Turk. J. Math. Comput. Sci.
7(2017) 73–82

http://dergipark.gov.tr/tjmcs
http://tjmcs.matder.org.tr

MATDER

Prolongations of Hypersurfaces to Vector Bundles

Mustafa Özkan[a,*], Erdoğan Esin[a]

[a]*Department of Mathematics, Faculty of Science, Gazi University, 06500 Ankara, Turkey.*

Received: 08-11-2017 • Accepted: 11-12-2017

Abstract. This paper introduces and studies the concept of prolongations of a hypersurface to a vector bundle. We develop the theory of hypersurfaces using the metric tensor which is the complete lift of the metric tensor of the initial manifold.

2010 AMS Classification: 53C99.

Keywords: Prolongation, vector bundle, hypersurface.

1. Introduction

Prolongations of hypersurfaces to tangent bundles were studied by Tani in [14]. Prolongations of geometric structures on manifold to tangent bundle, tangent bundle of order 2 and of order r were studied by several authors, see [3, 8–11]. Prolongations of tensor fields and connections to vector bundles have been studied by several authors, such as [1,4,5,12,16]. Also, in [6,7] the authors were studied prolongations of structures on manifold to vector bundle. In the spirit of this work, we study generalization to vector bundle of [14].

This paper is organized as follows: In §2, we provide the notation and background material concerning the basic concepts of tangent and vector bundles that will be used in the rest of this paper. In §3, we recall several results stated in [4, 5, 12, 16]. Such results are indispensable in the subsequent sections. In §4, we define the vertical and complete lifts of the vector fields defined along a surface. In particular, we consider two kinds of lifts of the normal vector field of the surface as vector fields normal to the prolonged surface. In §5, we recall some formulas for surfaces, and then we use them in §6 to obtain fundamental formulas containing so-called the second fundamental tensor for prolonged surfaces. In the last section, we formulate the Gauss, Weingarten and structure equations, i.e., Gauss, Codazzi and Ricci equations, for the prolonged surface in the form of lifts corresponding to the equations of the surface given in the base space. In particular, we observe that our results on vector bundles coincide with the earlier results of [14] in which the tangent bundle TM was taken as the vector bundle.

*Corresponding Author
Email addresses: ozkanm@gazi.edu.tr (M. Özkan), eresin@gazi.edu.tr (E. Esin)

2. Background Material

Let M be a differentiable manifold and T_pM be the tangent space at a point p in M. TM is the tangent bundle of M with its projection $\pi_M : TM \to M$. We denote by $\Im^r_s(M)$ the space of C^∞ tensor fields of type (r, s). We denote by $\Im(M)$ the tensor algebra on M. In other words, $\Im(M) = \sum_{r,s} \Im^r_s(M)$.

Let M be an $m-$dimensional differentiable manifold of class C^∞ and V be a coordinate neighborhood in M and (x^i) be certain local coordinates defined in V. We introduce a system of coordinates $(x^i, \dot{x}^i)$ in $\pi_M^{-1}(V)$ with cartesian coordinates $(\dot{x}^i)$ in each tangent space T_pM, $p \in V$, with respect to the natural frame $\left(\frac{\partial}{\partial x^i}\right)$ of local coordinates (x^i). We then call $(x^i, \dot{x}^i)$ the coordinates induced in $\pi_M^{-1}(V)$ from (x^i), or simply the induced coordinates in $\pi_M^{-1}(V)$ [14, 15].

Consider an $(m-1)-$dimensional manifold S with its imbedding $\imath : S \to M$. The differential mapping $d\imath$ from TS into TM is called the tangential map of $\imath$. Such tangential map is denoted by D. Note that D induces its tangential map $dD : TTS \to TTM$, which is denoted by $\tilde{D}$. Recall that TS is a $2(m-1)-$dimensional submanifold of TM and TTS is a $4(m-1)-$dimensional submanifold of of TTM. By setting

$$T(S, M) = \pi_M^{-1}(\imath(S)) \qquad \text{and} \qquad T(TS, TM) = \pi_{TM}^{-1}(D(TS))\,,$$

we obtain

$$\imath : S \to M \qquad \text{and} \qquad d\imath = D : TS \to T(S, M) \subset TM\,.$$

Note that $\imath$ has the local expressions

$$x^i = x^i(u^\alpha),$$

in terms of local coordinates (x^i) where (u^α) are local coordinates of S. Therefore, the local expressions of D are given by

$$\begin{cases} x^i = x^i(u^\alpha), \\ \dot{x}^i = D^i_\alpha \dot{u}^\alpha, \quad D^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha} \end{cases}$$

with respect to the local coordinates $(x^i, \dot{x}^i)$ and $(u^\alpha, \dot{u}^\alpha)$ induced from (x^i) and (u^α), respectively. In the following we sometimes identify S with its image $\imath(S)$ and TS with its image $D(TS)$ [14].

Let us denote the tensor algebra associated to $T(S, M)$ by $\Im(S, M)$. The tangential map D induces an isomorphism from $\Im(S)$ into $\Im(S, M)$. Similarly, dD induces an isomorphism from $\Im(TS)$ to $\Im(T(S, M))$. A mapping $\bar{X}$ assigning to each $p \in S$ a tangent vector at p of M, i.e.,

$$\bar{X}_p \in T_pM\,,$$

is called a vector field defined along S. Note that $\Im^1_0(S, M)$ is the set of all vector fields along S. Similarly a mapping $\bar{T}$ assigning to each $p \in S$ a tensor of type (r, s) at p

$$\bar{T}_p \in T_pM \otimes \cdots \otimes T_pM \otimes T^*_pM \otimes \cdots \otimes T^*_pM,$$

is called a tensor field of type (r, s) along S. Here, $T^*_p(M)$ is the dual space of T_pM. If the space of tensor fields of type (r, s) along S is denoted by $\Im^r_s(S, M)$, it can be seen that $\Im^0_0(S, M) = \Im^0_0(S)$. In the rest of this paper, the elements of $\Im(S)$ will be denoted by f, X, ω and so on. On the other hand, the elements of $\Im(S, M)$ will be denoted by $\bar{f}, \bar{X}, \bar{\omega}$ and so on [14].

Let E be an $(m+n)-$dimensional vector bundle in which $n \geq m$ with the projection $\pi : E \to M$. For a local coordinate system (V, x^i) on M, let $(\pi^{-1}(V), x^i, y^a)$ be the induced local coordinate system on E defined by $x^i(v) = x^i(\pi(v))$ and $v = y^a(v)\delta_a$ for all $v \in \pi^{-1}(V)$, where $1 \leq i \leq m$, $1 \leq a \leq n$. Here, $\delta_1, ..., \delta_n$ are adapted sections. For the special case $E = TM$, we consider the induced local coordinate system $(\pi_M^{-1}(V), x^i, \dot{x}^i)$ on TM where $x^i = x^i(\pi_M(v))$ and $\dot{x}^i(v) = dx^i(v)$ for all $v \in \pi^{-1}(V)$ [13].

Consider a surjective vector bundle morphism ${}^0I : E \to TM$ satisfying $\pi_M \circ {}^0 I = \pi$. Let p be a point in M, then there are components $\Omega^i_a(p)$ such that

$${}^0I(z) = \Omega^i_a(p) z^a \left.\frac{\partial}{\partial x^i}\right|_p$$

where $z \in E_p = \pi^{-1}(p)$. Moreover ${}^0I$ has the following local expressions in terms of local coordinates $\left(x^i, y^a\right)$ and $\left(x^i, \dot{x}^i\right)$ [16]:

$${}^0I(x^i, y^a) = (x^i, \Omega^i_a y^a)\,.$$

Proposition 2.1. *Let $\widehat{{}^{0}I} : T(S, M) \to TS$ be a vector bundle morphism. $\widehat{{}^{0}I}$ has the local expression*

$$\widehat{{}^{0}I}(u^\alpha, \dot{x}^i) = (u^\alpha, \hat{\Omega}_i^\alpha \dot{x}^i)\,,$$

with respect to the local coordinates $\left(u^\alpha, \dot{x}^i\right)$ and $(u^\alpha, \dot{u}^\alpha)$. Therefore we obtain

$$D_\alpha^i \hat{\Omega}_i^\alpha = \delta_i^i\,,$$

where δ_i^i is the Kronecker delta.

Let (E, π, M) be a vector bundle over M and $S \subset M$ be a submanifold of M. We set $\bar{E} = \pi^{-1}(S) \subset E$ and denote by $\bar{\pi} : \bar{E} \longrightarrow S$ the restriction of π to $\bar{E}$. We then obtain a new vector bundle $(\bar{E}, \bar{\pi}, S)$, called the restriction of E to S. Each fiber $\bar{E}_x$, $x \in S$, is equal to the corresponding fiber E_x. The local coordinates of $\bar{E}$ are (u^α, y^a) and $\dim \bar{E} = (m-1) + n$ [2].

Let $(\tilde{E}, \tilde{\pi}, S)$ subvector bundle of (E, π, M) such that $\tilde{E} = TS$ with local coordinates $(u^\alpha, v^h)_{\substack{1 \le \alpha \le m-1 \\ 1 \le h \le n-1}}$ of $\tilde{E}$ whenever $E = TM$. Then $\dim \tilde{E} = (m-1) + (n-1)$.

Let

$$B : \tilde{E} \to E$$

be the inclusion map. Then B has the following local expressions with respect to the local coordinates $(u^\alpha, \dot{u}^\alpha)$ and $\left(x^i, y^a\right)$:

$$B(u^\alpha, \dot{u}^\alpha) = (x^i(u^\alpha), B_h^a v^h).$$

In particular, B induces its tangential map $dB : T\tilde{E} \to TE$ which is denoted by $\tilde{B}$. By setting $T(\tilde{E}, E) = \pi_E^{-1}(B(\tilde{E}))$ we obtain

$$\begin{aligned} B :& \quad \tilde{E} \to \bar{E} \subset E \\ dB = \tilde{B} :& \quad T\tilde{E} \to T(\tilde{E}, E) = T\bar{E} \subset TE. \end{aligned}$$

Let $\widetilde{{}^{o}I} : \tilde{E} \to TS$ be a surjective vector bundle morphism such that $\widetilde{{}^{o}I} = id_{TS}$ in the special case when $\pi_S \circ \widetilde{{}^{o}I} = \tilde{\pi}$ and $E = TM$. Then there are components $\tilde{\Omega}_h^\alpha(p)$ such that

$$\widetilde{{}^{o}I}(u) = \tilde{\Omega}_h^\alpha(p) u^h \left.\frac{\partial}{\partial u^\alpha}\right|_p \quad (\text{for } u \in \tilde{E}_p).$$

In terms of local coordinates (u^α, v^h) of $\tilde{E}$, $\widetilde{{}^{o}I}$ has local expressions

$$\widetilde{{}^{o}I}(u^\alpha, v^h) = (u^\alpha, \tilde{\Omega}_h^\alpha v^h).$$

In the following, we assume all manifolds, tensor fields, maps and linear connections to be differentiable and of class C^∞.

3. Vertical and Complete Lifts

In this section, two kinds of isomorphism of $\Im(M)$ into $\Im(E)$ or $\Im(S)$ into $\Im(\tilde{E})$, which are called vertical and complete lifts, will be given.

Let f, X, ω and F be a function, a vector field, a 1−form and a tensor field of type $(1, 1)$ in M, respectively. We denote respectively by f^V, X^V, ω^V and F^V, their vertical lifts and by f^C, X^C, ω^C and F^C, their complete lifts. For a function f in M, we have by definition

$$f^V = f \circ \pi \tag{3.1}$$

and

$$f^C = \partial_i f \Omega_a^i y^a, \quad (\partial_i = \partial/\partial x^i) \tag{3.2}$$

with respect to the induced coordinates. Moreover these lifts have the properties [4,5,16]:

$$\begin{aligned}
&(fX)^V = f^V X^V, \quad (fX)^C = f^C X^V + f^V X^C, \\
&X^V f^V = 0, \quad X^V f^C = X^C f^V = (Xf)^V, \quad X^C f^C = (Xf)^C, \\
&\omega^V X^V = 0, \quad \omega^V X^C = \omega^C X^V = (\omega X)^V, \quad \omega^C X^C = (\omega X)^C, \\
&F^V X^C = (FX)^V, \quad F^C X^C = (FX)^C, \\
&[X,Y]^C = \left[X^C, Y^C\right], \quad [X,Y]^V = \left[X^C, Y^V\right] = \left[X^V, Y^C\right].
\end{aligned} \tag{3.3}$$

For a tensor field of the form $T = P \otimes Q$ where P and Q are arbitrary tensor fields, its vertical and complete lifts are given respectively by

$$\begin{aligned} T^V &= (P \otimes Q)^V = P^V \otimes Q^V \\ T^C &= (P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C \end{aligned} \tag{3.4}$$

[4,5]. For the later use we note here the following formulas [4,5]:

$$\begin{aligned}
&T^C\left(X^C, Y^C\right) = (T(X,Y))^C, \\
&T^C\left(X^C, Y^V\right) = T^V\left(X^C, Y^C\right) = (T(X,Y))^V, \\
&T^C\left(X^V, Y^V\right) = T^V\left(X^V, Y^C\right) = T^V\left(X^V, Y^V\right) = 0
\end{aligned} \tag{3.5}$$

where X, Y are arbitrary vector fields in M and $T \in \Im^0_2(M)$.

Remark 3.1. Let $\overset{*}{X}$ and $\overset{*}{Y}$ are arbitrary elements of $\tilde{\Im}^1_0(E)$ such that $\overset{*}{X}\hat{f}^C = \overset{*}{Y}\hat{f}^C$ for all $\hat{f} \in \Im^0_0(M)$, then $\overset{*}{X} = \overset{*}{Y}$. Let $\overset{*}{\omega}$ and $\overset{*}{\eta}$ are arbitrary elements of $\tilde{\Im}^0_1(E)$ such that $\overset{*}{\omega}(\hat{X}^C) = \overset{*}{Y}(\hat{X}^C)$ for all $\hat{X} \in \Im^1_0(M)$, than $\overset{*}{\omega} = \overset{*}{\eta}$. Consequently any arbitrary elements of $\tilde{\Im}(E)$ are completely determined by its action on the set of $\hat{f}^C$ and $\hat{X}^C$, $\hat{f}$ and $\hat{X}$ being arbitrary elements of $\Im^0_0(M)$ and $\Im^1_0(M)$, respectively [4,5].

4. Vertical and Complete Lifts of $\Im^r_s(S,M)$ to E

In this section, we give two kinds of isomorphism of $\Im(S,M)$ into $\Im(\bar{E})$, so-called vertical and complete lifts from $\Im(S,M)$ to E.

We define now the lifts of elements of $\Im^r_s(S,M)$ to E.

Definition 4.1. Let $\bar{f}$ be a function defined on S. The vertical lift $\bar{f}^{\bar{V}}$ of $\bar{f}$ to E is defined by

$$\bar{f}^{\bar{V}} = \bar{f} \circ \tilde{\pi}. \tag{4.1}$$

Proposition 4.2. *The restriction of $\hat{f}^V$ to $\tilde{E}$ coincides with $\bar{f}^{\bar{V}}$, where $\hat{f}$ is an arbitrary element of $\Im^0_0(M)$.*

In order to define the complete lift for an arbitrary point p of S, we consider a sufficiently small neighborhood U of p in M. In U we can construct a function $\hat{f}$ such that $\hat{f}$ coincides with $\bar{f}$ on the connected component $(U \cap S)^0$ in $U \cap S$ containing p. We remark that a local extension $\hat{f}$ satisfies $\partial_\alpha \hat{f} = \partial_\alpha \bar{f}$ along $(U \cap S)^0$.

Then the complete lift of $\hat{f}$ to $\pi^{-1}(U)$ is defined as

$$\hat{f}^C = \partial_i \hat{f} \Omega^i_a y^a$$

in the local coordinates in $\pi^{-1}(U)$. We see that the restriction of $\hat{f}^C$ to $\pi^{-1}((U \cap S)^0)$ is independent of the choice of $\hat{f}$. In fact, if we denote by # the operation of taking restriction to $\pi^{-1}((U \cap S)^0)$, we have

$$\begin{aligned} \#\hat{f}^C &= \#(\partial_i \hat{f} \Omega^i_a y^a) = \partial_i \hat{f} \Omega^i_a B^a_h v^h = \partial_i \hat{f} D^i_\alpha \hat{\Omega}^\alpha_i \Omega^i_a B^a_h v^h \\ &= \partial_\alpha \hat{f} \tilde{\Omega}^\alpha_h v^h = \partial_\alpha \bar{f} \tilde{\Omega}^\alpha_h v^h \end{aligned} \tag{4.2}$$

in $\pi^{-1}((U \cap S)^0)$.

Definition 4.3. We can define a function which coincides with $\#\hat{f}^C$ in each coordinates neighborhood. We denote it by $\bar{f}^{\bar{C}}$ and call the complete lift of $\bar{f}$ to E.

Definition 4.4. Let $\bar{X}$ be an element of $\mathfrak{I}_0^1(S, M)$. The $\bar{X}_p$ being a tangent vector at $p \in S$ we shall define the vertical lift $\bar{X}^{\bar{V}}$ to E by

$$\bar{X}^{\bar{V}} \hat{f}^C = (\bar{X}\hat{f})^{\bar{V}} \tag{4.3}$$

along S, where $\hat{f}$ is an arbitrary element of $\mathfrak{I}_0^0(M)$.

Definition 4.5. Let $\bar{X}$ be an element of $\mathfrak{I}_0^1(S, M)$. The $\bar{X}_p$ being a tangent vector at $p \in S$, we shall define the complete lift $\bar{X}^{\bar{C}}$ to E by

$$\bar{X}^{\bar{C}} \hat{f}^C = (\bar{X}\hat{f})^{\bar{C}} \tag{4.4}$$

along S, where $\hat{f}$ is an arbitrary element of $\mathfrak{I}_0^0(M)$.

Proposition 4.6. *Definition 4.5 is equivalent to the one that $\bar{X}^{\bar{C}}$ is defined to be restriction on $\pi^{-1}((U \cap S)^0)$ of $\hat{X}^C$, where $\hat{X}$ is a local extension of $\bar{X}$ in U.*

Proof. To prove this, it is sufficient to prove

$$\bar{X}^{\bar{C}} \hat{f}^C = (\#\hat{X}^C)\hat{f}^C \quad \text{for} \quad \hat{f} \in \mathfrak{I}_0^0(M)$$

due to Remark 3.1. For an arbitrary vector field $\bar{X}$, $\bar{X}\hat{f}$ is a function which takes the value

$$(\bar{X}\hat{f})_p = \bar{X}_p \hat{f}$$

at p. Therefore we have

$$\bar{X}^{\bar{C}} \hat{f}^C = (\bar{X}\hat{f})^{\bar{C}} = \#(\hat{X}\hat{f})^C = \#(\hat{X}^C \hat{f}^C) = (\#\hat{X}^C)\hat{f}^C.$$

□

Definition 4.7. Let $\bar{\omega}$ be an element of $\mathfrak{I}_1^0(S, M)$. We define the vertical lift $\bar{\omega}^{\bar{V}}$ to E by

$$\bar{\omega}^{\bar{V}}(\bar{X}^{\bar{C}}) = \bar{\omega}(\bar{X})^{\bar{V}}$$

where $\bar{X}$ is an arbitrary element of $\mathfrak{I}_0^1(S, M)$.

Definition 4.8. Let $\bar{\omega}$ be an element of $\mathfrak{I}_1^0(S, M)$. We define the complete lift $\bar{\omega}^{\bar{V}}$ to E by

$$\bar{\omega}^{\bar{C}}(\bar{X}^{\bar{C}}) = \bar{\omega}(\bar{X})^{\bar{C}}$$

where $\bar{X}$ is an arbitrary element of $\mathfrak{I}_0^1(S, M)$.

Proposition 4.9. *Definition 4.8 is equivalent to the one that $\bar{\omega}^{\bar{C}}$ is defined to be restriction on $\pi^{-1}((U \cap S)^0)$ of $\hat{\omega}^C$, where $\hat{\omega}$ is a local extension of $\bar{\omega}$ in U.*

Proof. Let $\bar{X}$ be an arbitrary element of $\mathfrak{I}_0^1(S, M)$. If we consider a 1−form $\hat{\omega}$ in a sufficiently small neighborhood U of p such that $\hat{\omega}(\hat{X})$ is a local extension of $\bar{\omega}(\bar{X})$, i.e.,

$$\hat{\omega}(\hat{X})\Big|_{(U\cap S)^0} = \bar{\omega}(\bar{X})$$

where $\hat{X}$ is a local extension of $\bar{X}$. We call such $\hat{\omega}$ a local extension of $\bar{\omega}$ to U. Then we have the complete lift $\bar{\omega}^{\bar{C}}$

$$\bar{\omega}^{\bar{C}}(\bar{X}^{\bar{C}}) = \bar{\omega}(\bar{X})^{\bar{C}} = \#(\hat{\omega}(\hat{X})^C) = \#(\hat{\omega}^C(\hat{X}^C))$$

in $\pi^{-1}((U \cap S)^0)$, since $\bar{\omega}(\bar{X})^{\bar{C}}$ is defined by the restriction of the complete lift of a local extension. That is, $\hat{\omega}^C$ is a local extension of $\bar{\omega}^{\bar{C}}$ in the above sense. □

We now extend these lifts to a linear mapping from $\mathfrak{I}(S, M)$ to $\mathfrak{I}(\bar{E})$ under the condition:

$$\begin{aligned} \left(\bar{P} \otimes \bar{Q}\right)^{\bar{V}} &= \bar{P}^{\bar{V}} \otimes \bar{Q}^{\bar{V}}, \\ \left(\bar{P} \otimes \bar{Q}\right)^{\bar{C}} &= \bar{P}^{\bar{C}} \otimes \bar{Q}^{\bar{V}} + \bar{P}^{\bar{V}} \otimes \bar{Q}^{\bar{C}} \end{aligned} \tag{4.5}$$

where $\bar{P}$ and $\bar{Q}$ are arbitrary elements of $\mathfrak{I}(S, M)$.

We shall now sum up some properties of lifts derived immediately from the definitions.

Proposition 4.10. *The lift of $\Im_0^0(S, M)$ to E and the lift of $\Im_0^0(S)$ to $\tilde{E}$ are related by*

$$\bar{f}^{\bar{V}} = \bar{f}^{V}, \quad \bar{f}^{\bar{C}} = \bar{f}^{C} \quad for \quad \bar{f} \in \Im_0^0(S, M) = \Im_0^0(S). \tag{4.6}$$

That is

$$\hat{f}^{V} \circ D = (\hat{f} \circ \imath)^{V}, \quad \hat{f}^{C} \circ D = (\hat{f} \circ \imath)^{C} \quad for \quad \hat{f} \in \Im_0^0(M). \tag{4.7}$$

The lifts of vector fields tangent to S are tangent to $\tilde{E}$, i.e.,

$$(DX)^{\bar{V}} = \tilde{B}(X^{V}), \quad (DX)^{\bar{C}} = \tilde{B}(X^{C}) \quad for \quad X \in \Im_0^1(S). \tag{4.8}$$

Proof. (4.6) and (4.7) are easily seen from (3.1), (3.2), (4.1) and (4.2). As for (4.8), by virtue of (4.3), (4.4), (4.6) and (4.7), we have

$$\begin{aligned} \tilde{B}(X^{V})\hat{f}^{C} &= dB(X^{V})\hat{f}^{C} = X^{V}(\hat{f}^{C} \circ B) = X^{V}(\hat{f} \circ \imath)^{C} \\ &= (X(\hat{f} \circ \imath))^{V} = (d\imath(X)\hat{f})^{\bar{V}} = (DX)^{\bar{V}}\hat{f}^{C} \end{aligned}$$

and

$$\begin{aligned} \tilde{B}(X^{C})\hat{f}^{C} &= dB(X^{C})\hat{f}^{C} = X^{C}(\hat{f}^{C} \circ B) = X^{C}(\hat{f} \circ \imath)^{C} \\ &= (X(\hat{f} \circ \imath))^{C} = (d\imath(X)\hat{f})^{\bar{C}} = (DX)^{\bar{C}}\hat{f}^{C} \end{aligned}$$

for an arbitrary element f of $\Im_0^0(M)$. Consequently from Remark 3.1, we have (4.8). □

From definition of the lifts of elements of $\Im_s^r(S, M)$, we have the formulas similar to (3.3) and (3.4). Summing up, we have the following formulas.

Let $\bar{f}$ and $\bar{X}$ be arbitrary elements of $\Im_0^0(S)$ and $\Im_0^1(S, M)$ respectively, then we have

$$\begin{aligned} \hat{T}^{C}(\bar{X}^{\bar{C}}, \bar{Y}^{\bar{C}}) &= \hat{T}(\bar{X}, \bar{Y})^{C}, \\ \hat{T}^{C}(\bar{X}^{\bar{V}}, \bar{Y}^{\bar{C}}) &= \hat{T}^{V}(\bar{X}^{\bar{C}}, \bar{Y}^{\bar{C}}) = \hat{T}(\bar{X}, \bar{Y})^{V}, \\ \hat{T}^{C}(\bar{X}^{\bar{V}}, \bar{Y}^{\bar{V}}) &= \hat{T}^{V}(\bar{X}^{\bar{C}}, \bar{Y}^{\bar{V}}) = \hat{T}^{V}(\bar{X}^{\bar{V}}, \bar{Y}^{\bar{V}}) = 0 \end{aligned} \tag{4.9}$$

along S where $\hat{T} \in \Im_2^0(M)$.

5. Formulas for Surfaces

Let there be given a Riemannian metric $\hat{G}$ in M. If we denote by g the induced metric on S from $\hat{G}$, then by definition we have

$$g(X, Y) = \hat{G}(DX, DY) \quad \text{for} \quad X, Y \in \Im_0^1(S).$$

We consider the Riemannian covariant differentiation $\hat{\nabla}$ determined by $\hat{G}$ in M. Then we have along S

$$\hat{\nabla}_{DX}DY = T_X Y + N_X Y \quad \text{for} \quad X, Y \in \Im_0^1(S), \tag{5.1}$$

where $T_X Y$ and $N_X Y$ are tangential and normal parts of $\hat{\nabla}_{DX}DY$, respectively. Then the correspondence T which assigns $T_X Y$ to a pair of two vector fields X and Y defines a covariant differentiation along S. Thus we introduce a connection ∇ on S by the condition

$$D\nabla_X Y = T_X Y \tag{5.2}$$

where X and Y are arbitrary elements of $\Im_0^1(S)$. We can easily verify that ∇ thus defined is a Riemannian connection with respect to the induced metric g and we call ∇ the connection induced on S from $\hat{\nabla}$, or simply the induced connection on S. $N_X Y$ being normal to S, we can put

$$N_X Y = h(X, Y)N, \tag{5.3}$$

Let N be the normal vector field and h being a certain tensor field of type $(0, 2)$ on S. We call h the second fundamental tensor field and we define the tensor field H of type $(1, 1)$ by

$$g(HX, Y) = h(X, Y).$$

If we denote by R the curvature tensor field for the induced connection ∇, the equations of Weingarten, Gauss and Codazzi for S in M are written respectively as

$$\hat{\nabla}_{DX}N = -D(HX) \tag{5.4}$$

$$g(R(X,Y)Z,W) = \hat{G}(\hat{R}(DX,DY)DZ,DW) + g((HX)h(Y,Z) - (HY)h(X,Z),W) \tag{5.5}$$
$$\hat{G}(\hat{R}(DX,DY)N,DW) = g(\nabla_X HY - \nabla_Y HX, W) \tag{5.6}$$

where X, Y and Z are arbitrary elements of $\Im^1_0(S)$.

S is said to be totally umbilical if there exists a scalar field $\breve{m}$ such that

$$h(X,Y) = \breve{m}g(X,Y)$$

for arbitrary elements X, Y of $\Im^1_0(S)$. We call $\breve{m}$ the mean curvature of S, and have

$$\breve{m} = \frac{1}{m-1} TraceH.$$

If a totally umbilical hypersurface has the vanishing mean curvature, it is said to be totally geodesic.

6. The Induced metric and connection on $\tilde{E}$

Let $\hat{G}$ be the Riemannian metric given in M. Then the complete lift $\hat{G}^C$ of $\hat{G}$ is the pseudo-Riemannian metric in E ([5] Prop. 5.). We say that two vector field $\overset{*}{X}$ and $\overset{*}{Y}$ are orthogonal on $\tilde{E}$ with respect to $\hat{G}^C$, if

$$\hat{G}^C(\overset{*}{X}, \overset{*}{Y}) = 0$$

on $\tilde{E}$ and we say that $\overset{*}{N}$ is a normal vector field to $\tilde{E}$, if

$$\hat{G}^C(\overset{*}{N}, \tilde{B}\tilde{X}) = 0 \quad \text{for} \quad \tilde{X} \in \Im^1_0(\tilde{E}).$$

If we denote a mapping which assigns to each $p \in S$ to a normal vector N_p to S by N, N is a vector field along S. We can define its vertical lift $N^{\bar{V}}$ and complete lift $N^{\bar{C}}$ to E according to section 4. Then we find that for each point $x \in \tilde{E}$, $(N^{\bar{V}})_x$ and $(N^{\bar{C}})_x$ are normal vectors to $\tilde{E}$ with respect to $\hat{G}^C$ and they are self-orthogonal but mutually orthogonal, i.e.,

$$\begin{aligned} \hat{G}^C(N^{\bar{V}}, \tilde{B}X^C) &= \hat{G}^C(N^{\bar{C}}, \tilde{B}X^C) = 0 \\ \hat{G}^C(N^{\bar{C}}, N^{\bar{C}}) &= \hat{G}^C(N^{\bar{V}}, N^{\bar{V}}) = 0 \\ \hat{G}^C(N^{\bar{V}}, N^{\bar{C}}) &= 1 \end{aligned} \tag{6.1}$$

for $X \in \Im^1_0(S)$. These are direct consequences of (4.9). Moreover we can choose $\left\{N^{\bar{V}}, N^{\bar{C}}\right\}$ as the basis of normal space to $\tilde{E}$.

If we denote by $\tilde{g}$ the induced metric on $\tilde{E}$ from $\hat{G}^C$ then we have

$$\tilde{g}(X^C, Y^C) = \hat{G}^C(\tilde{B}X^C, \tilde{B}Y^C) \quad \text{for} \quad X, Y \in \Im^1_0(S). \tag{6.2}$$

The complete lift $\hat{\nabla}^C$ of $\hat{\nabla}$ to E is by definition an affine connection in E characterized by the property

$$\hat{\nabla}^C_{\hat{X}^C} \hat{Y}^C = (\hat{\nabla}_{\hat{X}} \hat{Y})^C \quad \text{for} \quad \hat{X}, \hat{Y} \in \Im^1_0(M), \tag{6.3}$$

from which we also have

$$\hat{\nabla}^C_{\hat{X}^C} \hat{Y}^C = (\hat{\nabla}_{\hat{X}} \hat{Y})^V \quad \text{for} \quad \hat{X}, \hat{Y} \in \Im^1_0(M) \tag{6.4}$$

[16].

Proposition 6.1 ([5]). *If $\hat{\nabla}$ is the Riemannian connection with respect to $\hat{G}$, $\hat{\nabla}^C$ is the Riemannian connection of E with respect to the pseudo-Riemannian metric $\hat{G}^C$.*

Similarly the complete lift ∇^C of the induced connection ∇ on S is the Riemannian connection with respect to g^C.

Denoting by $\tilde{\nabla}$ the connection induced on $\tilde{E}$ from $\hat{\nabla}^C$, along $\tilde{E}$ we have

$$\hat{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C = \tilde{B}(\tilde{\nabla}_{X^C} Y^C) + N_{X^C} Y^C \quad \text{for} \quad X, Y \in \Im^1_0(S), \tag{6.5}$$

where $N_{X^C} Y^C$ is the normal part of $\hat{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C$. Then we can put

$$N_{X^C} Y^C = \tilde{h}(X^C, Y^C) N^{\bar{V}} + \tilde{k}(X^C, Y^C) N^{\bar{C}}, \tag{6.6}$$

where $\tilde{h}$ and $\tilde{k}$ are certain tensor fields of type $(0,2)$ which are called the second fundamental tensor fields with respect to $N^{\bar{V}}$ and $N^{\bar{C}}$, respectively.

Proposition 6.2. *The connection $\tilde{\nabla}$ induced on $\tilde{E}$ from $\hat{\nabla}^C$ is the complete lifts of the connection ∇ induced on S from $\hat{\nabla}$. That is to say, $\tilde{\nabla}$ is the Riemannian connection of $\tilde{E}$ with respect to g^C satisfying the condition*

$$\tilde{\nabla}_{X^C} Y^C = (\nabla_X Y)^C \quad for \quad X, Y \in \Im^1_0(S).$$

Proof. First we shall show

$$\hat{\nabla}^C_{\tilde{B}X^C} \tilde{B} Y^C = (\hat{\nabla}_{DX} DY)^{\bar{C}} \quad \text{for} \quad X, Y \in \Im^1_0(S). \tag{6.7}$$

Recalling the definitions of lifts of DX in section 4, we introduce vector fields $\hat{X}$ and $\hat{Y}$ coincide respectively with DX and DY on $(U \cap S)^\circ$. Then we have

$$\hat{\nabla}^C_{\tilde{B}X^C} \tilde{B} Y^C = \hat{\nabla}^C_{(DX)^{\bar{C}}} (DY)^{\bar{C}} = \sharp\hat{\nabla}^C_{\hat{X}^C} \hat{Y}^C = \sharp(\hat{\nabla}_{\hat{X}} \hat{Y})^C = (\hat{\nabla}_{DX} DY)^{\bar{C}},$$

since $\hat{\nabla}_{\hat{X}} \hat{Y}$ is a vector field on U which coincides with $\hat{\nabla}_{DX} DY$ on $(U \cap S)^\circ$. By the same reason we have the following formulas which will be used in the next section.

$$\begin{aligned} \hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{C}} &= (\hat{\nabla}_{DX} N)^{\bar{C}} \\ & \qquad\qquad \text{for} \quad X \in \Im^1_0(S). \\ \hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{V}} &= (\hat{\nabla}_{DX} N)^{\bar{V}} \end{aligned} \tag{6.8}$$

Now by virtue of (4.8), (4.9), (5.1), (5.3) and (6.7), we get

$$\hat{\nabla}^C_{\tilde{B}X^C} \tilde{B} Y^C = \tilde{B}(\nabla_X Y)^C + h^C(X^C, Y^C) N^{\bar{V}} + h^V(X^C, Y^C) N^{\bar{C}}. \tag{6.9}$$

On the other hand by (6.5) and (6.6) we have

$$\hat{\nabla}^C_{\tilde{B}X^C} \tilde{B} Y^C = \tilde{B}(\tilde{\nabla}_{X^C} Y^C) + \tilde{h}(X^C, Y^c) N^{\bar{V}} + \tilde{k}(X^C, Y^c) N^{\bar{C}}. \tag{6.10}$$

Therefore we obtain

$$(\nabla_X Y)^C = \tilde{\nabla}_{X^C} Y^C.$$

□

Remark 6.3. It is known that if $\hat{R}$ is the curvature tensor field of $\hat{\nabla}$, then $\hat{R}^C$ is the curvature tensor field of $\hat{\nabla}^C$ ([5]). Similarly, the complete lift of the curvature tensor field of ∇^C. Therefore from this Proposition, the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ $(= \nabla^C)$ is the complete lift of the curvature tensor field of ∇.

Moreover from (6.9) and (6.10) we have

Proposition 6.4. *The complete and vertical lifts of the second fundamental tensor field of S are the second fundamental tensor fields with respect to $N^{\bar{V}}$ and $N^{\bar{C}}$, respectively.*

$\tilde{E}$ is said to be totally umbilic if and only if at each point of $\tilde{E}$, there exists differentiable functions λ and μ such

$$h^V(\tilde{X}, \tilde{Y}) = \lambda \tilde{g}(\tilde{X}, \tilde{Y}) \quad h^C(\tilde{X}, \tilde{Y}) = \mu \tilde{g}(\tilde{X}, \tilde{Y})$$

for any $\tilde{X}, \tilde{Y} \in \Im^1_0(\tilde{E})$. Then we find

$$\begin{aligned} \lambda &= \tfrac{1}{m+n-2} TraceH^V = 0 \\ \mu &= \tfrac{1}{m+n-2} TraceH^C \end{aligned} \tag{6.11}$$

in terms of local coordinates. If both λ and μ vanish, $\tilde{E}$ is said to be totally geodesic. The main curvature vector field $\tilde{M}$ of $\tilde{E}$ is defined by

$$\tilde{M} = \lambda N^{\bar{V}} + \mu N^{\bar{C}},$$

which is independent of the basis chosen in the space normal to $\tilde{E}$. The mean curvature $\tilde{m}$ of $\tilde{E}$ in E is defined to be the magnitude of the mean curvature vector field, (i.e. $\tilde{m} = \hat{G}^C(\tilde{M}, \tilde{M})$).

Proposition 6.5. *If $\tilde{E}$ is totally umbilic, then S is totally geodesic. $\tilde{E}$ is totally geodesic if and only if S is totally geodesic in M.*

Proof. If we assume that $\tilde{E}$ is totally umbilic, then the second fundamental tensor of S always vanishes, because $Traceh^V$ is zero. Conversely, if S is totally geodesic, $\tilde{E}$ is also totally geodesic from Proposition 6.4. □

Proposition 6.6. *The mean curvature of $\tilde{E}$ vanishes.*

Proof. If we denote the mean curvature by $\tilde{m}$, by the virtue of (6.1) we have

$$\tilde{m} = \hat{G}^C(\tilde{M}, \tilde{M}) = 2\lambda\mu, \quad (m+n-2)\lambda = TraceH^V, \quad (m+n-2)\mu = TraceH^C.$$

Now from (6.11), we have Proposition 6.6. □

7. The Structure Equation of $\tilde{E}$

In this section we shall investigate the equations of Gauss and Weingarten and the structure equations (i.e., the equations of Gauss, Codazzi and Ricci) on $\tilde{E}$ in E. These are written in the following form.

$$\left.\begin{array}{l} \hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{V}} = -\tilde{B}H^V X^C \\ \hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{C}} = -\tilde{B}H^C X^C \end{array}\right. \quad \text{for} \quad X \in \Im^1_0(S), \tag{7.1}$$

$$\begin{array}{rcl} \tilde{g}(\tilde{R}(X^C, Y^C)Z^C, W^C) & = & \hat{G}^C(\hat{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C, \tilde{B}W^C) \\ & & +\tilde{g}((H^C X^C)h^V(Y^C, Z^C) + (H^V X^C)h^C(Y^C, Z^C) \\ & & -(H^C Y^C)h^V(X^C, Z^C) - (H^V Y^C)h^C(X^C, Z^C), W^C) \end{array} \tag{7.2}$$

where $\tilde{R}$ is the Riemannian curvature tensor field of ∇^C on $\tilde{E}$.

$$\begin{array}{rcl} \hat{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{V}} & = & \tilde{B}(\nabla^C_{X^C} H^V Y^C - \nabla^C_{Y^C} H^V X^C), \\ \hat{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{C}} & = & \tilde{B}(\nabla^C_{X^C} H^C Y^C - \nabla^C_{Y^C} H^C X^C), \\ \hat{R}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}X^C & = & 0. \end{array} \tag{7.3}$$

Proof. By virtue of (3.3), (4.8), (5.4) and (6.8), (7.1) is reduced to

$$\hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{V}} = (\hat{\nabla}_{DX} N)^{\bar{V}} = -(DHX)^{\bar{V}} = -\tilde{B}(HX)^V = -\tilde{B}H^V X^C,$$

and

$$\hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{C}} = (\hat{\nabla}_{DX} N)^{\bar{C}} = -(DHX)^{\bar{C}} = -\tilde{B}(HX)^C = -\tilde{B}H^C X^C.$$

Next as for (7.2), we first note that $\tilde{R}$ is the complete lift of R, then we find

$$\tilde{R}(X^C, Y^C)Z^C = R^C(X^C, Y^C)Z^C \quad \text{for} \quad X, Y \in \Im^1_0(S)$$

because of Remark 6.3.

We shall drive the formulas for the later use

$$\begin{array}{rcl} R^C(X^C, Y^C)Z^C & = & (R(X,Y)Z)^C, \\ \hat{R}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C & = & (\hat{R}(DX, DY)DZ)^{\bar{C}}, \\ \hat{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{V}} & = & (\hat{R}(DX, DY)N)^{\bar{V}}, \\ \hat{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{C}} & = & (\hat{R}(DX, DY)N)^{\bar{C}}, \\ \hat{R}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}X^C & = & (\hat{R}(N,N)DX)^{\bar{V}} = 0. \end{array} \tag{7.4}$$

In fact, from (3.3) and (6.3) we have,

$$\begin{array}{rcl} R^C(X^C, Y^C)Z^C & = & \nabla^C_{X^C}\nabla^C_{Y^C}Z^C - \nabla^C_{Y^C}\nabla^C_{X^C}Z^C - \nabla^C_{[X^C, Y^C]}Z^C \\ & = & \nabla^C_{X^C}(\nabla_Y Z)^C - \nabla^C_{Y^C}(\nabla_X Z)^C - \nabla^C_{[X,Y]^C}Z^C \\ & = & (\nabla_X \nabla_Y Z)^C - (\nabla_Y \nabla_X Z)^C - (\nabla_{[X,Y]}Z)^C \\ & = & (R(X,Y)Z)^C. \end{array}$$

By making use of (6.6) and (6.7), the others are also obtained.

Now we have

$$\begin{aligned}\tilde{g}(\tilde{R}(X^C,Y^C)Z^C,W^C) &= \tilde{g}(R(X^C,Y^C)Z^C,W^C) = \tilde{g}((R(X,Y)Z)^C,W^C)\\ &= \hat{G}^C((\hat{R}(DX,DY)DZ)^{\bar{C}},(DW)^{\bar{C}})\\ &\quad +\tilde{g}(((HX)h(Y,Z)-(HY)h(X,Z))^C,W^C)\\ &= \hat{G}^C(\hat{R}^C(\tilde{B}X^C,\tilde{B}Y^C)\tilde{B}Z^C,\tilde{B}W^C)\\ &\quad +\tilde{g}((H^CX^C)h^V(Y^C,Z^C)+(H^VX^C)h^C(Y^C,Z^C)\\ &\quad -(H^CY^C)h^V(X^C,Z^C)-(H^VY^C)h^C(X^C,Z^C),W^C)\end{aligned}$$

from (5.4).

As for (7.3), by making use of the equations of Codazzi and Ricci (5.5), (5.6) and (7.4), we have

$$\begin{aligned}\hat{R}^C(\tilde{B}X^C,\tilde{B}Y^C)N^{\bar{V}} &= (\hat{R}(DX,DY)N)^{\bar{V}}\\ &= (D(\nabla_X HY-\nabla_Y HX))^{\bar{V}}\\ &= \tilde{B}(\nabla_X HY-\nabla_Y HX)^V\\ &= \tilde{B}(\nabla^C_{X^C}H^VY^C-\nabla^C_{Y^C}H^VX^C)\end{aligned}$$

$$\begin{aligned}\hat{R}^C(\tilde{B}X^C,\tilde{B}Y^C)N^{\bar{C}} &= (\hat{R}(DX,DY)N)^{\bar{C}}\\ &= (D(\nabla_X HY-\nabla_Y HX))^{\bar{C}}\\ &= \tilde{B}(\nabla_X HY-\nabla_Y HX)^C\\ &= \tilde{B}(\nabla^C_{X^C}H^CY^C-\nabla^C_{Y^C}H^CX^C)\end{aligned}$$

and

$$\hat{R}^C(N^{\bar{V}},N^{\bar{C}})\tilde{B}X^C = 0.$$

□

Thus we have (7.1), (7.2) and (7.3), which are the equations of Weingarten, Gauss and Codazzi and Ricci on $\tilde{E}$ in E, respectively.

References

[1] Gancarzewicz, J., Rahmani, N., *Horizontal lift of linear connections to vector bundles associated with the principal bundle of linear frames*, Colloquia Mathematics Societatis Janos Bolyai, **56**(1989), 273–284. 1

[2] Leon, M.D., Rodrigues, P.R., Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies 158, North-Holland, 1989. 2

[3] Omran, T., Sharffuddin, A., Husain, S.I., *Lifts of structures on manifolds*, Publications De L'institut Math. J., **36(50)**(1984), 93–97. 1

[4] Özkan, M., Esin, E., *Vertical lifts of tensor fields to vector bundle*, Algebras Groups Geom., **25(1)**(2008), 21–30. 1, 3, 3, 3.1

[5] Özkan, M., *Complete lifts of tensor fields to vector bundle*, Int. J. Math. Sci. App., **1(2)**(2012), 593–599. 1, 3, 3, 3.1, 6, 6.1, 6.3

[6] Özkan, M., *Complete lifts of structures on manifold to vector subbundle*, GUJS, **25(3)**(2012), 661–664. 1

[7] Özkan, M., Keçilioğlu, O., *Complete lift of a structure satisfying $F^3 + F = 0$ to vector subbundle*, Hadronic J., **38(3)**(2015), 295–305. 1

[8] Özkan, M., *Prolongations of golden structures to tangent bundles*, Differential Geometry-Dynamical Systems, **16**(2014), 227–238. 1

[9] Özkan, M., Çıtlak, A.A., Taylan, E., *Prolongations of golden structure to tangent bundle of Order* 2, GUJS, **28(2)**(2015), 253–258. 1

[10] Özkan, M., Yılmaz, F., *Prolongations of golden structures to tangent bundles of order r*. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat, **65(1)**(2016), 35–47. 1

[11] Özkan, M., Taylan, E., Çıtlak, A. A., *On lifts of silver structure*, Journal of Science and Arts, **2(39)**(2017), 223–234. 1

[12] Poor, W.A., Differential Geometric Structures, McGraw-Hill, New York, 1981. 1

[13] Saunders, D.J., The Geometry of Jet Bundles, Cambridge University Press, Cambridge, 1989. 2

[14] Tani, M., *Prolongations of hypersurfaces to tangent bundles*, Kodai Math. Semp. Rep., **21**(1969), 85–96. 1, 2

[15] Yano, K., Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973. 2

[16] Yıldırım, M., Esin, E., *Vertical and complete lifts to vector bundle*, Algebras Groups Geom., **24(1)**(2007), 75–84. 1, 2, 3, 6