0,u),f(0,v)\big)=1.\end{array}\right\}$$

<u>*Case (i)*</u>: $y,v\in[0,\frac{1}{2}]$.

$$g(0,y)\preccurlyeq g(0,v)\Leftrightarrow y^2\ge v^2. \tag{39}$$

From $d\big((0,x),f(0,y)\big)=1$, we get

$$x=\frac{y^2}{2}. \tag{40}$$

Similarly, from $d\big((0,u),f(0,v)\big)=1$, we get

$$u=\frac{v^2}{2}. \tag{41}$$

From (39), (40) and (41), we obtain $(0,x)\preccurlyeq(0,u)$.

<u>*Case (ii)*</u>: $g(0,1)\preccurlyeq g(0,0)$.

If $d\big((0,x),f(0,1)\big)=1$, we obtain

$$x=\frac{1}{4}. \tag{42}$$

If $d\big((0,u),f(0,0)\big)=1$, we have

$$u=0. \tag{43}$$

From (42) and (43), we obtain $(0,x)=\left(0,\frac{1}{4}\right)\preccurlyeq(0,0)=(0,u)$.

<u>*Case (iii)*</u>: $g(0,2)\preccurlyeq g(0,0)$.

From $d\big((0,x),f(0,2)\big)=1$ and $d\big((0,u),f(0,0)\big)=1$, we have $x=\frac{1}{4}$ and $u=0$. Therefore $(0,x)\preccurlyeq(0,u)$. Hence from all the above cases, we have *f* is g-proximally increasing on *A*.

Now, we define functions $\psi,\varphi,\theta: [0,\infty)\to: [0,\infty)$ by

$$\psi(t) = \begin{cases} t & if \quad t \in [0,1] \\ \frac{3}{2}t^2 & if \quad t > 1\,, \end{cases} \qquad \varphi(t) = \begin{cases} \frac{3}{4}t & if \quad t \in [0,1] \\ \frac{1}{2}t^2 & if \quad t > 1 \end{cases} \quad \text{and} \quad \theta(t) = \begin{cases} \frac{1}{4}t & if \quad t \in [0,1] \\ \frac{1}{8}t^2 & if \quad t > 1. \end{cases}$$

With these ψ, φ and θ, we verify that *f* and *g* satisfy the inequality (2). In the verification of the inequality (2), the following three cases are possible.

<u>Case (i)</u> : $x, y \in [0, \frac{1}{2}]$ such that $g(0,x) \preccurlyeq g(0,y)$.

$$\psi\Big(d\big(f(0,x), f(0,y)\big)\Big) = \psi\left(d\left(\left(1, \frac{x^2}{2}\right), \left(1, \frac{y^2}{2}\right)\right)\right) = \psi\left(\sqrt{\frac{x^2}{2} - \frac{y^2}{2}}\right) = \sqrt{\frac{x^2}{2} - \frac{y^2}{2}} = \frac{\sqrt{2}}{2}\sqrt{x^2 - y^2}$$

$$= \frac{3}{4}\sqrt{2x^2 - 2y^2} - \frac{1}{4}\sqrt{2x^2 - 2y^2}$$

$$= \varphi\Big(d\big(g(0,x), g(0,y)\big)\Big) - \theta\Big(d\big(g(0,x), g(0,y)\big)\Big).$$

<u>Case (ii)</u> : $g(0,1) \preccurlyeq g(0,0)$.

$$\psi\Big(d\big(f(0,1), f(0,0)\big)\Big) = \psi\left(d\left(\left(1, \frac{1}{4}\right), (1,0)\right)\right) = \psi\left(\sqrt{\frac{1}{4}}\right) = \sqrt{\frac{1}{4}} = \frac{1}{2} = \frac{3}{4} - \frac{1}{4}$$

$$= \varphi\Big(d\big(g(0,1), g(0,0)\big)\Big) - \theta\Big(d\big(g(0,1), g(0,0)\big)\Big).$$

<u>Case (iii)</u> : $g(0,2) \preccurlyeq g(0,0)$.

$$\psi\Big(d\big(f(0,2), f(0,0)\big)\Big) = \psi\left(d\left(\left(1, \frac{1}{4}\right), (1,0)\right)\right) = \psi\left(\sqrt{\frac{1}{4}}\right) = \sqrt{\frac{1}{4}} = \frac{1}{2} \le \frac{3}{4} = \frac{1}{2}\sqrt{2^2} - \frac{1}{8}\sqrt{2^2}$$

$$= \varphi\Big(d\big(g(0,2), g(0,0)\big)\Big) - \theta\Big(d\big(g(0,2), g(0,0)\big)\Big).$$

Therefore *f* and *g* satisfy the inequality (2). Also, it is trivial to see that condition *H* holds. Hence *f* and *g* satisfy all the hypotheses of Theorem 3.2 and (0,0) is the unique proximity coincidence point of *f* and *g* in *A*.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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