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Euler-Riesz Difference Sequence Spaces

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Abstract. Başar and Braha [9], introduced the sequence spaces $\breve{\ell}_\infty$, $\breve{c}$ and $\breve{c}_0$ of Euler- Cesáro bounded, convergent and null difference sequences and studied their some properties. The main purpose of this study is to introduce the sequence spaces $[\ell_\infty]_{e.r}, [c]_{e.r}$ and $[c_0]_{e.r}$ of Euler- Riesz bounded, convergent and null difference sequences by using the composition of the Euler mean E_1 and Riesz mean R_q with backward difference operator Δ. Furthermore, the inclusions $\ell_\infty \subset [\ell_\infty]_{e.r}, c \subset [c]_{e.r}$ and $c_0 \subset [c_0]_{e.r}$ strictly hold, the basis of the sequence spaces $[c_0]_{e.r}$ and $[c]_{e.r}$ is constucted and alpha-, beta- and gamma-duals of these spaces are determined. Finally, the classes of matrix transformations from the Euler- Riesz difference sequence spaces to the spaces ℓ_∞, c and c_0 are characterized.

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1. Preliminaries, Background and Notation

In this section, we give some basic definitions and notations for which we refer to [7, 12, 17, 23].

By a sequence space, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N} = \{0, 1, 2, ...\}$. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1 < p < \infty$.

We shall assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $0 < r < 1$, and use the convention that any term with negative subscript is equal to naught.

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

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By (λ, μ), we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda, \mu)$ if and only if the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

Let X be a sequence space and A be an infinite matrix. The sequence space

$$X_A = \{x = (x_k) \in w : Ax \in X\}$$

is called the domain of A in X which is a sequence space.

A sequence space λ with a linear topology is called a $K-$ *space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A $K-$ space is called an $FK-$ *space* provided λ is a complete linear metric space. An $FK-$ space whose topology is normable is called a $BK-$ *space*. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} ||x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)|| = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and is written as $x = \sum \alpha_k b_k$.

Given a $BK-$space $\lambda \supset \phi$, we denote the nth section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$, and we say that x has the property
AK if $\lim_{n\to\infty} ||x - x^{[n]}||_\lambda = 0$ (abschnittskonvergenz),
AB if $\sup_{n\in\mathbb{N}} ||x^{[n]}||_\lambda < \infty$ (abschnittsbeschränktheit),
AD if $x \in \phi$ (closure of $\phi \subset \lambda$) (abschnittsdichte),
KB if the set $\{x_k e^{(k)}\}$ is bounded in λ (koordinatenweise beschränkt),
where $e^{(k)}$ is a sequence whose only non-zero term is a 1 in kth place for each $k \in \mathbb{N}$. If one of these properties holds for every $x \in \lambda$ then we say that the space λ has that property [16, 23]. It is trivial that AK implies AD and AK iff $AB + AD$. For example, c_0 and ℓ_p are $AK-$spaces and, c and ℓ_∞ are not $AD-$spaces.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in w$.

Let us give the definition of some triangle limitation matrices which are needed in the text. Δ denotes the backward difference matrix $\Delta = (\Delta_{nk})$ and $\Delta' = (\Delta'_{nk})$ denotes the transpose of the matrix Δ, the forward difference matrix, which are defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

$$\Delta'_{nk} = \begin{cases} (-1)^{n-k} & , \quad n \leq k \leq n+1, \\ 0 & , \quad 0 \leq k < n \text{ or } k > n+1, \end{cases}$$

for all $k, n \in \mathbb{N}$; respectively.

Then, let us define the Euler mean $E_1 = (e_{nk})$ of order one and Riesz mean $R_q = (r_{nk})$ with respect to the sequence $q = (q_k)$

$$e_{nk} = \begin{cases} \frac{\binom{n}{k}}{2^n} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad r_{nk} = \begin{cases} \frac{q_k}{Q_n} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$ and where (q_k) is a sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$. Their inverses $E_1^{-1} = (g_{nk})$ and $R_q^{-1} = (h_{nk})$ are given by

$$g_{nk} = \begin{cases} \binom{n}{k}(-1)^{n-k}2^k & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad h_{nk} = \begin{cases} (-1)^{n-k}\frac{Q_k}{q_n} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad \textit{otherwise}, \end{cases}$$

for all $k, n \in \mathbb{N}$.

We define the matrix $\tilde{B} = (\tilde{b}_{nk})$ by the composition of the matrices E_1, R_q and Δ as

$$\tilde{b}_{nk} = \begin{cases} \frac{\binom{n}{k} q_k}{2^n Q_n} & , \quad 0 \leq k \leq n, \\ 0 & , k > n, \end{cases} \tag{1.2}$$

for all $k, n \in \mathbb{N}$.

In the literature, the notion of difference sequence spaces was introduced by Kızmaz [18], who defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : \Delta' x = (x_k - x_{k+1}) \in X\}$$

for $X \in \{\ell_\infty, c, c_0\}$. The difference space bv_p, consisting of all sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k-1})$ is in the sequence space ℓ_p, was studied in the case $0 < p < 1$ by Altay and Başar [5] and in the case $1 \leq p \leq \infty$ by Başar and Altay [6], and Çolak et al. [13]. Kirişçi and Başar [19] have introduced and studied the generalized difference sequence space

$$\hat{X} = \{x = (x_k) \in w : B(r, s)x \in X\},$$

where X denotes any of the spaces ℓ_∞, c, c_0 and ℓ_p with $1 \leq p < \infty$, and $B(r, s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R} \setminus \{0\}$. Following Kirişçi and Başar [19], Sönmez [21] has examined the sequence space $X(B)$ as the set of all sequences whose $B(r, s, t)-$ trasforms are in the space $X \in \{\ell_\infty, c, c_0, \ell_p\}$, where $B(r, s, t)$ denotes the triple band matrix $B(r, s, t) = \{b_{nk}\{r, s, t\}\}$ defined by

$$b_{nk}\{r, s, t\} = \begin{cases} r & , \quad n = k \\ s & , \quad n = k+1 \\ t & , \quad n = k+2 \\ 0 & , \quad otherwise \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R} \setminus \{0\}$. Quite recently, Başar has studied the spaces $\tilde{\ell}_p$ of $p-$absolutely $\tilde{B}-$summable sequences, in [8]. In [11], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences whose $S-$transforms are in the space $\ell(p)$. Also, many authors have constructed new sequence spaces by using matrix domain of infinite matrices. For instance, e_0^r and e_c^r in [1], e_p^r and e_∞^r in [3], $e_0^r(u, p)$, $e_c^r(u, p)$ in [14], $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ in [20], $c_0(\Delta_\lambda^m)$, $c^r(\Delta_\lambda^m)$ and $\ell_\infty(\Delta_\lambda^m)$ in [15], $r_0^t(p)$, $r_c^t(p)$ and $r_\infty^t(p)$ in [2], $r^q(p, \Delta)$ in [10]. Finally, the new technique for deducing certain topological properties, for example $AB-$, $KB-$, $AD-$properties, solidity and monotonicity etc., and determining the $\beta-$ and $\alpha-$duals of the domain of a triangle matrix in a sequence space is given by Altay and Başar [4].

Then, as a natural continuation of Başar [8], Başar and Braha [9] introduce the spaces $\breve{\ell}_\infty, \breve{c}$ and $\breve{c}_0$ of Euler-Cesáro bounded, convergent and null difference sequences by using the composition of the Euler mean E_1 and Cesáro mean C_1 of order one with backward difference operator Δ.

In the present paper, we introduce the $[\ell_\infty]_{e.r}, [c]_{e.r}$ and $[c_0]_{e.r}$of Euler-Riesz bounded, convergent and null difference sequences by using the composition of the Euler mean E_1 and Riesz mean R_q with respect to the sequence $q = (q_k)$ with backward difference operator Δ and prove that the inclusions $\ell_\infty \subset [\ell_\infty]_{e.r}, c \subset [c]_{e.r}$ and $c_0 \subset [c_0]_{e.r}$ strictly hold. We show that the spaces $[c_0]_{e.r}$ and $[c]_{e.r}$ turn out to be the separable BK spaces such that $[c]_{e.r}$ does not possess any of the following: AK property and monotonicity. Furthermore, we investigate some properties and compute alpha-, beta- and gamma-duals of these spaces. Afterwards, we characterize some matrix classes related to Euler-Riesz sequence spaces.

2. The Euler-Riesz Sequence Spaces

In this section, we give some new sequence spaces and investigate their certain properties.

$$\begin{aligned}[c_0]_{e.r} &= \left\{x=(x_k)\in w : \lim_{n\to\infty}\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}x_k=0\right\}\\ [c]_{e.r} &= \left\{x=(x_k)\in w : \lim_{n\to\infty}\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}x_k \ exists\right\}\\ [\ell_\infty]_{e.r} &= \left\{x=(x_k)\in w : \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}x_k\right|<\infty\right\}\end{aligned}$$

With the notation (1.2), we may redefine the spaces $[c_0]_{e.r}$,$[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ as fallows:

$$[c_0]_{e.r}=(c_0)_{\tilde{B}},\quad [c]_{e.r}=c_{\tilde{B}} \text{ and } [\ell_\infty]_{e.r}=(\ell_\infty)_{\tilde{B}}.$$

In the case $(q_k)=e=(1,1,1,\ldots)$; the sequence spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ are, respectively, reduced to the sequence spaces $\check{c}_0$, $\check{c}$ and $\check{\ell}_\infty$ which are introduced by Başar and Braha [9]. Define the sequence $y=(y_k)$, which will be frequently used, as the $\tilde{B}-$transform of a sequence $x=(x_k)$, i.e.,

$$y_k=\sum_{j=0}^{k}\frac{\binom{k}{j}q_j}{2^kQ_k}x_j,\quad k\in\mathbb{N}. \tag{2.1}$$

Throughout the text, we suppose that the sequences $x=(x_k)$ and $y=(y_k)$ are connected with the relation (2.1). One can obtain by a straightforward calculation from (2.1) that

$$x_k=\frac{1}{q_k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}2^jQ_jy_j,\quad k\in\mathbb{N}. \tag{2.2}$$

Theorem 2.1. *The sets $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ are linear spaces with coordinatewise addition and scalar multiplication that are $BK-$spaces with norm $||x||_{[c_0]_{e.r}}=||x||_{[c]_{e.r}}=||x||_{[\ell_\infty]_{e.r}}=||\tilde{B}x||_\infty$*

Proof. The proof of the first part of the theorem is a routine verification, and so we omit it. Furthermore, since (2.1) holds, c_0, c and ℓ_∞ are $BK-$spaces with respect to their natural norm, and the matrix $\tilde{B}$ is a triangle, Theorem 4.3.2 of Wilansky [23] implies that the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ are $BK-$spaces. □

Therefore, one can easily check that the absolute property does not hold on the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$, because $||x||_{[c_0]_{e.r}}\neq|||x|||_{[c_0]_{e.r}}$, $||x||_{[c]_{e.r}}\neq|||x|||_{[c]_{e.r}}$ and $||x||_{[\ell_\infty]_{e.r}}\neq|||x|||_{[\ell_\infty]_{e.r}}$ for at least one sequence in the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$, where $|x|=(|x_k|)$. This says that $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ are the sequence spaces of nonabsolute type.

Theorem 2.2. *$[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ are linearly isomorphic to the spaces c_0, c and ℓ_∞, respectively, i.e., $[c_0]_{e.r}\cong c_0$, $[c]_{e.r}\cong c$ and $[\ell_\infty]_{e.r}\cong\ell_\infty$.*

Proof. To prove this theorem, we should show the existence of a linear bijection between the spaces $[c_0]_{e.r}$ and c_0. Consider the transformation S defined, with the notation of (2.1), from $[c_0]_{e.r}$ to c_0 by $y=Sx=\tilde{B}x$. The linearity of S is clear. Further, it is obvious that $x=\theta$ whenever $Sx=\theta$ and hence S is injective, where $\theta=(0,0,0,...)$.

Let $y\in c_0$ and define the sequence $x=\{x_n\}$ by

$$x_n=\frac{1}{q_n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}2^kQ_ky_k;\ \text{for all } n\in\mathbb{N}.$$

Then, we have

$$\begin{aligned}\lim_{n\to\infty}(\tilde{B}x)_n &= \lim_{n\to\infty}\left[\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}x_k\right]\\ &= \lim_{n\to\infty}\left[\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}\frac{1}{q_k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}2^jQ_jy_j\right]\\ &= \lim_{n\to\infty}y_n = 0\end{aligned}$$

which says us that $x \in [c_0]_{e.r}$. Additionally, we observe that

$$\begin{aligned}||x||_{[c_0]_{e.r}} &= \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}\frac{1}{q_k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}2^jQ_jy_j\right|\\ &= \sup_{n\in\mathbb{N}}|y_n| = \|y\|_\infty < \infty.\end{aligned}$$

Consequently, S is surjective and is norm preserving. Hence, S is a linear bijection which therefore says us that the spaces $[c_0]_{e.r}$ and c_0 are linearly isomorphic, as desired.

It is clear that if the spaces $[c_0]_{e.r}$ and c_0 are replaced by the spaces $[c]_{e.r}$ and c or $[\ell_\infty]_{e.r}$ and ℓ_∞ respectively, then we obtain the fact that $[c]_{e.r} \cong c$ and $[\ell_\infty]_{e.r} \cong \ell_\infty$. This completes the proof. □

We wish to exhibit some inclusion relations concerning with the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$, in the present section. Here and after, by λ we denote any of the sets $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ and μ denotes any of the spaces c_0, c or ℓ_∞.

Theorem 2.3. *The inclusions $\mu \subset \lambda$ hold.*

Proof. Let $x = (x_k) \in \mu$. Then, since it is immediate that

$$\begin{aligned}\|x\|_\lambda = \|\widetilde{B}x\|_\infty &= \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^nQ_n}x_k\right|\\ &\leq \|x\|_\infty \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^n} = \|x\|_\infty.\end{aligned}$$

The inclusion $\mu \subset \lambda$ holds.

□

Theorem 2.4. *The space $[c_0]_{e.r}$ has $AK-$property.*

Proof. Let $x = (x_k) \in [c_0]_{e.r}$ and $x^{[n]} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$. Hence,

$$x - x^{[n]} = \{0, 0, ..., 0, x_{n+1}, x_{n+2}, ...\} \Rightarrow \|x - x^{[n]}\|_{[c_0]_{e.r}} = \|(0, 0, ..., 0, x_{n+1}, x_{n+2}, ...)\|$$

and since $x \in [c_0]_{e.r}$,

$$\|x - x^{[n]}\|_{[c_0]_{e.r}} = \sup_{k\geq n+1}\left|\sum_{j=0}^{k}\frac{\binom{k}{j}q_j}{2^kQ_k}x_j\right|$$

Then the space $[c_0]_{e.r}$ has $AK-$property. □

Since the isomorphism S, defined in Theorem 2.1, is surjective, the inverse image of the basis of the spaces c_0 and c are the basis of the new spaces $[c]_{e.r}$ and $[c_0]_{e.r}$, respectively. Since the space ℓ_∞ has no Schauder basis, $[\ell_\infty]_{e.r}$ has no Schauder basis. Therefore, we have the following theorem without proof.

Theorem 2.5. *Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the space $[c_0]_{e.r}$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases}\frac{\binom{n}{k}(-1)^{n-k}2^kQ_k}{q_n} &, \quad 0 \leq k < n,\\ 0 &, \quad k \geq n.\end{cases}$$

Let $\lambda_k = (\tilde{B}x)_k$ *for all* $k \in \mathbb{N}$. *Then the following assertions are true:*

(i): *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $[c_0]_{e.r}$ *and any* $x \in [c_0]_{e.r}$ *has a unique representation of the form*

$$x = \sum_k \lambda_k b^{(k)}.$$

(ii): *The set* $\{e, b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $[c]_{e.r}$ *and any* $x \in [c]_{e.r}$ *has a unique representation of the form*

$$x = le + \sum_k [\lambda_k - l]\, b^{(k)},$$

where $l = \lim_{k\to\infty}(\tilde{B}x)_k$.

Remark 2.6. It is well known that every Banach space X with a Schauder basis is separable.

From Theorem 2.5 and Remark 2.6, we can give the following corollary:

Corollary 2.7. *The spaces* $[c_0]_{e.r}$ *and* $[c]_{e.r}$ *are separable.*

3. Duals of The New Sequence Spaces

In this section, we state and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$ duals of the sequence spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ of non-absolute type.

The set $S(\lambda, \mu)$ defined by

$$S(\lambda, \mu) = \left\{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\right\} \tag{3.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can eaisly observe for a sequence space ν with $\lambda \supset \nu \supset \mu$ that the inclusions

$$S(\lambda, \mu) \subset S(\nu, \mu) \text{ and } S(\lambda, \mu) \subset S(\lambda, \nu)$$

hold. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

For giving the alpha-, beta- and gamma-duals of the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ of non-absolute type, we need the following Lemma;

Lemma 3.1. *[22]*

(i): $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k\in K} a_{nk} \right| < \infty.$$

(ii): $A \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} |a_{nk}| < \infty. \tag{3.2}$$

(iii): $A \in (c : c)$ *if and only if (3.2) holds, and*

$$\exists(\alpha_k) \in w \text{ such that } \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for all } k \in \mathbb{N}, \tag{3.3}$$

$$\exists\alpha \in \mathbb{C} \text{ such that } \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha.$$

Now, we may give the theorems determining the $\alpha-, \beta-$ and $\gamma-$duals of the Euler-Riesz sequence spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$.

Theorem 3.2. *Define the set a_q as follows:*

$$a_q = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} \binom{n}{k} (-1)^{n-k} 2^k \frac{a_n}{q_n} Q_k \right| < \infty \right\}.$$

Then, $\{[c_0]_{e.r}\}^\alpha = \{[c]_{e.r}\}^\alpha = \{[\ell_\infty]_{e.r}\}^\alpha = a_q$.

Proof. We give the proof for the space $[c_0]_{e.r}$. We chose the sequence $a = (a_k) \in w$. We can easily derive with (2.2) that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k \frac{a_n}{q_n} Q_k y_k = (By)_n, \quad (n \in \mathbb{N}); \tag{3.4}$$

where $B = (b_{nk})$ is defined by the formula

$$b_{nk} = \begin{cases} \binom{n}{k} (-1)^{n-k} 2^k \frac{a_n}{q_n} Q_k & , \quad (0 \le k \le n) \\ 0 & , \quad (k > n) \end{cases}, (n, k \in \mathbb{N}).$$

It follows from (3.4) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in [c_0]_{e.r}$ if and only if $By \in \ell_1$ whenever $y \in c_0$. This gives the result that $\{[c_0]_{e.r}\}^\alpha = a_q$. □

Theorem 3.3. *The matrix $D(r) = (d_{nk})$ is defined by*

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \binom{j}{k} (-1)^{j-k} 2^k \frac{a_j}{q_j} Q_k & , \quad (0 \le k \le n) \\ 0 & , \quad (k > n) \end{cases} \tag{3.5}$$

for all $k, n \in \mathbb{N}$. Then, $\{[c_0]_{e.r}\}^\beta = b_1 \cap b_2$ and $\{[c]_{e.r}\}^\beta = b_1 \cap b_2 \cap b_3$ where

$$\begin{aligned} b_1 &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_k |d_{nk}| < \infty \right\}, \\ b_2 &= \left\{ a = (a_k) \in w : \lim_{n \to \infty} d_{nk} = \alpha_k \right\}, \\ b_3 &= \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k d_{nk} \text{ exists} \right\}. \end{aligned}$$

Proof. We give the proof for the space $[c_0]_{e.r}$. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{1}{q_k} Q_j y_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{k}{j} (-1)^{k-j} 2^j \frac{a_k}{q_k} Q_j \right] y_k = (Dy)_n, \end{aligned} \tag{3.6}$$

where $D = (d_{nk})$ defined by (3.5).

Thus, we decude by (3.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in [c_0]_{e.r}$ if and only if $Dy \in c$ whenever $y = (y_k) \in c_0$. Therefore, we derive from (3.2) and (3.3) that

$$\lim_{n \to \infty} d_{nk} \text{ exists for each } k \in \mathbb{N},$$

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}| < \infty$$

which shows that $\{[c_0]_{e.r}\}^\beta = b_1 \cap b_2$. □

Theorem 3.4. $\{[c_0]_{e.r}\}^\gamma = \{[c]_{e.r}\}^\gamma = b_1$.

Proof. This is obtained in the similar way used in the proof of Theorem 3.3. □

4. Matrix Transformations Related to The New Sequence Spaces

In this section, we characterize the matrix transformations from the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$ into any given sequence space μ and from the sequence space μ into the spaces $[c_0]_{e.r}$, $[c]_{e.r}$ and $[\ell_\infty]_{e.r}$

Since $[c_0]_{e.r} \cong c_0$ (or $[c]_{e.r} \cong c$ and $[\ell_\infty]_{e.r} \cong \ell_\infty$), we can say: The equivalence "$x \in [c_0]_{e.r}$ (or $x \in [c]_{e.r}$ and $x \in [\ell_\infty]_{e.r}$), if and only if $y \in c_0$ (or $y \in c$ and $y \in \ell_\infty$)" holds.

In what follows, for brevity, we write,

$$\tilde{a}_{nk} := \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k \frac{Q_k}{q_n} a_{nk}$$

for all $k, n \in \mathbb{N}$.

Theorem 4.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$e_{nk} := \tilde{a}_{nk} \tag{4.1}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then,

- **(i):** *$A \in ([c_0]_{e.r} : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [c_0]_{e.r}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (c_0 : \mu)$.*
- **(ii):** *$A \in ([c]_{e.r} : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{([c]_{e.r}\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (c : \mu)$.*
- **(iii):** *$A \in ([\ell_\infty]_{e.r} : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\ell_\infty]_{e.r}\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (\ell_\infty : \mu)$.*

Proof. We prove only Part (i). Let μ be any given sequence space. Suppose that (4.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $[c_0]_{e.r}$ and c_0 are linearly isomorphic.

Let $A \in ([c_0]_{e.r} : \mu)$ and take any $y = (y_k) \in c_0$. Then EB exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in b_1 \cap b_2$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in c_0$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$.

We have that $Ey = Ax$ which leads us to the consequence $E \in (c_0 : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c_0]_{e.r}\}^{\beta}$ for each $n \in \mathbb{N}$ and $E \in (c_0 : \mu)$, and take any $x = (x_k) \in [c_0]_{e.r}$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{Q_j}{q_k} a_{kj} \right] y_k$$

for all $n \in \mathbb{N}$, that $Ey = Ax$ and this shows that $A \in ([c_0]_{e.r} : \mu)$. This completes the proof of Part (i). □

Theorem 4.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} := \sum_{j=0}^{k} \frac{\binom{k}{j} q_j}{2^k Q_k} a_{jk} \text{ for all } k, n \in \mathbb{N}.$$

Let μ be any given sequence space. Then,

- **(i):** *$A \in (\mu : [c_0]_{e.r})$ if and only if $B \in (\mu : c_0)$.*
- **(ii):** *$A \in (\mu : [c]_{e.r})$ if and only if $B \in (\mu : c)$.*
- **(iii):** *$A \in (\mu : [\ell_\infty]_{e.r})$ if and only if $B \in (\mu : \ell_\infty)$.*

Proof. We prove only Part (iii). Let $z = (z_k) \in \mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=0}^{k} \frac{\binom{k}{j} q_j}{2^k Q_k} \left(\sum_{k=0}^{m} a_{jk} z_k \right) \quad \text{for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = \{\tilde{B}(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in [\ell_\infty]_{e.r}$ whenever $z \in \mu$ if and only if $Bz \in \ell_\infty$ whenever $z \in \mu$. This completes the proof of Part (iii). □

The folowing results were taken from Stieglitz and Tietz [22]:

$$\lim_k a_{nk} = 0 \text{ for all } n, \tag{4.2}$$

$$\lim_n |\sum_k a_{nk}| \text{ exist}, \tag{4.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |\lim_{n\to\infty} a_{nk}|, \tag{4.4}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0, \tag{4.5}$$

Lemma 4.3. *Let $A = (a_{nk})$ be an infinite matrix. Then*

- **(i):** $A = (a_{nk}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ *if and only if (3.2) holds.*
- **(ii):** $A = (a_{nk}) \in (c_0 : c_0)$ *if and only if (3.2) and (4.2) hold.*
- **(iii):** $A = (a_{nk}) \in (c : c_0)$ *if and only if (3.2), (4.2) and (4.5) hold.*
- **(iv):** $A = (a_{nk}) \in (\ell_\infty : c_0)$ *if and only if (4.5) holds.*
- **(v):** $A = (a_{nk}) \in (c_0 : c)$ *if and only if (3.2) and (3.3) hold.*
- **(vi):** $A = (a_{nk}) \in (c : c)$ *if and only if (3.2), (3.3) and (4.3) hold.*
- **(vii):** $A = (a_{nk}) \in (\ell_\infty : c)$ *if and only if (3.3) and (4.4) hold.*

Now, we can give the following results:

Corollary 4.4. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

- **(i):** $A \in ([c_0]_{e.r} : c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c_0]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) and (4.2) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
- **(ii):** $A \in ([c_0]_{e.r} : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c_0]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) and (3.3) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
- **(iii):** $A \in ([c_0]_{e.r} : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c_0]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) holds with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 4.5. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

- **(i):** $A \in ([c]_{e.r} : c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2), (4.2) and (4.5) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
- **(ii):** $A \in ([c]_{e.r} : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2), (3.3) and (4.3) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
- **(iii):** $A \in ([c]_{e.r} : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[c]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) holds with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 4.6. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

- **(i):** $A \in ([\ell_\infty]_{e.r} : c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\ell_\infty]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.5)holds with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
- **(ii):** $A \in ([\ell_\infty]_{e.r} : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\ell_\infty]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.3) and (4.4) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
- **(iii):** $A \in ([\ell_\infty]_{e.r} : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\ell_\infty]_{e.r}\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) holds with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 4.7. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

- **(i):** $A = (a_{nk}) \in (c_0 : [c_0]_{e.r})$ *if and only if (3.2) and (4.2) hold with* b_{nk} *instead of* a_{nk}.
- **(ii):** $A = (a_{nk}) \in (c : [c_0]_{e.r})$ *if and only if (3.2), (4.2) and (4.5) hold with* b_{nk} *instead of* a_{nk}.
- **(iii):** $A = (a_{nk}) \in (\ell_\infty : [c_0]_{e.r})$ *if and only if (4.5) holds with* b_{nk} *instead of* a_{nk}.
- **(iv):** $A = (a_{nk}) \in (c_0 : [c]_{e.r}) = (c : [c]_{e.r}) = (\ell_\infty : [c]_{e.r})$ *if and only if (3.2) and (3.3) hold with* b_{nk} *instead of* a_{nk}.
- **(v):** $A = (a_{nk}) \in (c : [c]_{e.r})$ *if and only if 3.2), (3.3) and (4.3) hold with* b_{nk} *instead of* a_{nk}.
- **(vi):** $A = (a_{nk}) \in (\ell_\infty : [c]_{e.r})$ *if and only if (3.3) and (4.4) hold with* b_{nk} *instead of* a_{nk}.

(vii): *$A = (a_{nk}) \in (c_0 : [\ell_\infty]_{e.r}) = (c : [\ell_\infty]_{e.r}) = (\ell_\infty : [\ell_\infty]_{e.r})$ if and only if (3.2) holds with b_{nk} instead of a_{nk}.*

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