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RESEARCH ARTICLE

Partner Curves in Three Dimensional Degenerated Space Form

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

Using elementary and effective methods we study cone curves and their associated curves or partner curves in the three dimensional lightlike cone $\mathbb{Q}^3$, which is called three dimensional degenerated space form of the four dimensional Minkowski space $\mathbb{E}_1^4$. We define the associated curve of the cone curve and also partner curves of some special curves, such as a Bertrand curve and a Mannheim curve. We consider the properties and relations of a curve and its associated curve or partner curve. Some geometric characterizations of these curves are also given.

Keywords: *Degenerated space form, cone curve, curvature and torsion, associated curve, partner curve.*

AMS Subject Classification (2020): *Primary: 53A35; Secondary: 53A40; 53B30; 53C50.*

1. Introduction.

Special curves, or characteristic curves in some sense, are very useful and significant object, which should be good understood not only in geometry and other mathematical fields, but also in a lot of practical sciences, such as industrial design, pattern recognition and intelligent system, graph and image processing, etc ([1]-[4], [20]-[21]). The curves theory of the Minkowski space is very fundamental and important in both physics and mathematics. In [8], a kind of special curves, cone curves are studied and the notion of the cone curvature functions and also some examples of the cone curves in the Minkowski space are given. The applications of the cone curves in the three dimensional Minkowski space (simply, Minkowski 3-space) are given in [9]-[11], [13]-[15] and [19].

In [12], using very elementary and effective methods, a necessary and sufficient condition of a Bertrand curve is given for the non flat three dimensional Riemannian space forms (simply, Riemannian 3-space forms). And an explicit expression of the partner curve of a Bertrand curve is obtained.

As the extended topic, using associated curve, a new necessary and sufficient condition of which a Frenet curve is a Mannheim curve or Mannheim partner curve in the three dimensional Euclidean space is given and relative conclusions are generalized for the curves which lie on the three dimensional Riemannian sphere or lie in the three dimensional hyperbolic space in [16]. From these conclusions we know that the Mannheim curve and Mannheim partner curve on the three dimensional Riemannian sphere or in the three dimensional hyperbolic space can not be as a curves mate as in the case of the three dimensional Euclidean space. And it is easy to see that the geometric characterizations of such curves are conformal invariant.

In this work, for the curves in the three dimensional lightlike cone, i.e. degenerated 3-space form, we consider the generalization of the notions of classical Bertrand curve and Mannheim curve. At first we define the Bertrand curve and Mannheim curve in the 3-dimensional lightlike cone with the algebraic condition. Using associated curve of the cone curve we state a new necessary and sufficient condition of which a cone curve is a Bertrand curve or Mannheim curve in the degenerated 3-space form. We also give some geometric characterizations of these curves. Especially we consider also the partner curves of a cone curve both on the de Sitter space $\mathbb{S}_1^3$ and in the hyperbolic space $\mathbb{H}^3$.

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2. Preliminaries.

In this section we recall some notations and concepts defined in [8]. Let $x : \mathbf{I} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a spacelike curve in the three dimensional lightlike cone $\mathbb{Q}^3$ (degenerated 3-space form) of the four dimensional Minkowski space (Minkowski 4-space) $\mathbb{E}_1^4$ with arc length parameter s. We put

$$y(s) = -\ddot{x}(s) - \frac{1}{2}\langle \ddot{x}(s), \ddot{x}(s)\rangle x(s), \tag{2.1}$$

and have

$$\langle y(s), y(s)\rangle = \langle x(s), x(s)\rangle = \langle y(s), \dot{x}(s)\rangle = 0, \quad \langle x(s), y(s)\rangle = 1. \tag{2.2}$$

Definition 2.1. Let $x = x(s) : \mathbf{I} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a spacelike curve in $\mathbb{Q}^3$ with arc length parameter s. Then $y(s)$, defined by (2.1), is also a curve in $\mathbb{Q}^3$ and called associated curve (or dual curve) of the curve $x(s)$.

We choose vector field $\beta(s)$ such that $\{x(s), \alpha(s), \beta(s), y(s)\}$ forms a standard asymptotic orthonormal basis of the Minkowski 4-space $\mathbb{E}_1^4$. Then the Frenet formulas of the curve $x = x(s) : \mathbf{I} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ can be written as

$$\begin{cases} \dot{x}(s) = \alpha(s), \\ \dot{\alpha}(s) = \kappa(s)x(s) - y(s), \\ \dot{\beta}(s) = \tau(s)x(s), \\ \dot{y}(s) = -\kappa(s)\alpha(s) - \tau(s)\beta(s). \end{cases} \tag{2.3}$$

Definition 2.2. The functions $\kappa(s)$ and $\tau(s)$ in (2.3) are called the (first) cone curvature and cone torsion (or second cone curvature) of the curve $x(s)$ in $\mathbb{Q}^3 \in \mathbb{E}_1^4$. The frame field $\{x(s), \alpha(s), \beta(s), y(s)\}$ is called the cone Frenet frame of the curve $x(s)$.

We consider the associated curve $y(s)$ of $x(s) \subset \mathbb{Q}^3 \subset \mathbb{E}_1^4$. Define $\tilde{x}(\tilde{s}) := y(s)$ and denote the arc length parameter of $\tilde{x}(\tilde{s})$ by $\tilde{s}$, and the cone Frenet frame of $\tilde{x}(\tilde{s})$ by $\{\tilde{x}(s), \tilde{\alpha}(s), \tilde{\beta}(s), \tilde{y}(s)\}$. From (2.3) we have

$$\tilde{\alpha}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s} = -\kappa(s)\alpha(s) - \tau(s)\beta(s). \tag{2.4}$$

Let

$$\tilde{\alpha} = \alpha\cos\theta + \beta\sin\theta, \qquad \theta = \theta(s). \tag{2.5}$$

For convenience we put

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s} = \sqrt{\kappa^2 + \tau^2}. \tag{2.6}$$

Then we have

$$\cos\theta = \frac{-\kappa}{\sqrt{\kappa^2 + \tau^2}}, \qquad \sin\theta = \frac{-\tau}{\sqrt{\kappa^2 + \tau^2}}, \qquad \tan^{-1}\theta = \frac{\kappa}{\tau}. \tag{2.7}$$

From the definition of the cone Frenet frame and (2.5)-(2.7), by a direct calculation, we get

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{s}^2} = \frac{\theta'}{\sqrt{\kappa^2 + \tau^2}}(-\sin\theta + \beta\cos\theta) - \frac{\cos\theta}{\sqrt{\kappa^2 + \tau^2}}y - x \tag{2.8}$$

and

$$\langle \frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{s}^2}, \frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{s}^2}\rangle = \frac{\theta'^2 - 2\kappa}{\kappa^2 + \tau^2}.$$

Then

$$-\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{s}^2} - \frac{1}{2}\langle \frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{s}^2}, \frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{s}^2}\rangle\tilde{x} = \frac{-\theta'}{\sqrt{\kappa^2 + \tau^2}}(-\alpha\sin\theta + \beta\cos\theta) + x - \frac{1}{2}\left(\frac{\theta'^2}{\kappa^2 + \tau^2}\right)y. \tag{2.9}$$

Therefore, together with (2.1), we have

$$\tilde{y} = x - \frac{1}{2}\left(\frac{\theta'^2}{\kappa^2 + \tau^2}\right)y - \frac{\theta'}{\sqrt{\kappa^2 + \tau^2}}(-\alpha\sin\theta + \beta\cos\theta), \tag{2.10}$$

and

$$\tilde{\beta} = \frac{\theta'}{\sqrt{\kappa^2 + \tau^2}}y + (-\alpha\sin\theta + \beta\cos\theta). \tag{2.11}$$

Thus we get the following conclusions.

Theorem 2.1. Let $x(s) \subset \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a cone curve with arc length parameter s. Put

$$y(s) = -\ddot{x}(s) - \frac{1}{2}\langle \ddot{x}(s), \ddot{x}(s)\rangle x(s), \tag{2.12}$$

then the following statements are equivalent.

(1) The curvature $\kappa(s)$ and the torsion $\tau(s)$ of $x(s)$ satisfy $\frac{\tau}{\kappa}$ = constant.
(2) The tangent vector field of curve $x(s)$ intersects tangent vector field of its associate curve $y(s)$ at a constant angle.
(3) The associate curve of $x(s)$ is $y(s)$, and the associate curve of $y(s)$ is $x(s)$.

Proof. From (2.5) and (2.7) we know that the statements (1) and (2) are equivalent. From (2.10) we know that the statements (2) and (3) are equivalent. □

3. Cone Bertrand curve and partner.

For any constants a, c and μ such that $ac \neq 0$ and $2ac + \mu^2 = 0$, we consider a new cone curve $\bar{x}(\bar{s})$ in $\mathbb{Q}^3$:

$$\bar{x}(\bar{s}) := cx + ay + \mu\beta, \tag{3.1}$$

where $\bar{s}$ is the arc length parameter of $\bar{x}(\bar{s})$. From (3.1) we have

$$\bar{\alpha}\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = (c - a\kappa)\alpha - a\tau\beta + \mu\tau x. \tag{3.2}$$

We put

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \sqrt{(c - a\kappa)^2 + a^2\tau^2} \tag{3.3}$$

and

$$\bar{\alpha} = \alpha\cos\varphi + \beta\sin\varphi + \lambda x. \tag{3.4}$$

Then we have

$$\cos\varphi = \frac{c - a\kappa}{\sqrt{(c - a\kappa)^2 + a^2\tau^2}}, \qquad \sin\varphi = \frac{-a\tau}{\sqrt{(c - a\kappa)^2 + a^2\tau^2}}, \qquad \tan^{-1}\varphi = \frac{c - a\kappa}{-a\tau}, \tag{3.5}$$

and

$$\lambda = \frac{\mu\tau}{\sqrt{(c - a\kappa)^2 + a^2\tau^2}}. \tag{3.6}$$

Therefore we obtain the following conclusion:

Theorem 3.1. Let $x(s) \subset \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a cone curve with arc length parameter s. Put

$$y(s) = -\ddot{x}(s) - \frac{1}{2}\langle \ddot{x}(s), \ddot{x}(s)\rangle x(s) \tag{3.7}$$

and

$$\bar{x}(\bar{s}) := \bar{x}(s) = cx(s) + ay(s) + \mu\beta(s). \tag{3.8}$$

Where the frame field $\{x(s), \alpha(s), \beta(s), y(s)\}$ is the cone Frenet frame of the cone curve $x(s)$. Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ of $x(s)$ satisfy $a\kappa + b\tau = c$ if and only if the tangent vector field of the curve $x(s)$ intersects the tangent vector field of the curve $\bar{x}(s)$ at a constant angle, where a, b, c, μ are constant and $ac \neq 0$, $2ac + \mu^2 = 0$.

Proof. From (3.5) we know that $a\kappa + b\tau = c$ if and only if φ is constant. □

According to the classical curves theories and notions in the Euclidean 3-space, we give the following definition for the *Bertrand curves* in $\mathbb{Q}^3 \subset \mathbb{E}_1^4$ ([8]).

Definition 3.1. Let $x(s)$ be a proper curve in $\mathbb{Q}^3 \subset \mathbb{E}^4_1$ with arc length parameter s. Then $x(s)$ is called a Bertrand curve in $\mathbb{Q}^3$ if the curvature function $\kappa(s)$ and the torsion function $\tau(s)$ of $x(s)$ satisfy $a\kappa(s) + b\tau(s) = c$ for some constants a, b, c and $ab \neq 0$. The curve $\bar{x}(s)$ defined by (3.8) is called the cone partner curve (or cone mate) of the Bertrand curve $x(s)$.

Remark 3.1. In the Definition 3.1 we use an algebraic condition to avoid the use of other notions.

For a Bertrand curve $x(s)$ with $c - a\kappa = b\tau$, the relations (3.3) - (3.6) become

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \sqrt{b^2 + a^2}(\varepsilon\tau), \qquad \varepsilon = \pm 1,\ \varepsilon\tau > 0, \tag{3.9}$$

$$\bar{\alpha}(s) = \alpha(s)\cos\varphi + \beta(s)\sin\varphi + \lambda x(s), \tag{3.10}$$

$$\cos\varphi = \frac{\varepsilon b}{\sqrt{b^2 + a^2}}, \qquad \sin\varphi = \frac{-\varepsilon a}{\sqrt{b^2 + a^2}}, \qquad \tan^{-1}\varphi = \frac{b}{-a}, \qquad \lambda = \frac{\varepsilon\mu}{\sqrt{b^2 + a^2}}. \tag{3.11}$$

By a direct calculation we have

$$\ddot{\bar{x}}\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = (\kappa\cos\varphi + \tau\sin\varphi)x - y\cos\varphi + \lambda\alpha,$$

$$\ddot{\bar{x}} = \frac{(\kappa\cos\varphi + \tau\sin\varphi)x - y\cos\varphi + \lambda\alpha}{\sqrt{b^2 + a^2}(\varepsilon\tau)} = \frac{(b\kappa - a\tau)x - by + \mu\alpha}{\tau(a^2 + b^2)},$$

$$\langle \ddot{\bar{x}}, \ddot{\bar{x}} \rangle = \frac{-2\kappa}{\tau^2(a^2 + b^2)},$$

$$\begin{aligned} \bar{\kappa}(\bar{s}) &= -\frac{1}{2}\langle \dot{\bar{\alpha}}(\bar{s}), \dot{\bar{\alpha}}(\bar{s}) \rangle = -\frac{1}{2}\langle \ddot{\bar{x}}(\bar{s}), \ddot{\bar{x}}(\bar{s}) \rangle \\ &= \frac{\kappa}{\tau^2(a^2 + b^2)}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \bar{y}(\bar{s}) &= \bar{y}(s) = -\ddot{\bar{x}}(s) - \frac{1}{2}\langle \ddot{\bar{x}}(s), \ddot{\bar{x}}(s) \rangle \bar{x}(s) \\ &= -\frac{(b\kappa - a\tau)x - by + \mu\alpha}{\tau(a^2 + b^2)} + \frac{\kappa}{\tau^2(a^2 + b^2)}(cx + ay + \mu\beta) \\ &= \left[(a^2 + b^2)\tau^2\right]^{-1}\left[a(\kappa^2 + \tau^2)x - \mu\tau\alpha + \mu\kappa\beta + cy\right], \end{aligned} \tag{3.13}$$

$$\bar{\beta}(\bar{s}) = \bar{\beta}(s) = [(a^2 + b^2)\tau^2]^{-1}[-\mu\kappa x - a\tau\alpha + (b\tau + 2a\kappa)\beta - \mu y]. \tag{3.14}$$

Then

$$\begin{cases} \bar{x}(\bar{s}) = cx(s) + ay(s) + \mu\beta(s), \\ \bar{\alpha}(s) = \alpha(s)\cos\varphi + \beta(s)\sin\varphi + \lambda x(s), \\ \bar{\beta}(\bar{s}) = [(a^2 + b^2)\tau^2]^{-1}[-\mu\kappa x - a\tau\alpha + (b\tau + 2a\kappa)\beta - \mu y], \\ \bar{y}(\bar{s}) = \left[(a^2 + b^2)\tau^2\right]^{-1}\left[a(\kappa^2 + \tau^2)x - \mu\tau\alpha + \mu\kappa\beta + cy\right]. \end{cases} \tag{3.15}$$

Remark 3.2. Really the curve (3.8) can be written as

$$\bar{x}(\bar{s}) := \bar{x}(s) = cx(s) + ay(s) + \mu_1\alpha(s) + \mu_2\beta(s), \tag{3.16}$$

where $2ac + \mu_1^2 + \mu_2^2 = 0$.

4. Cone Mannheim curve and partner.

Theorem 4.1. Let $x(s) \subset \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a cone curve with arc length parameter s. Put

$$y(s) = -\ddot{x}(s) - \frac{1}{2}\langle \ddot{x}(s), \ddot{x}(s)\rangle x(s) \tag{4.1}$$

and

$$\bar{x}(\bar{s}) := \bar{x}(s) = cx(s) + ay(s) + \mu\beta(s). \tag{4.2}$$

Where the frame field $\{x(s), \alpha(s), \beta(s), y(s)\}$ is the cone Frenet frame of the cone curve $x(s)$. The curvature function $\kappa(s)$ and torsion function $\tau(s)$ of $x(s)$ satisfy $c\kappa(s) = a(\kappa^2(s) + \tau^2(s))$ if and only if the intersection angle φ between the tangent vector fields of the curve $x(s)$ and $\bar{x}(s)$ satisfies

$$\tan\varphi(s) = -\frac{\kappa(s)}{\tau(s)},$$

where a, c, μ are constant and $ac \neq 0$, $2ac + \mu^2 = 0$.

Proof. From $c\kappa = a(\kappa^2 + \tau^2)$ we have

$$-\frac{\kappa}{\tau} = \frac{-a\tau}{c - \alpha\kappa}.$$

Therefore by (3.5) we get the conclusion of the theorem. □

Definition 4.1. Let $x(s)$ be a proper curve in $\mathbb{Q}^3 \subset \mathbb{E}^4_1$ with arc length parameter s. Then $x(s)$ is called a Mannheim curve in $\mathbb{Q}^3$ if the curvature function $\kappa(s)$ and the torsion function $\tau(s)$ of $x(s)$ satisfy $c\kappa = a(\kappa^2 + \tau^2)$ for some constants a, c and $ac \neq 0$. The curve $\bar{x}(s)$ defined by (4.2) is called the cone partner curve (or cone mate) of the Mannheim curve $x(s)$.

For the Mannheim curve, (3.3) - (3.6) become

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \sqrt{c(c - a\kappa)}, \tag{4.3}$$

$$\bar{\alpha} = \alpha\cos\varphi + \beta\sin\varphi + \lambda x, \tag{4.4}$$

$$\cos\varphi = \frac{c - a\kappa}{\sqrt{c(c - a\kappa)}}, \qquad \sin\varphi = \frac{-a\tau}{\sqrt{c(c - a\kappa)}}, \qquad \tan^{-1}\varphi = \frac{c - a\kappa}{-a\tau}, \tag{4.5}$$

$$\lambda = \frac{\mu\tau}{\sqrt{c(c - a\kappa)}}. \tag{4.6}$$

By a direct calculation we have

$$\ddot{\bar{x}}\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \dot{\varphi}(-\alpha\sin\varphi + \beta\cos\varphi) - y\cos\varphi + \dot{\lambda}x + \lambda\alpha,$$

$$\begin{aligned}
\ddot{\bar{x}} &= \frac{\dot{\varphi}(-\alpha\sin\varphi + \beta\cos\varphi) - y\cos\varphi + \lambda\alpha}{\sqrt{(c - a\kappa)^2 + a^2\tau^2}} \\
&= \frac{\dot{\varphi}[a\tau\alpha + (c - a\kappa)\beta] - (c - a\kappa)y + \mu\tau\alpha}{c(c - a\kappa)} \\
&= \frac{\dot{\varphi}a\tau + \mu\tau}{c(c - a\kappa)}\alpha + \frac{\dot{\varphi}}{c}\beta - \frac{1}{c}y \\
&= c^{-1}\left[\tau(a\dot{\varphi} + \mu)(c - a\kappa)^{-1}\alpha + \dot{\varphi}\beta - y\right],
\end{aligned}$$

$$\langle \ddot{\bar{x}}, \ddot{\bar{x}}\rangle = \frac{\tau^2(a\dot{\varphi} + \mu)^2 + \dot{\varphi}^2(c - a\kappa)^2}{c^2(c - a\kappa)^2},$$

$$\begin{aligned}
\bar{\kappa}(\bar{s}) &= -\frac{1}{2}\langle \dot{\bar{\alpha}}(\bar{s}), \dot{\bar{\alpha}}(\bar{s})\rangle = -\frac{1}{2}\langle \ddot{\bar{x}}(\bar{s}), \ddot{\bar{x}}(\bar{s})\rangle \\
&= \frac{\tau^2(a\dot{\varphi} + \mu)^2 + \dot{\varphi}^2(c - a\kappa)^2}{-2c^2(c - a\kappa)^2},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}\bar{y}(\bar{s}) &= \bar{y}(s) = -\ddot{\bar{x}}(s) - \frac{1}{2}\langle\ddot{\bar{x}}(s), \ddot{\bar{x}}(s)\rangle\bar{x}(s) = -\ddot{\bar{x}}(s) + \bar{\kappa}(\bar{s})\bar{x}(s) \\ &= -c^{-1}\left[\tau(a\dot{\varphi} + \mu)(c - a\kappa)^{-1}\alpha + \dot{\varphi}\beta - y\right] + \bar{\kappa}(cx + ay + \mu\beta) \\ &= c\bar{\kappa}x - c^{-1}\tau(a\dot{\varphi} + \mu)(c - a\kappa)^{-1}\alpha + (\mu\bar{\kappa} - c^{-1}\dot{\varphi})\beta + (a\bar{\kappa} + c^{-1})y,\end{aligned} \tag{4.8}$$

$$\bar{\beta}(\bar{s}) = \bar{\beta}(s) = B_1(s)x(s) + B_2(s)\alpha(s) + B_3(s)\beta(s) + B_4(s)y(s). \tag{4.9}$$

Where

$$\begin{aligned}-1 &= \beta_1 B_1(s) + \beta_2 B_2(s) + \beta_3 B_3(s) + \beta_4 B_4(s). \\ &= \begin{vmatrix} c & 0 & \mu & a \\ \lambda & \cos\varphi & \sin\varphi & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ c\bar{\kappa} & -c^{-1}\tau(a\dot{\varphi} + \mu)(c - a\kappa)^{-1} & \mu\bar{\kappa} - c^{-1}\dot{\varphi} & a\bar{\kappa} + c^{-1} \end{vmatrix}.\end{aligned} \tag{4.10}$$

Then

$$\begin{cases} \bar{x}(\bar{s}) = cx(s) + ay(s) + \mu\beta(s), \\ \bar{\alpha}(s) = \alpha(s)\cos\varphi + \beta(s)\sin\varphi + \lambda x(s), \\ \bar{\beta}(\bar{s}) = B_1(s)x(s) + B_2(s)\alpha(s) + B_3(s)\beta(s) + B_4(s)y(s), \\ \bar{y}(\bar{s}) = c\bar{\kappa}x - c^{-1}\tau(a\dot{\varphi} + \mu)(c - a\kappa)^{-1}\alpha + (\mu\bar{\kappa} - c^{-1}\dot{\varphi})\beta + (a\bar{\kappa} + c^{-1})y. \end{cases} \tag{4.11}$$

5. Spherical and hyperbolic Bertrand partner curve.

Now we construct a new curve on $\mathbb{S}^3_1$ or in $\mathbb{H}^3$ using the curve $x(s) \subset \mathbb{Q}^3 \subset \mathbb{E}^4_1$ and its associated curve $y(s)$. For any constants a, c such that $ac \neq 0$, we consider the curve $\bar{x}(\bar{s})$ on $\mathbb{S}^3_1$ $(ac > 0)$ or in $\mathbb{H}^3$ $(ac < 0)$:

$$\bar{x}(\bar{s}) := \frac{1}{\sqrt{2|ac|}}(cx + ay), \tag{5.1}$$

where $\bar{s}$ is the arc length parameter of $\bar{x}(\bar{s})$. From (5.1) we have

$$\bar{\alpha}\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \frac{1}{\sqrt{2|ac|}}\left[(c - a\kappa)\alpha - a\tau\beta\right]. \tag{5.2}$$

We put

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \frac{\sqrt{(c - a\kappa)^2 + a^2\tau^2}}{\sqrt{2|ac|}} \tag{5.3}$$

and

$$\bar{\alpha} = \alpha\cos\varphi + \beta\sin\varphi. \tag{5.4}$$

Then we have

$$\cos\varphi = \frac{c - a\kappa}{\sqrt{(c - a\kappa)^2 + a^2\tau^2}}, \qquad \sin\varphi = \frac{-a\tau}{\sqrt{(c - a\kappa)^2 + a^2\tau^2}}, \qquad \tan^{-1}\varphi = \frac{c - a\kappa}{-a\tau}. \tag{5.5}$$

Therefore we obtain the following conclusion:

Theorem 5.1. Let $x(s) \subset \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a cone curve with arc length parameter s. Put

$$y(s) = -\ddot{x}(s) - \frac{1}{2}\langle\ddot{x}(s), \ddot{x}(s)\rangle x(s) \tag{5.6}$$

and

$$\bar{x}(\bar{s}) := \bar{x}(s) = \frac{1}{\sqrt{2|ac|}}\left[cx(s) + ay(s)\right]. \tag{5.7}$$

Then the curvature function $\kappa(s)$ and the torsion function $\tau(s)$ of $x(s)$ satisfy $a\kappa + b\tau = c$, i.e. a Bertrand curve in $\mathbb{Q}^3$, if and only if the tangent vector field of the curve $x(s)$ intersects the tangent vector field of the curve $\bar{x}(s)$ at a constant angle, where a, c are constant and $ac \neq 0$.

Proof. From (5.5) we know that $a\kappa + b\tau = c$ if and only if φ is constant. $\square$

Definition 5.1. Let $x(s)$ be a Bertrand curve in $\mathbb{Q}^3 \subset \mathbb{E}^4_1$ with arc length parameter s. Then the curve $\bar{x}(s)$ defined by (5.7) is called the spherical partner curve $(ac > 0)$ (spherical mate) of $x(s)$ or the hyperbolic partner curve $(ac < 0)$ (hyperbolic mate) of $x(s)$.

Remark 5.1. Since

$$\langle \bar{x}, \bar{x} \rangle = \frac{2ac}{2|ac|} = \varepsilon = \pm 1,$$

the partner curve $\bar{x}(\bar{s})$ of the cone curve $x(s)$ is the curve lie on the de Sitter space $\mathbb{S}^3_1$ $(ac > 0)$ or in the hyperbolic space $\mathbb{H}^3$ $(ac < 0)$.

Remark 5.2. To avoid Crossref, we omit the introduction of the spherical curve on de Sitter space and hyperbolical curve in hyperbolic space.

- For the spherical curve $x(s)$ on $\mathbb{S}^3_1 \subset \mathbb{E}^4_1$, see [12], section 2, page 472.
- For the hyperbolical curve $x(s)$ in $\mathbb{H}^3 \subset \mathbb{E}^4_1$, see [12], section 3, page 476.

Remark 5.3. The Mannheim partner curve can be characterized with the same methods. We end here to avoid lengthy.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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