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# A Note on The Weighted Wiener Index and The Weighted Quasi-Wiener Index

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**Abstract**

In this study, we consider the weighted Wiener index and the weighted quasi-Wiener index of simple connected weighted graphs and we find some bounds for the weighted Wiener index and the weighted quasi-Wiener index of the weighted graphs. Moreover, we obtain some results by using these bounds for weighted and unweighted graphs.

## 1. INTRODUCTION

A weighted graph is a graph that has a numeric label associated with each edge, called the weight of edge. In many applications, the edge weights are usually represented by nonnegative integers or square matrices. In this paper, we generally deal with simple connected weighted graphs where the edge weights are positive definite square matrices.

Let $G=(V,E)$ be a simple connected weighted graph on $n$ vertices. Let $w_{ij}$ be the positive definite weight matrix of order $t$ of the edge $ij$ and assume that $w_{ij}=w_{ji}$. The weight of a vertex $i\in V$ defined as $w_i=\sum_{j:j\sim i} w_{ij}$, where the notation $j\sim i$ to mean that $j$ is adjacent to $i$.

Unless otherwise specified, by a weighted graph we mean a graph with each edge weight is a positive definite square matrix.

The weighted distance between vertices $i$ and $j$ of a weighted graph $G$, denoted by $D_w(i,j)$, is defined to be the sum of the weights of edges in the shortest path from $i$ to $j$. Also, the weighted Wiener index $W(G,w)$ of a weighted graph $G$ is defined as

$$W(G,w)=\frac{1}{2}\sum_{i\in V}\sum_{j\in V}\mu_1(D_w(i,j))=\sum_{i<j}\mu_1(D_w(i,j)),$$

where $\mu_1(D_w(i,j))$ is the largest eigenvalue of $D_w(i,j)$.

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The weighted quasi-Wiener index $W^*(G,w)$ of a weighted graph *G* is defined as

$$W^*(G,w) = nt\sum_{i=1}^{(n-1)t}\frac{1}{\mu_i},$$

where $\mu_1 \geq \mu_2 \geq ... \geq \mu_{nt-t} \geq \mu_{nt-t+1} = \mu_{nt-t+2} = ... = \mu_{nt} = 0$ are the eigenvalues of weighted Laplacian matrix $L(G)$ of *G*.

Wiener index and quasi-Wiener index of unweighted graphs have been researched to a great extent in the literature. This paper is organized as follows. In the Section 2, an upper bound for the weighted Wiener index is obtained and some results are presented weighted and unweighted graphs. In Section 3, some bounds for the weighted quasi-Wiener index are found. Besides, some results for number weighted and unweighted graphs are given.

## 2. AN UPPER BOUND ON THE WEIGHTED WIENER INDEX

**Lemma 2.1.** Let *G* be a simple connected weighted graph and $\mu_1 \geq \mu_2 \geq ... \geq \mu_t$ be the eigenvalues of $D_w(i,j)$. Then

$$tr\left(D_w(i,j)^2\right) \geq \mu_1\left(D_w(i,j)^2\right),$$

where $tr\left(D_w(i,j)^2\right)$ is the trace of $D_w(i,j)^2$.

**Proof.** We clearly have

$$\begin{aligned} tr\left(D_w(i,j)^2\right) &= \sum_{i=1}^{t}\mu_i^2 \\ &\geq \mu_1^2 = \mu_1\left(D_w(i,j)^2\right). \end{aligned}$$

**Theorem 2.2.** Let *G* be a simple connected weighted graph. Then

$$W(G,w) \leq \frac{n}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} tr\left(D_w(i,j)^2\right)}. \quad (1)$$

**Proof.** By the definition of weighted Wiener index, we get

$$W(G,w)^2 = \left(\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_1(D_w(i,j))\right)^2$$

From Cauchy-Schwartz inequality and Lemma 2.1, we have

$$\leq \frac{n}{4}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\mu_1(D_w(i,j))\right)^2$$

$$\leq \frac{n^2}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_1\left(D_w(i,j)^2\right)$$

$$\leq \frac{n^2}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} tr\left(D_w(i,j)^2\right),$$

and then

$$W(G,w) \leq \frac{n}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} tr\left(D_w(i,j)^2\right)}.$$

Hence the theorem is proved.

**Corollary 2.3.** If $G$ be a simple connected weighted graph, where each edge weight $w_{ij}$ is a positive number, then

$$W(G,w) \leq \frac{n}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} D_w(i,j)^2}\ .$$

**Proof.** For number weighted graphs, where the edge weights $w_{ij}$ are positive number, we have $tr\left(D_w(i,j)^2\right) = D_w(i,j)^2$ for all *i, j*. Using Theorem 2.2 we get the required result.

**Corollary 2.4.** If $G$ be a simple connected unweighted graph, then

$$W(G,w) \leq \frac{n}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} d(i,j)^2}\ ,$$

where $d(i,j)$ is the length of the shortest path from *i* to *j*.

**Proof.** For an unweighted graph, $D_w(i,j) = d(i,j)$ for all *i; j*. Using Corollary 2.3 we get the required result.

## 3. SOME BOUNDS ON THE WEIGHTED QUASI-WIENER INDEX

### 3.1. Lower Bounds on The Weighted Quasi-Wiener Index

**Theorem 3.1.1.** Let $G$ be a simple connected weighted graph. Then

$$W^*(G,w) \geq \frac{n(n-1)t^2}{\mu_1}. \qquad (2)$$

**Proof.** By the definition of weighted quasi-Wiener index, we get

$$W^*(G,w) = nt\sum_{i=1}^{(n-1)t}\frac{1}{\mu_i}$$
$$= nt\left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \ldots + \frac{1}{\mu_{nt-t}}\right)$$

$$\geq nt\left(\frac{1}{\mu_1}+\frac{1}{\mu_1}+\ldots+\frac{1}{\mu_1}\right)$$
$$= nt\frac{nt-t}{\mu_1}$$
$$= \frac{n(n-1)t^2}{\mu_1}.$$

Thus

$$W^*(G,w) \geq \frac{n(n-1)t^2}{\mu_1},$$

and the theorem follows.

**Corollary 3.1.2.** If $G$ be a simple connected weighted graph, where each edge weight $w_{ij}$ is a positive number, then

$$W^*(G,w) \geq \frac{n(n-1)}{\mu_1}.$$

**Proof.** For number weighted graphs, where the edge weights $w_{ij}$ are positive number, we have $W^*(G,w) = n\sum_{i=1}^{(n-1)}\frac{1}{\mu_i}$. Using Theorem 3.1.1 we get the required result.

**Theorem 3.1.3.** Let $G$ be a simple connected weighted graph. Then

$$W^*(G,w) \geq nt\frac{1}{\sum_{i=1}^{n}\sum_{k=1}^{t}\sum_{j:j\sim i}\mu_k\left(w_{ij}\right)}. \quad (3)$$

**Proof.** By the definition of weighted quasi-Wiener index, we get

$$\sum_{i=1}^{(n-1)t}\mu_i = tr[L(G)]$$
$$= \sum_{i=1}^{n} tr(w_i)$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{t}\mu_k(w_i)$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{t}\sum_{j:j\sim i}\mu_k\left(w_{ij}\right).$$

Thus,

$$W^*(G,w) = nt\sum_{i=1}^{(n-1)t}\frac{1}{\mu_i}$$

$$\geq nt\frac{1}{\sum_{i=1}^{(n-1)t}\mu_i}$$

$$\geq nt\frac{1}{\sum_{i=1}^{n}\sum_{k=1}^{t}\sum_{j:j\sim i}\mu_k\left(w_{ij}\right)}.$$

The proof is completed.

**Corollary 3.1.4.** If *G* be a simple connected weighted graph, where each edge weight $w_{ij}$ is a positive number, then

$$W^*(G,w)\geq n\frac{1}{\sum_{i=1}^{n}\sum_{j:j\sim i}w_{ij}}.$$

**Proof.** For number weighted graphs, we have $tr(w_i)=w_i=\sum_{j:j\sim i}w_{ij}$. Using Theorem 3.1.3 we get the required result.

**Corollary 3.1.5.** If *G* be a simple connected unweighted graph, then

$$W^*(G,w)\geq n\frac{1}{\sum_{i=1}^{n}d_i},$$

where $d_i$ is the degree of vertex *i*.

**Proof.** For an unweighted graph, $w_{ij}=1$ and $w_i=d_i$ for all *i; j*. Using Corollary 3.1.4 we get the required result.

**3.2. An Upper Bound on The Weighted Quasi-Wiener Index**

**Theorem 3.2.1.** Let *G* be a simple connected weighted graph. Then

$$W^*(G,w)\leq\frac{n(n-1)t^2}{\mu_1}\sqrt{\sum_{i=1}^{n}tr\left[w_i{}^2+\sum_{j:j\sim i}w_{ij}^2\right]}. \tag{4}$$

**Proof.** By the definition of weighted quasi-Wiener index, we get

$$W^*(G,w)^2=\left(nt\sum_{i=1}^{(n-1)t}\frac{1}{\mu_i}\right)^2.$$

From Cauchy-Schwartz inequality, we get

$$\le (n-1)n^2t^3 \sum_{i=1}^{(n-1)t} \frac{1}{\mu_i^{\,2}}$$

$$\le (n-1)^2 n^2 t^4 \frac{tr\left\lfloor L(G)^2 \right\rfloor}{\mu_1^2 \mu_2^2 \dots \mu_{(n-1)t}^2}.$$

Moreover, $(i,i)-$ th element of $L(G)^2$ is equal to $w_i^2 + \sum_{j:j\sim i} w_{ij}^2$ . Hence

$$\le (n-1)^2 n^2 t^4 \frac{\sum_{i=1}^{n} tr\left[ w_i^2 + \sum_{j:j\sim i} w_{ij}^2 \right]}{\mu_1^2 \mu_2^2 \dots \mu_{(n-1)t}^2}$$

$$\le \frac{(n-1)^2 n^2 t^4}{\mu_1^2} \sum_{i=1}^{n} tr\left[ w_i^2 + \sum_{j:j\sim i} w_{ij}^2 \right]$$

and thus

$$W^*(G,w) \le \frac{n(n-1)t^2}{\mu_1} \sqrt{\sum_{i=1}^{n} tr\left[ w_i^{\,2} + \sum_{j:j\sim i} w_{ij}^2 \right]}.$$

Hence the theorem is proved.

**Corollary 3.2.2.** If *G* be a simple connected weighted graph, where each edge weight $w_{ij}$ is a positive number, then

$$W^*(G,w) \le \frac{n(n-1)}{\mu_1} \sqrt{\sum_{i=1}^{n} \left( w_i^{\,2} + \sum_{j:j\sim i} w_{ij}^2 \right)}.$$

**Proof.** For number weighted graphs, where the edge weights $w_{ij}$ are positive number, we have $tr\left( w_i^{\,2} + \sum_{j:j\sim i} w_{ij}^2 \right) = w_i^{\,2} + \sum_{j:j\sim i} w_{ij}^2$ . Using Theorem 3.2.1 we get the required result.

**Corollary 3.2.3.** If *G* be a simple connected unweighted graph, then

$$W^*(G,w) \le \frac{n(n-1)}{\mu_1} \sqrt{\sum_{i=1}^{n} \left( d_i^{\,2} + d_i \right)},$$

where $d_i$ is the degree of vertex *i*.

**Proof.** For an unweighted graph, $w_{ij} = 1$ and $w_i = d_i$ for all *i; j*. Using Corollary 3.2.2 we get the required result.

## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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