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Some Weighted Martingale Inequalities on Rearrangement Invariant Quasi-Banach Function Spaces

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ABSTRACT

The Burkholder-Davis-Gundy’s inequalities and the sharp maximal function inequalities for martingale inequalities are established for rearrangement invariant quasi-Banach function spaces. Martingale inequalities very important in mathematic Martingale inequalities are worked by very mathematicians. We will establish some weighted Martingale inequalities for rearrangement invariant quasi-Banach function spaces.

Keywords: Banach space, Martingale inequalities, Sharp maximal function

Yeniden Düzenlenmiş Değişmez Quasi-Banach Fonksiyon Uzaylarında Ağırlıklı Martingale Eşitsizlikler

ÖZ

Daha önce yeniden düzenlenmiş değişmez quasi Banach fonksiyon uzaylarında Martingale eşitsizliği için sharp maksimal fonksiyon ve Burkholder-Davis-Gundy’s eşitsizlikleri kurulmuştu. Martingale eşitsizlikleri matematikte çok önemli bir yere sahiptir. Bir çok matematikçi tarafından üzerinde durulmuştur. Bizde yeniden düzenlenmiş değişmez quasi- Banach fonksiyon uzaylarında ağırlıklı Martingale eşitsizliklerini inşa edeceğiz.

Anahtar Kelimeler: Banach uzayı, Martingale eşitsizlikleri, Sharp maksimal fonksiyon

INTRODUCTION

Our aim in this paper is the generalization of some weighted martingale inequalities to rearrangement invariant quasi-Banach function spaces. The study of martingale inequalities on Banach function spaces have been further extended to Doob’s inequality, Davis inequalities and Doob’s decompositions in [1, 2]. Since the establishment of some important martingale inequalities in Lebesgue spaces, such as the Burkholder-Davis-Gundy’s inequalities [3], there are several generalizations on these inequalities to general function spaces. The martingale inequalities for sharp functions on Lebesgue spaces was obtained in [4-11]. For instance, the Burkholder-Davis-Gundy’s inequalities is generalized to Banach function spaces in [12, 13, 14, 15-18]. The Burkholder-Davis-Gundy’s inequalities and the sharp maximal function inequalities for martingale are established for rearrangement invariant quasi-Banach function spaces in [19, 20]. Furthermore, we are also interested in the some weighted martingale inequalities for sharp functions. In [21-25], the corresponding results in Orlicz spaces and in term of modular were obtained. The martingale inequalities of sharp functions on Lorentz spaces is recently showed in [26]. In this paper, we extend the Burkholder-Davis-Gundy’s inequalities and the martingale inequalities of sharp functions to quasi-Banach function spaces.

The result for martingale inequalities of sharp functions is also motivated by the following inequality for locally integrable functions on R^n

$$\|M_d(f)\|_{L^p} \leq C\|f^{\#}\|_{L^p}, \quad 0 < p < \infty$$

where $f \in L^p(R^n)$, $f^{\#}$ is the sharp maximal function of f and $M_d(f)$ is the dyadic maximal function of f [27, 28]. For the Burkholder-Davis-Gundy’s inequalities, they are valid for any quasi-Banach function spaces X with $1 < p_X \leq q_X < \infty$. The Burkholder Davis-Gundy’s inequalities cannot be generalized to quasi-Banach function spaces with $p_X < 1$. For instance, in [28, Proposition 2.16], a counterexample is given to show that the Burkholder-Davis-Gundy’s inequalities are not valid on L^p with $0 < p < 1$. Therefore, our result already cover the range $1 < p_X \leq q_X < \infty$, the only possible generalization is to the range $1 < p_X \leq q_X < \infty$.

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Auxillary Statements and Definitions

Let $(\Omega, \Sigma, \mathfrak{A})$ be a complete probability space. We denote the space of measurable function on $(\Omega, \Sigma, \mathfrak{A})$ by $\mathcal{K}$. Let $\mathcal{A} = (\mathcal{A}_n)_{n\geq 0}$ be a filtration. That is, $(\mathcal{A}_n)_{n\geq 0}$ is a nondecreasing sequence of sub σ algebras of Σ with $\Sigma = \sigma(\cup_{n\geq 0} \mathcal{A}_n)$.

Let $\mathcal{A}_{-1} = \mathcal{A}_0$.

The conditional expectation operators relative to $\mathcal{A}_n$ are denoted by $\mathbb{M}_n$. For any martingale $f = (f_n)_{n\geq 0}$ on Ω, write $d_i f = f_i - f_{i-1}$, $i \geq 0$. The maximal function, the square function (quadratic variation) and the conditional square function (conditional quadratic variation) of f are defined by

$$M_n(f) = \sup_{0\leq i\leq n}|f_i|,\ \ M(f) = \sup_{i\geq 0}|f_i|,$$
$$S_n(f) = (\textstyle\sum_{i=0}^{n}|d_i f|^2)^{1/2},$$
$$S(f) = (\textstyle\sum_{i=0}^{\infty}|d_i f|^2)^{1/2},$$
$$s_n(f) = (\textstyle\sum_{i=0}^{n}\mathbb{M}_{i-1}.\,|d_i f|^2)^{1/2},$$
$$s(f) = (\textstyle\sum_{i=0}^{\infty}\mathbb{M}_{i-1}.\,|d_i f|^2)^{1/2},$$

respectively.

Let $0 < r < \infty$. For any uniformly integrable martingale $f = (f_n)_{n\geq 0}$ on Ω, the sharp functions of f are defined by

$$f^{\#} = \sup_{n>0}\mathbb{M}_n|f - f_{n-1}|,$$
$$f_r^S = \sup_{n\geq 0}\left(\mathbb{M}_n[S^2(f) - S_{n-1}^2(f)]^{r/2}\right)^{1/r},$$
$$f_r^s = \sup_{n\geq 0}\left(\mathbb{M}_n[s^2(f) - s_{n-1}^2(f)]^{r/2}\right)^{1/r},$$

respectively. For any $f \in \mathcal{K}$, the distribution function of f is given by

$$\mathfrak{A}_f(\lambda) = \mathfrak{A}\{x \in \Omega: |f(x)| > \lambda\},\ \lambda \geq 0.$$

The decreasing rearrangement of f is defined by

$$f^*(t) = inf\{\lambda: \mathfrak{A}_f(\lambda) \leq t\},\ t \geq 0.$$

Definition 2.1. A quasi-Banach space $X \subset \mathcal{K}$ is called a rearrangement invariant quasi-Banach function space if there exists a quasi Banach function space $\bar{X}$ on (0,∞) with quasi-norm $\rho_{\bar{X}}: \mathcal{K}(0,\infty) \to [0,\infty]$ so that $\|f\|_x = \rho_{\bar{X}}(f^*)$, $f \in X$ where $\mathcal{K}(0,\infty)$ denote the set of Lebesgue measurable functions on (0,∞). Notice that whenever X is a rearrangement-invariant Banach function space studied in [4, Chapter 2, Definition 4.1] over an nonatomic measure space, the existence $\bar{X}$ of associated with X is guaranteed by the Luxemburg representation theorem [4, Chapter 2, Theorem 4.10]). Therefore, Definition 2.1 is a generalization of the notion of rearrangement-invariance to quasi Banach function spaces. We recall the definition of the Boyd indices on quasi-Banach function spaces. For any $s \geq 0$ and $f \in \mathcal{K}(0,\infty)$, define $(D_s f)(x) = f(sx), x \in (0,\infty)$.
Let $\|D_s\|_{\bar{X}\to\bar{X}}$ be the operator norm of D_s on $\bar{X}$

Definition 2.2. Let X be a quasi-Banach function space on $\mathcal{K}$. The lower Boyd index of X, p_X, and the upper Boyd index of X, q_X, are defined by

$$p_X = sup\left\{p: \exists C > 0\ \ such\ that\ \forall\ 0 \leq s < 1,\ \ \|D_s\|_{\bar{X}\to\bar{X}} \leq Cs^{-1/p}\right\}$$

and

$$q_X = inf\left\{q: \exists C > 0\ \ such\ that\ \forall\ 1 \leq s,\ \ \|D_s\|_{\bar{X}\to\bar{X}} \leq Cs^{-1/q}\right\}$$

respectively.

The following inequalities give the connection between the decreasing rearrangements of $\mathcal{K}(f)$ and $f^{\#}$, which plays a fundamental role on the proof of the martingale inequalities of sharp functions on quasi-Banach function spaces.

Definition 2.3. A weight function on a set Ω is a mapping from Ω to the real numbers w is nonnegative, almost everywhere positive function on Ω .

$$w: \Omega \to R$$

Proposition 2.1. Let $1 \leq r < \infty$. For any uniformly integrable martingale $f = (f_n)_{n\geq 0}$ on Ω, and $t > 0$, we have

$$\left(M(f)\right)^*(t) \leq 4(f^{\#})^*(t/2) + \left(M(f)\right)^*(2t), \tag{2.1}$$
$$\left(S(f)\right)^*(t) \leq 4(f_r^S)^*(t/2) + \left(S(f)\right)^*(2t), \tag{2.2}$$
$$\left(s(f)\right)^*(t) \leq 4(f_r^s)^*(t/2) + \left(s(f)\right)^*(2t), \tag{2.3}$$

For the proof of (2.1), the reader is referred to [22, Lemma 4]. The reader is refereed to [26, Lemma 1] for the proofs of (2.2) and (2.3). We report a result from Bagby and Kurtz which is originally used to establish the rearranged good λ inequality for maximal singular integral operator and Hardy-Littlewood maximal functions on R^n. The proof of the following proposition is given in [22, Theorem 3.6.9]. For completeness, we provide the proof of the following proposition.

Proposition 2.2. Let f, g be a pair of measurable functions on Ω . If f, g satisfy

$$f^*(t) \leq f^*(2t) + Cg^*(t/2), t > 0, \tag{2.4}$$

then

$$f^*(t) \leq 2Cg^*(t/2) + C\int_t^\infty g^*(s)\frac{ds}{s}, t > 0. \tag{2.5}$$

For the proof of Proposition 2.2, the reader is referred to see [20, Proposition 2.2.]
We obtain the martingale inequalities for sharp functions and the Burkholder Davis-Gundy's inequalities on quasi-Banach function spaces in this section. We begin with the martingale inequalities for sharp functions.

MAIN RESULT

Theorem 3.1. Let $1 \leq r < \infty$ and let w is nonnegative, almost every where positive function on Ω and let X be a rearrangement-invariant quasi Banach function space on $\mathcal{K}$ with $0 < p_X \leq q_X < \infty$. For any uniformly integrable martingale $f = (f_n)_{n\geq 0}$ on Ω, we have

$$\|M(f,w)\|_X \leq C\|(f,w)^\#\|_X, \tag{3.1}$$

$$\|S(f,w)\|_X \leq C\|(f,w)_r^S\|_X, \tag{3.2}$$

$$\|s(f,w)\|_X \leq C\|(f,w)_r^s\|_X, \tag{3.3}$$

Proof. We only give the proof for (3.1) because the proofs for (3.2) and (3.3) follow similarly. Proposition 2.1 assures that

$$(M(f,w))^*(t) \leq 4((f,w)^\#)^*(t/2) + (M(f,w))^*(2t),$$

In view of Proposition 2.2, we find that

$$(M(f,w))^*(t) \leq 2C((f,w)^\#)^*(t/2) + C\int_t^\infty((f,w)^\#)^*(s)w\frac{ds}{s}$$

As $(f^\#)^*$ is a decreasing function, this yields

$$(M(f,w))^*(t) \leq 2C((f,w)^\#)^*(t/2) + C\sum_{i=0}^\infty(f,w)^\#)^*(2^it) \tag{3.4}$$

According to Aoki-Rolewicz theorem [13, Theorem 1.3], we have a $\mathfrak{K} > 0$ such that $\rho_{\bar{X}}^{\mathfrak{K}}(.)$ satisfies the triangle inequality. Therefore, applying $\rho_{\bar{X}}^{\mathfrak{K}}(.)$ on both sides of (3.4), we find that

$$\rho_{\bar{X}}((M(f,w))^*)^{\mathfrak{K}} \leq 2C\rho_{\bar{X}}\left(\left(D_{1/2}(f,w)^\#\right)^*\right)^{\mathfrak{K}} + C\sum_{i=0}^\infty \rho_{\bar{X}}((D_{2^i}(f,w)^\#)^*)^{\mathfrak{K}}.$$

That is,

$$\|M(f,w)\|_X^{\mathfrak{K}} \leq C\rho_{\bar{X}}\left(D_{2^k}(f,w)^\#\right)^{\mathfrak{K}} + C\sum_{i=0}^\infty \rho_{\bar{X}}(D_{2^i}(f,w)^\#)^{\mathfrak{K}}.$$

Since $0 < p_X \leq q_X < \infty$, we have $0 < p < p_X \leq q_X < q < \infty$ such that for some $C > 0$

$$\|D_s\|_{\bar{X}\to\bar{X}} \leq Cs^{-1/p}, \quad 0 \leq s < 1,$$
$$\|D_s\|_{\bar{X}\to\bar{X}} \leq Cs^{-1/q}, \quad 1 \leq s.$$

The above bounds on the operator norms of D_s , $0 \leq s < \infty$ yield

$$\|M(f,w)\|_X^{\mathfrak{K}} \leq C\|(f,w)^\#\|_X + C\sum_{i=0}^\infty 2^{-i\mathfrak{K}/q}\|(f,w)^\#\|_X^{\mathfrak{K}} \leq C\|(f,w)^\#\|_X^{\mathfrak{K}}$$

for some $C > 0$ which gives our desired result.

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