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Higher Dimensional Leibniz-Rinehart Algebras

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Abstract

In this article, we delve into the realm of higher dimensional Leibniz-Rinehart algebras, exploring the intricate structures of Leibniz algebroids and their applications. By generalizing the concept of Lie algebroids and incorporating a Leibniz rule for the anchor map, the study sheds light on the fundamental principles underlying connections and underscores their significance. Through a comprehensive analysis of Leibniz-Rinehart algebras, this study paves the way for advancements and applications, offering a deeper understanding of the intricate relationship between algebraic and geometric structures.

1. Introduction

A Leibniz algebroid is a mathematical structure that generalizes the concept of a Lie algebroid by incorporating a Leibniz rule for the anchor map. Regarding the Lie bracket on algebroid sections, the anchor map in a Leibniz algebroid satisfies a Leibniz rule. This indicates that the Leibniz product of sections which joins the Lie bracket and the anchor map, is preserved by the anchor map. In generalized geometry, Leibniz algebroids are essential because they offer a framework for investigating connections, torsion, and curvature in a more expansive context. Leibniz algebroids form a basic subject of research in contemporary geometric and algebraic structures because of their applicability in many branches of mathematics and physics, such as string theory, mathematical physics, and differential geometry.

Leibniz algebroids generalize the concept of Lie algebroids by incorporating a Leibniz rule for the anchor map. This generalization allows for a broader class of structures to be studied, providing a more flexible framework for geometric and algebraic investigations.

Leibniz algebroids play a crucial role in the study of connections, torsion, and curvature in a generalized setting. By introducing appropriate structures on Leibniz algebroids, one can analyze geometric properties and derive meaningful results related to curvature and other geometric quantities. The study of Leibniz algebroids has applications in various areas of mathematics and theoretical physics. They are used in differential geometry, mathematical physics, and string theory to describe geometric structures and symmetries, making them essential tools for understanding fundamental principles in these fields. Leibniz algebroids provide a bridge between algebraic structures, such as Lie algebras, and geometric objects, such as vector bundles and tangent bundles. This connection allows for a deeper understanding of the interplay between algebraic and geometric concepts, leading to new insights and discoveries. Their study leads to advanced research topics in modern mathematics, including generalized geometry, Poisson geometry, and higher structures. Researchers use Leibniz algebroids to explore cutting-edge ideas and develop new theories that push the boundaries of mathematical knowledge.

The algebraization of Leibniz algebroids, known as Leibniz-Rinehart algebras, are mathematical structures that generalize the relationship between the Leibniz algebra of smooth vector fields on a manifold and the algebra of smooth functions. They are composed of a commutative algebra and a Leibniz algebra with extra structure. This concept provides a categorical framework for addressing problems related to left (right) Kähler quantization and reduction. Leibniz-Rinehart algebras play a crucial role in capturing infinitesimal symmetries and have applications in various mathematical areas, offering a deeper understanding of the relationship between algebraic and geometric structures. The study of Lebniz-Rinehart algebras opens up avenues for exploring connections between classical and quantum theories, paving the way for advancements in mathematical research and applications.

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2. Leibniz-Rinehart Algebras

Leibniz-Rinehart algebras are algebraic structures that combine the properties of Leibniz algebras and modules over a commutative ring. Specifically, a Leibniz-Rinehart algebra over a commutative algebra C consists of a Leibniz algebra $\mathfrak{g}$ together with a C-module structure on $\mathfrak{g}$ and a map ρ from $\mathfrak{g}$ to derivations of C that respects both the Leibniz algebra bracket and the module action. Leibniz-Rinehart algebras generalize Leibniz algebras by incorporating module structures, allowing for a richer interplay between algebraic and geometric properties. They play a significant role in various mathematical areas, offering a framework for studying differential operators, deformations, and geometric structures with algebraic underpinnings.
We indicate that for the rest of the paper, $\mathbb{K}$ is fixed as the ground field and C is fixed as a commutative algebra over $\mathbb{K}$. Here, $\mathrm{Der}(C)$ is the set of all $\mathbb{K}$-derivations over the set C, given by

$$Der(C) = \{D : C \longrightarrow C | D(cc') = cD(c') + D(c)c'\},$$

where it forms the set of all $\mathbb{K}-$linear transformations.

Definition 2.1. *[1] A Leibniz algebra $\mathfrak{g}$ over $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with a $\mathbb{K}$-bilinear map $[-,-] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Leibniz identity*

$$[\varsigma,[\varsigma',\varsigma'']] = [[\varsigma,\varsigma'],\varsigma''] + [\varsigma',[\varsigma,\varsigma'']],$$

for all $\varsigma,\varsigma',\varsigma'' \in \mathfrak{g}$.

Definition 2.2. *[1] Let $\mathfrak{g}$ be a Leibniz algebra and $\varphi := (d,D)$ be a pair of $\mathbb{K}$-linear maps $d,D : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that*

$$\begin{array}{rcl} D[\varsigma,\varsigma'] &=& [D(\varsigma),\varsigma'] - [D(\varsigma'),\varsigma], \\ d[\varsigma,\varsigma'] &=& [d(\varsigma),\varsigma'] + [\varsigma,d(\varsigma')], \\ [\varsigma,d(\varsigma')] &=& [\varsigma,D(\varsigma')], \end{array}$$

for all $\varsigma,\varsigma' \in \mathfrak{g}$. The pair $\varphi = (d,D)$ is called a biderivation of $\mathfrak{g}$.

The set of all biderivations of $\mathfrak{g}$ is denoted by $Bider(\mathfrak{g})$. Following [1], $Bider(\mathfrak{g})$ is endowed with a Leibniz algebra structure with respect to the bracket $[\varphi,\varphi']$ where

$$[\varphi,\varphi'] = (dd' - d'd, Dd' - d'D),$$

for all $\varphi = (d,D), \varphi' = (d',D') \in Bider(\mathfrak{g})$.

Definition 2.3. *[2] Let $\mathfrak{g},\mathfrak{g}'$ be Leibniz algebras. An action of $\mathfrak{g}$ on $\mathfrak{g}'$ is a pair of $\mathbb{K}$-bilinear maps,*

$$\mathfrak{g}\otimes\mathfrak{g}' \longrightarrow \mathfrak{g}',\ (\varsigma,\varsigma') \longmapsto [\varsigma,\varsigma'],\ \mathfrak{g}'\otimes\mathfrak{g} \longrightarrow \mathfrak{g}',\ (\varsigma',\varsigma) \longmapsto [\varsigma',\varsigma],$$

such that

$$\begin{array}{rcl} [\varsigma,[\varepsilon,\varepsilon']] &=& [[\varsigma,\varepsilon],\varepsilon'] + [\varepsilon,[\varsigma,\varepsilon']], \\ [\varsigma,[\varepsilon',\varepsilon]] &=& [[\varsigma,\varepsilon'],\varepsilon] + [\varepsilon',[\varsigma,\varepsilon]], \\ [\varepsilon',[\varsigma,\varepsilon]] &=& [[\varepsilon',\varsigma],\varepsilon] + [\varsigma,[\varepsilon',\varepsilon]], \\ [\varsigma,[\varsigma',\varepsilon']] &=& [[\varsigma,\varsigma'],\varepsilon'] + [\varsigma',[\varsigma,\varepsilon']], \\ [\varsigma',[\varsigma,\varepsilon']] &=& [[\varsigma',\varsigma],\varepsilon'] + [\varsigma,[\varsigma',\varepsilon']], \\ [\varsigma',[\varepsilon',\varsigma]] &=& [[\varsigma',\varepsilon'],\varsigma] + [\varepsilon',[\varsigma',\varsigma]], \end{array}$$

for all $\varsigma,\varepsilon \in \mathfrak{g}$, $\varsigma',\varepsilon' \in \mathfrak{g}'$.

Definition 2.4. *[3] A Leibniz-Rinehart algebra over $(\mathbb{K},C)$ is a Leibniz $\mathbb{K}$-algebra $\mathfrak{g}$ together with a structure of C-module on $\mathfrak{g}$ and the map, called anchor map, $\rho : \mathfrak{g} \longrightarrow \mathrm{Der}(C)$ which are simultaneously Leibniz algebra and C-module homomorphisms such that*

$$[\varsigma,c\varsigma'] = c[\varsigma,\varsigma'] + \rho(\varsigma)(c)\varsigma',$$

$$\rho[|\varsigma,\varsigma'|] = [\rho(\varsigma),\rho(\varsigma')],$$

for all $c \in C$, ς, $\varsigma' \in \mathfrak{g}$.

Let $(\mathfrak{g},\rho),(\mathfrak{g}',\rho')$ be Leibniz-Rinehart algebras. A Leibniz-Rinehart algebra homomorphism $f : (\mathfrak{g},\rho) \longrightarrow (\mathfrak{g}',\rho')$ consists of a simultaneously Leibniz $\mathbb{K}$-algebra and C-module homomorphism $f : \mathfrak{g} \longrightarrow \mathfrak{g}'$ such that $\rho' \circ f = \rho$. Consequently, we have the category of Leibniz-Rinehart algebras over $(\mathbb{K},C)$ which will be denoted here by $\mathfrak{L}\mathfrak{b}\mathfrak{R}(C)$.

Example 2.5.

1. *If $\rho = 0$, then a Leibniz-Rinehart algebra $\mathfrak{g}$ is a Leibniz C-algebra.*
2. *If $C = \mathbb{K}$, then $Der(C) = 0$, and a Leibniz-Rinehart algebra $\mathfrak{g}$ is a Leibniz algebra.*
3. *Every Lie-Rinehart algebra [4]-[9] is a Leibniz-Rinehart algebra, in fact there is an inclusion functor $inc : \mathfrak{L}\mathfrak{R}(C) \hookrightarrow \mathfrak{L}\mathfrak{b}\mathfrak{R}(C)$ from the category of Lie-Rinehart algebras, which is left adjoint to the Liezation functor that assigns to a Leibniz-Rinehart algebra $(\mathfrak{g},\rho)$ the Lie-Rinehart algebra $\mathfrak{g}_{\mathrm{Lie}} = \mathfrak{g}/\mathfrak{g}^{\mathrm{ann}}$, where $\mathfrak{g}^{\mathrm{ann}} = \langle\{[\varsigma,\varsigma] : \varsigma \in \mathfrak{g}\}\rangle$, and anchor map $\widetilde{\rho} : \mathfrak{g}_{\mathrm{Lie}} \longrightarrow \mathrm{Der}(C)$ induced from ρ.*

4. *A right NP-algebra over a ring $\mathbb{K}$ is an algebra P which is an associative and Leibniz algebra and satisfies the right Poisson identity*
$$[p', p\cdot p''] = p\cdot [p', p''] + [p', p]\cdot p'',$$
for all $p, p', p'' \in P$. (The notion introduced in [10] as a noncommutative analogue of classical Poisson algebras).
Let P be commutative. Define $\rho : P \longrightarrow Der(P)$, by $p \longmapsto [p, \_]$, for all $p \in P$. Then we have
$$\begin{aligned}[p', p\cdot p''] = &\quad p\cdot [p', p''] + [p', p]\cdot p'' \\ = &\quad p\cdot [p', p''] + \rho(p')(p)\, p'',\end{aligned}$$
which makes (P, ρ) a Leibniz-Rinehart algebra.

Poisson algebras are of significant importance in mathematics and theoretical physics for several reasons, Poisson algebras are closely related to symplectic geometry, where they provide a framework for studying classical mechanical systems. The Poisson bracket structure on a Poisson algebra captures the essential properties of symplectic manifolds, allowing for the formulation of Hamiltonian dynamics and symplectic geometry in a purely algebraic setting. Moreover, Poisson algebras play a crucial role in the process of quantization, which is the mathematical procedure of transitioning from classical mechanics to quantum mechanics. By understanding the Poisson bracket structure of a system, one can derive quantum operators that correspond to classical observables, leading to a deeper understanding of quantum systems. And, in geometric quantization, Poisson algebras provide a bridge between classical and quantum mechanics by quantizing symplectic manifolds. This process involves associating a Hilbert space to the space of functions on a symplectic manifold, with the Poisson bracket structure guiding the quantization procedure. In addition to the study of Poisson algebras also involves investigating Poisson cohomology, which captures the algebraic structure of Poisson brackets. Poisson cohomology provides insights into the underlying geometry of Poisson manifolds and plays a role in understanding the deformation theory of Poisson structures.Therefore, the following example is significant.

5. *[3] A Leibniz algebroid over a vector bundle E over a base manifold M is an anchor $\rho : E \longrightarrow M$ together with an $\mathbb{R}$-bilinear Leibniz bracket on the $C^\infty(M)$-module $Sec(E)$ of smooth sections of E, which satisfy*
$$[\varsigma, f\varsigma'] = f[\varsigma, \varsigma'] + \rho(\varsigma)(f)\varsigma',$$
for all $f \in C^\infty(M)$, $\varsigma, \varsigma' \in Sec(E)$.
6. *Let $(\mathfrak{g}, \rho)$ be a Leibniz-Rinehart algebra. Then $C \rtimes \mathfrak{g}$ is a Leibniz-Rinehart algebra with the bracket*
$$[(c,\varsigma), (c', \varsigma')] = \left(\rho(\varsigma)(c') - \rho(\varsigma')(c), [\varsigma, \varsigma']\right)$$
and
$$\widetilde{\rho}(c, \varsigma) = \rho(\varsigma),$$
for all $(c,\varsigma), (c', \varsigma') \in C \rtimes \mathfrak{g}$.
7. *Let $T : Der(C) \longrightarrow Der(C)$, be a $\mathbb{K}$-linear and a C-module homomorphism such that*
$$T(D_1)T(D_2) = T(T(D_1)D_2) = T(D_1, T(D_2)),$$
for all D_1, $D_2 \in Der(C)$. Then $Der(C)$ is a Leibniz $\mathbb{K}$-algebra with the bracket
$$\begin{aligned}[\![,]\!] : Der(C) \times Der(C) &\longrightarrow Der(C) \\ (D_1, D_2) &\longmapsto [\![D_1, D_2]\!] = T(D_1)D_2 - D_2T(D_1),\end{aligned}$$
which satisfies
$$[\![D_1, cD_2]\!] = c[\![D_1, D_2]\!] + T(D_1)(c)D_2,$$
for all $c \in C$, D_1, $D_2 \in Der(C)$. On the other hand, we have
$$\begin{aligned}T[\![D_1, D_2]\!] &= T(T(D_1)D_2 - D_2T(D_1)) \\ &= T(T(D_1)D_2) - T(D_2T(D_1)) \\ &= T(D_1)T(D_2) - T(D_2)T(D_1) \\ &= [T(D_1), T(D_2)],\end{aligned}$$
for all $D_1, D_2 \in Der(C)$, which makes $Der(C)$ a Leibniz-Rinehart algebra with the anchor map T.

The action of algebra refers to the application of algebraic structures and operations to analyze mathematical objects and solve problems. Algebraic structures such as groups, rings, fields, and vector spaces provide a framework for understanding symmetries, transformations, and relationships within mathematical systems. The action of algebra is essential in various areas of mathematics and its applications, including physics, cryptography, computer science, and engineering, where algebraic techniques are used for modeling, problem-solving, and optimization. Additionally, algebraic concepts serve as the theoretical foundation for many branches of mathematics, uncovering deep connections and fundamental principles underlying mathematical structures and phenomena.The following definition provides the action of $\mathfrak{g}$, the Leibniz-Rinehart algebra, on R, the Leibniz C-algebra.

Definition 2.6. *Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra and Υ be a Leibniz C-algebra. An action of $\mathfrak{g}$ on Υ is a pair of $\mathbb{K}$-bilinear maps*
$$\begin{aligned}&\mathfrak{g}\otimes\Upsilon \longrightarrow \Upsilon, &\quad& \Upsilon\otimes\mathfrak{g} \longrightarrow \Upsilon \\ &(\varsigma, \upsilon) \longmapsto [\varsigma, \upsilon] && (\upsilon, \varsigma) \longmapsto [\upsilon, \varsigma]\end{aligned}$$
which define a Leibniz action of $\mathfrak{g}$ on Υ in the category of Leibniz $\mathbb{K}$-algebras such that
$$[\upsilon, c\varsigma] = c[\upsilon, \varsigma],$$
$$[\varsigma, c\upsilon] = c[\varsigma, \upsilon] + \rho\varsigma(c)\,\upsilon,$$
for all $c \in C$, $\varsigma \in \mathfrak{g}$, $\upsilon \in \Upsilon$.

Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra and Υ be an abelian Leibniz C-algebra (i.e. a Leibniz algebra with trivial bracket) on which $\mathfrak{g}$ acts in the category of $\mathbb{K}$-algebras. Then Υ is called a Leibniz-Rinehart representation or a right module over $\mathfrak{g}$. We denote the category of Leibniz-Rinehart modules over $\mathfrak{g}$ by $\mathscr{MOD}_{(\mathfrak{g},C)}$.

Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra and Υ be a Leibniz-Rinehart representation over $\mathfrak{g}$. An abelian extension of $\mathfrak{g}$ by Υ is a split exact sequence

$$0 \longrightarrow \Upsilon \longrightarrow \mathfrak{g}' \overset{s}{\curvearrowleft}\!\!\longrightarrow \mathfrak{g} \longrightarrow 0$$

where $\mathfrak{g}'$ is a Leibniz-Rinehart algebra such that the action, which defined by

$$[\varsigma, \upsilon] = [s(\varsigma), i(\upsilon)],$$

$$[\upsilon, \varsigma] = [i(\upsilon), s(\varsigma)],$$

for all $\varsigma \in \mathfrak{g}, \upsilon \in \Upsilon$, of $\mathfrak{g}$ on Υ induced by the extension is the prescribed one.

Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra, Υ be a Leibniz C-algebra with an action of $\mathfrak{g}$ on Υ. Consider the set $\Upsilon \oplus \mathfrak{g}$ and the bracket

$$[(\upsilon, \varsigma), (\upsilon', \varsigma')] = ([\upsilon, \upsilon'] + [\varsigma, \upsilon'] + [\upsilon, \varsigma'], [\varsigma, \varsigma']),$$

for all $\upsilon, \upsilon' \in \Upsilon$, $\varsigma, \varsigma' \in \mathfrak{g}$. $\Upsilon \oplus \mathfrak{g}$ is a Leibniz-Rinehart algebra with anchor map

$$\begin{aligned} \widetilde{\rho} \quad : \quad & \Upsilon \oplus \mathfrak{g} \longrightarrow Der(C), \\ & \widetilde{\rho}(\upsilon, \varsigma) = \rho(\varsigma) \end{aligned}$$

. This constructed Leibniz-Rinehart algebra will be called as the semi-direct product of Υ and $\mathfrak{g}$ which will be denoted by $\Upsilon \rtimes \mathfrak{g}$.

Indeed, $\widetilde{\rho}$, is a Leibniz algebra and C-module homomorphism. On the other hand,

$$\begin{aligned} [(\upsilon, \varsigma), c(\upsilon', \varsigma')] &= [(\upsilon, \varsigma), (c\upsilon', c\varsigma')] \\ &= ([\upsilon, c\upsilon'] + [\varsigma, c\upsilon'] + [\upsilon, c\varsigma'], [\varsigma, a\varsigma']) \\ &= (c[\upsilon, \upsilon'] + c[\varsigma, \upsilon'] + \rho(\varsigma)(c)\upsilon' + c[\upsilon, \varsigma'], c[\varsigma, \varsigma'] + \rho(\varsigma)(c)\varsigma') \\ &= (c([\upsilon, \upsilon'] + [\varsigma, \upsilon'] + [\upsilon, \varsigma']) + \rho(\varsigma)(c)\upsilon', c[\varsigma, \varsigma'] + \rho(\varsigma)(c)\varsigma') \\ &= (a([\upsilon, \upsilon'] + [\varsigma, \upsilon'] + [\upsilon, \varsigma'], [\varsigma, \varsigma']) + (\rho(\varsigma)(c)\upsilon', \rho(\varsigma)(c)\varsigma') \\ &= a([\upsilon, \varsigma], [\upsilon', \varsigma']) + \widetilde{\rho}(\upsilon, \varsigma)(c)(\upsilon', \varsigma'), \end{aligned}$$

for all $(\upsilon, \varsigma), (\upsilon', \varsigma') \in \Upsilon \rtimes \mathfrak{g}$, $c \in C$, as required.

If Υ is abelian then the canonical embeddings $i_\Upsilon : \Upsilon \longrightarrow \Upsilon \rtimes \mathfrak{L}$, $i_\mathfrak{g} : \mathfrak{g} \longrightarrow \Upsilon \rtimes \mathfrak{g}$, as well as the canonical projection $p_\mathfrak{g} : \Upsilon \rtimes \mathfrak{L} \longrightarrow \mathfrak{L}$ are Leibniz-Rinehart homomorphisms. Consequently, we have the abelian extension

$$\Upsilon \overset{i_\Upsilon}{\rightarrowtail} \Upsilon \rtimes \mathfrak{g} \overset{p_\mathfrak{g}}{\longrightarrow} \mathfrak{g}$$

which splits by $i_\mathfrak{g} : \mathfrak{g} \longrightarrow \Upsilon \rtimes \mathfrak{g}$. The induced representation structure on the kernel from the sequence coincides with the previous one.

Definition 2.7. *Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra and Υ be a representation of $\mathfrak{g}$. A derivation from $\mathfrak{g}$ to Υ consists of a map $\delta : \mathfrak{g} \longrightarrow \Upsilon$ such that*

$$\delta(c\varsigma) = c\delta(\varsigma),$$

$$\delta([\varsigma, \varsigma']) = [\delta(\varsigma), \varsigma'] + [\varsigma, \delta(\varsigma')],$$

for all $c \in C$, $\varsigma, \varsigma' \in \mathfrak{g}$.

The set of all derivations from $\mathfrak{g}$ to Υ gives rise to an C-module structure which will be denoted by $Der_C(\mathfrak{g}, \Upsilon)$.

Theorem 2.8. *There is a 1-1 correspondence between the elements of $Der_C(\mathfrak{g}, \Upsilon)$ and the Leibniz-Rinehart homomorphisms $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g} \rtimes \Upsilon$, for which $p_\mathfrak{g} \circ \sigma = id_\mathfrak{g}$.*

Proof. A map σ satisfying $p_\mathfrak{g} \circ \sigma = id_\mathfrak{g}$ gives rise to a derivation $\delta_\sigma = p_\Upsilon \circ \sigma : \mathfrak{g} \longrightarrow \Upsilon$. On the other hand, for a given derivation $\delta : \mathfrak{g} \longrightarrow \Upsilon$. we have the Leibniz-Rinehart homomorphism $\sigma_\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \rtimes \Upsilon$, $\varsigma \mapsto (\varsigma, \sigma(\varsigma))$, for all $\varsigma \in \mathfrak{g}$. The maps $\sigma \mapsto \delta_\sigma$, $\delta \mapsto \sigma_\delta$ are inverse to each other, as required. □

Let Υ be a Leibniz C-algebra and $\mathfrak{g}$ be a Leibniz-Rinehart algebra. Let $DO(C, \mathfrak{g}, \Upsilon)$ be the vector space of pairs (φ, ς) where $\varphi = (d, D) \in Bider_\mathbb{K}(\Upsilon)$ and $\varsigma \in \mathfrak{g}$ such that

$$d(c\upsilon) = cd(\upsilon) + \rho(\varsigma)(c)\upsilon,$$

$$D(c\upsilon) = cD(\upsilon),$$

for all $c \in C$, $\upsilon \in \Upsilon$. Then the componentwise operations make $DO(C, \mathfrak{g}, \Upsilon)$ a C-module and Leibniz $\mathbb{K}$-algebra. In addition, $DO(C, \mathfrak{g}, \Upsilon)$ is a Leibniz-Rinehart algebra where the anchor map defined as the composition of

$$DO(C, \mathfrak{g}, \Upsilon) \overset{pr}{\longrightarrow} \mathfrak{g} \overset{\rho}{\longrightarrow} Der(C),$$

Indeed,

$$[(\varphi, \varsigma), c(\varphi', \varsigma')] = c[(\varphi, \varsigma), (\varphi', \varsigma')] + (\varphi, \varsigma)(c)(\varphi', \varsigma'),$$

for all $c \in C$, (φ, ς), $(\varphi', \varsigma') \in DO(C, \mathfrak{g}, \Upsilon)$.

Let $Bider_C(\Upsilon)$ be the Leibniz C-algebra of all C-biderivations of the Leibniz C-algebra Υ. Then we have the following exact sequence

$$0 \longrightarrow Bider_C(\Upsilon) \xrightarrow{\beta} DO(C,\mathfrak{g},\Upsilon) \xrightarrow{p} \mathfrak{g},$$

where $p((d,D),\varsigma) = \varsigma$ and $\beta(d,D) = ((d,D),0)$, for all $(d,D) \in Bider_C(\Upsilon)$, $((d,D),\varsigma) \in DO(C,\mathfrak{g},\Upsilon)$.
Let $\mathfrak{g}$ has an action on Υ. We have the Leibniz-Rinehart algebra homomorphism $f : \mathfrak{g} \longrightarrow DO(C,\mathfrak{g},\Upsilon)$ which makes

$$\begin{array}{ccc} DO(C,\mathfrak{g},\Upsilon) & \xrightarrow{p} & \mathfrak{g} \\ {\scriptstyle f}\uparrow & \diagup\!\!\diagup & \\ \mathfrak{g} & & \end{array}$$

commutative.
For any Leibniz-Rinehart algebra homomorphism $f : \mathfrak{g} \longrightarrow DO(C,\mathfrak{g},\Upsilon)$, $\varsigma \longmapsto (\varphi^{\varsigma},\varsigma)$, the maps $\mathfrak{g}\times\Upsilon \longrightarrow \Upsilon$, $(\varsigma,\upsilon) \longmapsto [\varsigma,\upsilon] := d^{\varsigma}(\upsilon)$ and $\Upsilon\times\mathfrak{g} \longrightarrow \Upsilon$, $(\upsilon,\varsigma) \longmapsto [\upsilon,\varsigma] := D^{\varsigma}(\upsilon)$ defines an action of $\mathfrak{g}$ on Υ. Indeed,

$$\begin{aligned} [\varsigma, c\upsilon] &= d^{\varsigma}(c\upsilon) \\ &= d^{\varsigma}(\upsilon) + \varsigma(c)\upsilon \\ &= c[\varsigma,\upsilon] + \varsigma(c)\upsilon \end{aligned}$$

On the other hand, since

$$\begin{aligned} (d^{c\varsigma}, D^{c\varsigma}) &= f(c\varsigma) = cf(\varsigma) \\ &= c(d^{\varsigma}, D^{\varsigma}) \\ &= (cd^{\varsigma}, cD^{\varsigma}) \end{aligned}$$

we have

$$[\upsilon, c\varsigma] = cD^{\varsigma}(\upsilon) = c[\upsilon,\varsigma] \ ,$$

for all $c \in C$, $\upsilon \in \Upsilon, \varsigma \in \mathfrak{g}$, as required.
Various algebraic structures of crossed modules are given in [11]-[21]. Similarly, we have provided our definition of crossed module in the following.

3. Crossed Modules of Leibniz-Rinehart Algebras

A crossed module of Leibniz-Rinehart algebras over a base ring C consists of a Leibniz-Rinehart algebra $\mathfrak{g}$, a Leibniz C-algebra Υ, an action of $\mathfrak{g}$ on Υ, and a Leibniz algebra homomorphism ∂ from Υ to $\mathfrak{g}$ satisfying certain compatibility conditions. This concept generalizes the notion of crossed modules for Leibniz algebras and provides a framework for studying the interactions between Leibniz-Rinehart algebras. The classification of crossed modules of Leibniz-Rinehart algebras is closely related to the third cohomology of Leibniz-Rinehart algebras, highlighting the deep connection between algebraic structures and cohomological invariants in this setting. The study of crossed modules for Leibniz-Rinehart algebras offers insights into the algebraic and geometric properties of these structures, contributing to a deeper understanding of their behavior and applications in various mathematical contexts.

Definition 3.1. *A crossed module $\partial : \Upsilon \longrightarrow \mathfrak{g}$ in the category of Leibniz-Rinehart algebras, which will be called as Leibniz- Rinehart crossed module hereafter, is a homomorphism of Leibniz $\mathbb{K}$-algebras consisting of a Leibniz-Rinehart algebra $\mathfrak{g}$ and a Leibniz C-algebra Υ together with an action of $\mathfrak{g}$ on Υ such that*

$$\begin{aligned} &\partial[\varsigma,\upsilon] = [\varsigma,\partial(\upsilon)], \\ &\partial[\upsilon,\varsigma] = [\partial(\upsilon),\varsigma], \\ &[\partial(\upsilon'),\upsilon] = [\upsilon',\upsilon] = [\upsilon',\partial(\upsilon)], \\ &\partial(c\upsilon) = c\partial(\upsilon), \\ &\partial(\upsilon)(c) = 0, \end{aligned}$$

for all $\upsilon,\upsilon' \in \Upsilon$, $\varsigma \in \mathfrak{g}$, $c \in C$.

Let $(\mathfrak{g},\rho)$ be a Leibniz-Rinehart algebra. A Leibniz-Rinehart subalgebra $\mathscr{I}$ of $\mathfrak{g}$ is a Leibniz $\mathbb{K}$-subalgebra $\mathscr{I}$, which is a Leibniz-Rinehart algebra with anchor map induced from ρ. A Leibniz-Rinehart subalgebra $\mathscr{I}$ of $\mathfrak{g}$ is an ideal if $\mathscr{I}$ is an ideal of $\mathfrak{g}$ as Leibniz $\mathbb{K}$-algebra and the compositions,

$$\mathscr{I} \hookrightarrow \mathfrak{g} \xrightarrow{\rho} Der(C)$$

is trivial.

Example 3.2. *Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra and $\mathscr{I}$ is an ideal of $\mathfrak{g}$. Then $(\mathscr{I},\mathfrak{L},\mathfrak{inc.})$ is a crossed module with the actions of $\mathfrak{g}$ on $\mathscr{I}$ defined by*

$$\begin{array}{ccc} \mathfrak{g}\times\mathscr{I} & \longrightarrow & \mathscr{I} \\ (\varsigma,\upsilon) & \mapsto & [\varsigma,\upsilon] \end{array} \qquad \begin{array}{ccc} \mathscr{I}\times\mathfrak{g} & \longrightarrow & \mathscr{I} \\ (\upsilon,\varsigma) & \mapsto & [\upsilon,\varsigma]. \end{array}$$

Proposition 3.3. *If $\partial : \Upsilon \longrightarrow \mathfrak{g}$ is a crossed module then $Im(\partial)$ is an ideal of $\mathfrak{g}$*

Proof. Since $\partial : \Upsilon \longrightarrow \mathfrak{g}$ is a crossed module, we have

$$[\varsigma,\partial(\upsilon)] = \partial[\varsigma,\upsilon],$$

$$\partial[\upsilon,\varsigma] = [\partial(\upsilon),\varsigma],$$

for all $\upsilon \in \Upsilon$ and $\varsigma \in \mathfrak{g}$. Then $\partial[\varsigma,\upsilon]$, $\partial[\upsilon,\varsigma] \in Im(\partial)$, and $Im(\partial) \trianglelefteq \mathfrak{g}$ □

Example 3.4. *Let Υ be a representation over $\mathfrak{g}$. Then the zero morphism $0 : \Upsilon \longrightarrow \mathfrak{g}$ is a Leibniz-Rinehart crossed module.*

Proposition 3.5. *Let $\partial : \Upsilon \longrightarrow \mathfrak{g}$ be a Leibniz-Rinehart crossed module. Then we have the following:*
(i) $ker(\partial) \trianglelefteq \Upsilon$
(ii) $ker(\partial)$ is a $\mathfrak{g}/\partial(\Upsilon)$-module.

Proof. Direct checking. □

Under the light of this information, we can think Leibniz-Rinehart crossed modules as the generalizations of Leibniz-Rinehart algebras and ideals.

Example 3.6. *Let $\mathfrak{g}$ be a Leibniz-Rinehart algebra, $\theta : \Upsilon \longrightarrow \Upsilon'$ be a homomorphism of representations over $\mathfrak{g}$. We have the action of $\Upsilon' \rtimes \mathfrak{g}$ on Υ defined by*

$$\left[(\upsilon',\varsigma),\upsilon\right] = [\varsigma,\upsilon],\ \left[\upsilon,(\upsilon',\varsigma)\right] = [\upsilon,\varsigma],$$

for all $\varsigma \in \mathfrak{g}$, $\upsilon \in \Upsilon$ and $\upsilon' \in \Upsilon'$. Define

$$\begin{array}{rcl} \partial : \Upsilon & \longrightarrow & \Upsilon' \rtimes \mathfrak{g} \\ \upsilon & \longmapsto & (\theta(\upsilon), 0). \end{array}$$

Then $(\Upsilon, \Upsilon' \rtimes \mathfrak{g}, \partial)$ is a Leibniz-Rinehart crossed module with the defined action of $\Upsilon' \rtimes \mathfrak{g}$ on Υ .

4. Conclusion

In this section, the exploration of higher dimensional Leibniz-Rinehart algebras in our article has provided valuable insights into the intricate structures of Leibniz algebroids and their applications. By generalizing the concept of Lie algebroids and incorporating a Leibniz rule for the anchor map, the study has deepened our understanding of connections in algebraic structures. The findings not only pave the way for advancements in the field but also offer a bridge between algebraic and geometric concepts, leading to new insights and discoveries. The study of higher dimensional Leibniz-Rinehart algebras holds promise for further research in areas such as generalized geometry, Poisson geometry, and higher structures, contributing to the ongoing exploration of advanced mathematical theories and applications.

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