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On Linear Combinations of Harmonic Mappings Convex in the Horizontal Direction

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Abstract

The process of creating univalent harmonic mappings which are not analytic is not simple or straightforward. One efficient method for constructing desired univalent harmonic maps is by taking the linear combination of two suitable harmonic maps. In this study, we take into account two harmonic, univalent, and convex in the horizontal direction mappings, which are horizontal shears of $\Psi_m(z) = \frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$, and have dilatations $\omega_1(z) = z$, $\omega_2(z) = \frac{z+b}{1+bz}$, $b \in (-1,1)$. We obtain sufficient conditions for the linear combination of these two harmonic mappings to be univalent and convex in the horizontal direction. In addition, we provide an example to illustrate the result graphically with the help of Maple.

1. Introduction

In the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$, a continuous complex-valued function $f = u + iv$ is harmonic for the real harmonic functions u and v, may be expressed as $f = h + \overline{g}$ in which h and g are analytic in $\mathbb{E}$. Denote by H be the class of harmonic mappings f normalized by $h(0) = g(0) = h'(0) - 1 = 0$, where

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=1}^{\infty} b_m z^m.$$

The Jacobian of $f = h + \overline{g}$ is given by $J_f = |h'|^2 - |g'|^2$. In [1], it is proved that $J_f > 0$ in $\mathbb{E}$ if and only if $f \in H$ is locally univalent and sense-preserving. For every z in $\mathbb{E}$, the condition $J_f > 0$ is equal to the dilatation $\omega(z) = g'(z)/h'(z)$ satisfying $|\omega(z)| < 1$ (see [2, 3]).

We denote by $\mathbb{S}_H$ the class of all univalent, harmonic, and sense-preserving mappings $f = h + \overline{g} \in H$. Let $\mathbb{S}_H^0 = \{f \in \mathbb{S}_H : g'(0) = 0\} \subset \mathbb{S}_H$. A domain is said to be convex in the horizontal direction (CHD) (or convex in the vertical direction), if every line parallel to the real axis (or imaginary axis) intersects the domain either with a connected or empty intersection. If $f \in \mathbb{S}_H^0$ maps $\mathbb{E}$ onto a CHD domain, f is said to be a CHD mapping.
A function $f \in \mathbb{S}_H$ is CHD, if

$$h(z) - g(z) = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right) \quad \text{for } \gamma \in \left[\frac{\pi}{2}, \pi\right). \tag{1.1}$$

Let $\mathbb{S}_H(\gamma)$ be the class of all such mappings. Recently, Çakmak et al. [4] studied the convolutions of mappings in the class $\mathbb{S}_H(\gamma)$.
Construction of univalent harmonic mappings is not a very easy and straight forward task. In 1984, Clunie and Sheil-Small introduced a method, known as shear construction or shearing, for constructing a univalent harmonic mapping from a related conformal map. Following method described in the result of Clunie and Sheil-Small [2] creates harmonic mappings that are convex in one direction:

Lemma 1.1. *[2] A harmonic locally univalent function $f = h + \overline{g}$ maps $\mathbb{E}$ univalently onto a domain convex in a direction ϕ if and only if an analytic univalent function $h - e^{2i\phi}g$ maps $\mathbb{E}$ univalently onto a domain convex in the direction of ϕ.*

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Taking the linear combination of two appropriate harmonic maps is another method to create additional examples of non-analytic harmonic mappings. Recently, many researchers have studied this topic such as Dorff and Rolf [5], Long and Dorff [6], and Kumar et al. [7] investigated the linear combination of harmonic univalent mappings which are convex in the vertical direction (CVD). Dorff and Rolf [5] provided the conditions for the linear combination of harmonic mappings which are CVD and have same dilatation to be univalent and CVD. Long and Dorff [6] obtained the conditions (especially conditions of dilatation) for the linear combination of harmonic mappings f_m for $m = 1,2$ which satisfy $h_m + g_m = \frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$ $\left(\gamma_m \in \left[\frac{\pi}{2},\pi\right)\right)$ to be univalent and CVD. Wang et al. [8] proved the linear combinations of harmonic right half plane mappings which satisfy $h_m + g_m = \frac{z}{1-z}$ for $m = 1,2$ are CHD. Additionally, Demirçay [9], Demirçay and Yaşar [10] examined the conditions for the linear combination of harmonic mappings f_m for $m = 1,2$ which satisfy $h_m - g_m = \frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$ to be univalent and CHD.

It is clear from the aforementioned publications that the dilatation functions of the corresponding harmonic functions are significant in determining how their linear combinations behave. In this article, our primary goal is to use two harmonic mappings satisfying (1.1) with particular dilatations $\omega_1(z) = z$, $\omega_2(z) = \frac{z+b}{1+bz}$, $b \in (-1,1)$ to design univalent, sense-preserving, and CHD harmonic mappings. We derive adequate requirements for the univalent and CHD nature of the linear combination of these two harmonic mappings.

2. Preliminary Results

In this section, we state three results obtained by Demirçay [9] and Demirçay and Yaşar [10] and an efficient tool which is known as Cohn's Rule [11].

Theorem 2.1. *[9, 10] Let $f_m = h_m + \overline{g}_m \in \mathbb{S}_H(\gamma_m)$, for $m = 1,2$ and $\gamma_1,\gamma_2 \in \left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). Then $f_3 = \lambda f_1 + (1-\lambda)f_2 \in S_H$ and CHD for $0 \le \lambda \le 1$, if f_3 is locally univalent and sense-preserving.*

Lemma 2.2. *[9, 10] Let $f_m = h_m + \overline{g}_m \in \mathbb{S}_H(\gamma_m)$, for $m = 1,2$ and $\gamma_1,\gamma_2 \in \left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). If $\omega_m = \frac{g_m'}{h_m'}$ are dilatations of f_m, $m = 1,2$, respectively, then the dilatation ω of $f_3 = \lambda f_1 + (1-\lambda)f_2$ $(0 \le \lambda \le 1)$ is given by*

$$\omega = \frac{I}{II} \tag{2.1}$$

where

$$\begin{aligned} I &= \lambda\omega_1(1-\omega_2)\left(1+2z\cos\gamma_2+z^2\right) \\ &\quad +(1-\lambda)\omega_2(1-\omega_1)\left(1+2z\cos\gamma_1+z^2\right), \end{aligned}$$

and

$$\begin{aligned} II &= \lambda(1-\omega_2)\left(1+2z\cos\gamma_2+z^2\right) \\ &\quad +(1-\lambda)(1-\omega_1)\left(1+2z\cos\gamma_1+z^2\right). \end{aligned}$$

Theorem 2.3. *[9, 10] Let $f_m = h_m + \overline{g}_m \in \mathbb{S}_H(\gamma_m)$, for $m = 1,2$ and $\gamma_1,\gamma_2 \in \left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). If $\gamma_1 = \gamma_2$, then $f_3 = tf_1 + (1-\lambda)f_2 \in S_H$ and CHD for $0 \le \lambda \le 1$.*

Lemma 2.4. *(Cohn's Rule, see [11]) Suppose a polynomial*

$$r(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m \tag{2.2}$$

of degree m, and

$$r^*(z) = z^n \overline{r\left(\frac{1}{\overline{z}}\right)} = \overline{c}_m + \overline{c}_{m-1}z + \overline{c}_{m-2}z^2 + \cdots + \overline{c}_0 z^m.$$

Indicate the number of roots in r inside and on the unit circle, respectively, using the symbols s and t. If $|c_0| < |c_m|$, then

$$r_1 = \frac{\overline{c}_m r(z) - c_0 r^*(z)}{z}$$

has the number of roots inside and on the unit circle, respectively, $s_1 = s-1$ and $t_1 = t$.

3. Main Result

Theorem 3.1. *Suppose $f_m = h_m + \overline{g}_m \in \mathbb{S}_H(\gamma_m)$, for $m = 1,2$ and $\gamma_1,\gamma_2 \in \left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). If $\omega_1(z) = z$, $\omega_2(z) = \frac{z+b}{1+bz}$, $b \in (-1,1)$, then $f_3 = \lambda f_1 + (1-\lambda)f_2 \in S_H$ $(0 < \lambda < 1)$ and CHD provided $b(\gamma_1 - \gamma_2) > 0$.*

We require the following lemma in order to demonstrate our primary finding:

Lemma 3.2. *Let $b \in (-1,0)\cup(0,1)$, $\lambda \in (0,1)$, and $\gamma_1,\gamma_2 \in \left[\frac{\pi}{2},\pi\right)$. If $b(\gamma_1-\gamma_2) > 0$, then*

$$(i)\quad |1+b(1-2\lambda)| > |b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)|; \tag{3.1}$$

$$(ii)\quad |1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)| \tag{3.2}$$

$$> |(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2|. \tag{3.3}$$

Proof of (i). It is obvious that $1+b(1-2\lambda)>0$ holds for $b\in(-1,0)\cup(0,1)$ and $\lambda\in(0,1)$. Then, following inequalities needs to be proved

$$-(1+b(1-2\lambda))<b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2), \tag{3.4}$$

$$b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)<1+b(1-2\lambda). \tag{3.5}$$

That is,

$$-[1+b(1-2\lambda)]<b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)<0.$$

First, because $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$, then $b(\gamma_1-\gamma_2)>0$ is equivalent to $b(\cos\gamma_1-\cos\gamma_2)<0$. Therefore, for $0<\lambda<1$ we have

$$b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)<0.$$

Now, we contemplate two cases to prove the second inequality.
Case 1: If $b\in(0,1)$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$, then $b(\gamma_1-\gamma_2)>0$ implies $-1<\cos\gamma_1-\cos\gamma_2<0$. Thus,

$$b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)>-b\lambda(1-\lambda)>-[1+b(1-2\lambda)] \tag{3.6}$$

holds for $\lambda\in(0,1)$. (3.6) holds because of $b\left(\lambda^2-3\lambda+1\right)>-1$ for $\lambda\in(0,1)$ and $b\in(0,1)$.
Case 2: If $b\in(-1,0)$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$, then $b(\gamma_1-\gamma_2)>0$ implies $0<\cos\gamma_1-\cos\gamma_2<1$. Thus,

$$b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)>b\lambda(1-\lambda)>-[1+b(1-2\lambda)] \tag{3.7}$$

holds for $\lambda\in(0,1)$. (3.7) holds because of $b\left(\lambda^2+\lambda-1\right)<1$ for $\lambda\in(0,1)$ and $b\in(-1,0)$.
Proof of (ii). If $b(\gamma_1-\gamma_2)>0$, then in view of inequality (i) we know that

$$[1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)]>0$$

for $b\in(-1,0)\cup(0,1)$, $\lambda\in(0,1)$, $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$. So inequality (ii) is equivalent to the inequalities

$$\begin{aligned}&1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\\ >\;&(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2\\ >\;&-[1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)].\end{aligned} \tag{3.9}$$

Now, let

$$\begin{aligned}f(b,\lambda)\;:=\;&1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\\ &-[(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2]\\ =\;&(1+b)(1-\cos\gamma_1)\\ &+\lambda\left[(1+b(1-2\lambda))(\cos\gamma_1-\cos\gamma_2)+2b(\cos\gamma_1-1)\right].\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial f(b,\lambda)}{\partial b}\;=\;&1-\cos\gamma_1+[3\cos\gamma_1-\cos\gamma_2-2]\lambda\\ &+2[\cos\gamma_2-\cos\gamma_1]\lambda^2,\\ \frac{\partial f(b,\lambda)}{\partial\lambda}\;=\;&4b[\cos\gamma_2-\cos\gamma_1]\lambda+[(3b+1)\cos\gamma_1-(b+1)\cos\gamma_2-2b].\end{aligned}$$

Let $\frac{\partial f(b,\lambda)}{\partial b}=0$ and $\frac{\partial f(b,\lambda)}{\partial\lambda}=0$. Then we have

$$b=b_0=\frac{\cos\gamma_1-\cos\gamma_2}{2-\cos\gamma_1-\cos\gamma_2}$$

and

$$\lambda=\lambda_0=\frac{1}{2},\text{ and }\lambda=\lambda_1=\frac{1-\cos\gamma_1}{\cos\gamma_2-\cos\gamma_1}.$$

Since $\lambda_1\notin(0,1)$, it is obvious that

$$f(b,\lambda)\geq f(b_0,\lambda_0)=1-\frac{\cos\gamma_1}{2}-\frac{\cos\gamma_2}{2}>0$$

which implies that

$$\begin{aligned}&1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\\ >\;&(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2.\end{aligned}$$

Thus, inequality (3.8) is proved.
Next, let

$$\begin{aligned} I &:= (1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2 \\ &\quad +[1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)] \\ &= (1+b)(1+\cos\gamma_1)+\lambda\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right]. \end{aligned}$$

Let

$$g(b):=(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1).$$

Since

$$g'(b)=-\cos\gamma_1-\cos\gamma_2-2<0,$$

g is decreasing for $b\in(-1,0)\cup(0,1)$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$.
Also, $g(-1)=2\cos\gamma_2+2>0$ and $g(1)=-2\cos\gamma_1-2<0$.
If $g(b)<0$, then

$$\begin{aligned} I &= (1+b)(1+\cos\gamma_1)+\lambda\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right] \\ &> (1+b)(1+\cos\gamma_1)+\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right] \\ &> (1-b)(1+\cos\gamma_2) \\ &> 0. \end{aligned}$$

If $g(b)>0$, then

$$\begin{aligned} I &= (1+b)(1+\cos\gamma_1)+\lambda\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right] \\ &> (1+b)(1+\cos\gamma_1) \\ &> 0. \end{aligned}$$

Thus, the proof is complete.
Proof of Theorem 3.1. In view of Theorem 2.1, it suffices to show that $|\omega|<1$ in $\mathbb{E}$. If $b=0$, then $\omega_2(z)=\omega_1(z)=z$. If we substitute these into (2.1), we get $\omega=z$. If $\gamma_1=\gamma_2$, then this case was proved in Theorem 2.3. Thus, we just need to consider the case $b(\gamma_1-\gamma_2)>0$.

Setting $\omega_1(z)=z$ and $\omega_2(z)=\frac{z+b}{1+bz}$ in (2.1), we get

$$\begin{aligned} \omega(z) &= \frac{\lambda z\left(1-\frac{z+b}{1+bz}\right)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)(1-z)\left(1+2z\cos\gamma_1+z^2\right)\frac{z+b}{1+bz}}{\lambda\left(1-\frac{z+b}{1+bz}\right)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)(1-z)\left(1+2z\cos\gamma_1+z^2\right)} \\ &= \frac{\lambda z(1-b)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)\left(1+2z\cos\gamma_1+z^2\right)(z+b)}{\lambda(1-b)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)\left(1+2z\cos\gamma_1+z^2\right)(1+bz)} \\ &= \frac{r(z)}{r^*(z)}, \end{aligned}$$

where

$$\begin{aligned} r(z) &= (1-b\lambda)z^3+[2\lambda(1-b)\cos\gamma_2+2(1-\lambda)\cos\gamma_1+b(1-\lambda)]z^2 \\ &\quad +[1-b\lambda+2b(1-\lambda)\cos\gamma_1]z+b(1-\lambda) \\ &: = c_3z^3+c_2z^2+c_1z+c_0 \end{aligned}$$

and

$$\begin{aligned} r^*(z) &= b(1-\lambda)z^3+[1-b\lambda+2b(1-\lambda)\cos\gamma_1]z^2 \\ &\quad +[2\lambda(1-b)\cos\gamma_2+2(1-\lambda)\cos\gamma_1+b(1-\lambda)]z+(1-b\lambda) \\ &= z^3\overline{p\left(\frac{1}{\overline{z}}\right)}. \end{aligned}$$

Thus if z_0 is a zero of r and $z_0\neq0$, then $1/\overline{z_0}$ is a zero of r^*, we can rewrite

$$\omega(z)=\frac{(z+\eta)(z+\xi)(z+\zeta)}{(1+\overline{\eta}z)\left(1+\overline{\xi}z\right)\left(1+\overline{\zeta}z\right)}.$$

It is known that, the function $\varphi(z)=\frac{z+\delta}{1+\overline{\delta}z}$ for $|\delta|\leq1$ maps closed unit disk $\overline{\mathbb{E}}$ onto itself. If we show that $|\eta|\leq1$, $|\xi|\leq1$, $|\zeta|\leq1$ has a modulus that is strictly less than one for at least one of them, then $|\omega|<1$ in $\mathbb{E}$. As $|c_3|=|1-b\lambda|>|c_0|=|b(1-\lambda)|$ grips for all $-1<b<0,\ 0<b<1,$ and $0<\lambda<1$, applying Lemma 2.4 to r, and thus it suffices to prove that all the roots of r_1 lie inside or on the unit circle where

$$r_1(z)=\frac{c_3r(z)-c_0r^*(z)}{z}=(1-b)\widetilde{r_1}(z)$$

and

$$\begin{aligned}\widetilde{r_1}(z) &= [1+b(1-2\lambda)]z^2 \\ &+[2(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+2\lambda(1-b\lambda)\cos\gamma_2]z \\ &+[b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)] \\ &:= \widetilde{c_2}z^2+\widetilde{c_1}z+\widetilde{c_0}.\end{aligned}$$

By Lemma 3.2 of (i), we have $|\widetilde{c_2}|>|\widetilde{c_0}|$. Then applying again Lemma 2.4 on $\widetilde{r_1}$, we get

$$r_2(z)=\frac{b_2\widetilde{r_1}(z)-b_0\widetilde{r_1^*}(z)}{z}=-4b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\widetilde{r_2}(z),$$

and

$$\begin{aligned}\widetilde{r_2}(z) &= [1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)]z \\ &+[(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2] \\ &:= \widetilde{\widetilde{c_1}}z+\widetilde{\widetilde{c_0}}.\end{aligned}$$

By the Lemma 3.2 of (ii), we have $\left|\widetilde{\widetilde{c_1}}\right|>\left|\widetilde{\widetilde{c_0}}\right|$. Hence, the zeros of $\widetilde{r_2}$, r_2, $\widetilde{r_1}$, and r_1 lie in $|z|<1$. Thus, $|\omega|<1$.

Example 3.3. *Let $\gamma_1=\frac{5\pi}{6}$, then*

$$h_1(z)-g_1(z)=-i\log\left(\frac{1+ze^{i\frac{5\pi}{6}}}{1+ze^{-i\frac{5\pi}{6}}}\right).$$

Suppose $\omega_1(z)=z$, then we get

$$h_1'(z)-g_1'(z)=\frac{1}{(1-z)\left(1+z^2-\sqrt{3}z\right)}.$$

Using

$$\frac{g_1'(z)}{h_1'(z)}=z,$$

then integration gives

$$h_1(z)=\frac{1+\sqrt{3}}{2}\ln\left(\frac{1-z\sqrt{3}+z^2}{1-2z+z^2}\right)+\tan^{-1}(2z-\sqrt{3})+\frac{\pi}{3},$$

and

$$g_1(z)=\frac{1+\sqrt{3}}{2}\ln\left(\frac{1-z\sqrt{3}+z^2}{1-2z+z^2}\right)-\tan^{-1}(2z-\sqrt{3})-\frac{\pi}{3}.$$

Also, let $\gamma_2=\frac{\pi}{2}$ and $\omega_2(z)=\frac{2z+1}{2+z}$. Then

$$h_2(z)-g_2(z)=-\frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right).$$

Thus, we yield

$$h_2(z)=\frac{3}{4}\ln\left(\frac{1+z^2}{1-2z+z^2}\right)+\frac{\tan^{-1}(z)}{2},$$

and

$$g_2(z)=\frac{3}{4}\ln\left(\frac{1+z^2}{1-2z+z^2}\right)-\frac{\tan^{-1}(z)}{2}.$$

Then using Theorem 3.1, we can conclude that $f_3=\lambda f_1+(1-\lambda)f_2\in S_H$ and CHD. The images of the concentric circles which have radius 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9 in the unit disk $\mathbb{E}$ under f_3 with $\lambda=0,\frac{1}{2},1$, respectively, are shown in Figures 3.1, 3.2, and 3.3.

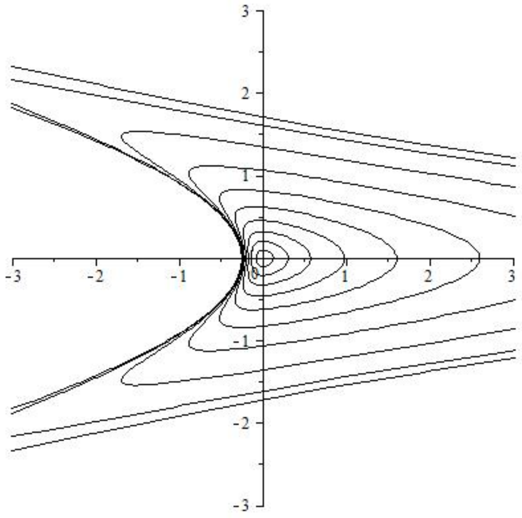

Figure 3.1: The image of $\mathbb{E}$ under f_3 with $\lambda = 0$

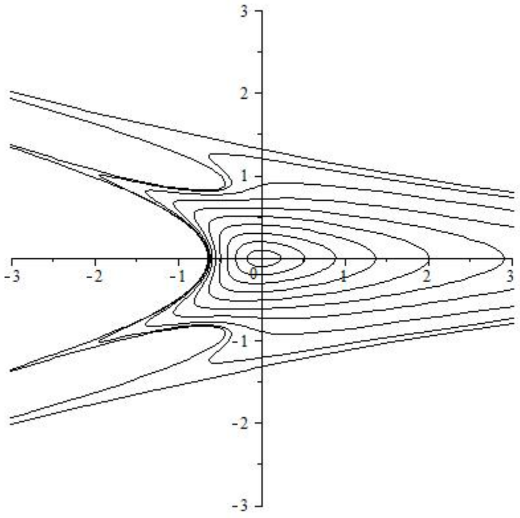

Figure 3.2: The image of $\mathbb{E}$ under f_3 with $\lambda = 1/2$

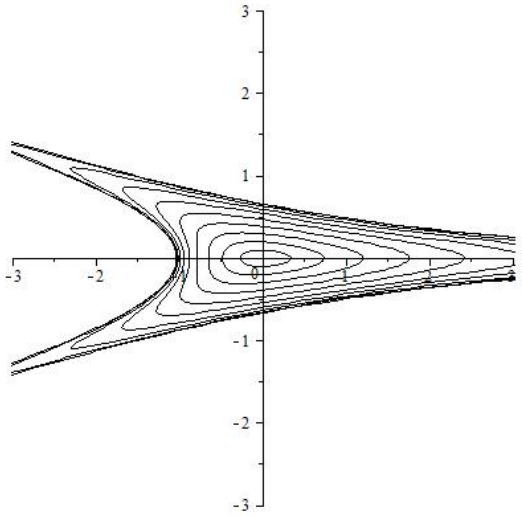

Figure 3.3: The image of $\mathbb{E}$ under f_3 with $\lambda = 1$

4. Conclusion

Fluid flow issues have been studied and resolved using harmonic mapping techniques (see [12]). Specifically, while working with planner fluid dynamical issues, the study of univalent harmonic mappings with unique geometric properties like convexity and convexity in one direction occurs naturally for addressing dynamical planner fluid problems. On the other hand, creating univalent harmonic mappings which are not analytic is not a very easy and straight forward task. To generate new examples of non-analytic desired univalent harmonic mappings, a linear combination of two suitable harmonic mappings can be helpful. In this paper, we considered two harmonic mappings $f_m = h_m + \overline{g}_m$ for $m = 1, 2$ which satisfy $h_m - g_m = \frac{1}{2i\sin\gamma_m} \log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$ for $\gamma_1, \gamma_2 \in \left[\frac{\pi}{2}, \pi\right)$ and have dilatations $\omega_1(z) = z$ and $\omega_2(z) = \frac{z+b}{1+bz}$ for $b \in (-1, 1)$. Our main result is if $b(\gamma_1 - \gamma_2) > 0$ then the linear combination $f_3 = \lambda f_1 + (1-\lambda) f_2$ for $0 < \lambda < 1$ is univalent and CHD. In addition, we provided an example to illustrate the result graphically with the help of Maple.

In our forthcoming research endeavor, we intend to explore the conditions for linear combination and convolution of harmonic mappings involving singular inner functions to be univalent and CHD.

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Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

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