Çankaya University Journal of Science and Engineering
Volume 12, No. 2 (2015) 020–032

On the Existence of Solutions for Boundary Value Problems in Banach Spaces

Hero Salahifard[1], S. Mansour Vaezpour[1], Abdolrahman Razani[2]

[1]*Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran.,*
[2]*Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.,*
e-mail: salahifard@gmail.com, vaez@aut.ac.ir, razani@sci.ikiu.ac.ir

Abstract: In this paper, by applying the theory of condensing multimaps and the topological degree, we deal with the existence of solutions for boundary value problems with second order differential inclusions in different cases where the underlying space is a Banach space. Indeed, we investigate the existence of solutions for the BVP

$$\begin{cases} x''(t) \in F(t, x(t)) & t \in I = [0,1], \\ x(0) = x(1) = 0, \end{cases}$$

where X is a real Banach space and the multifunction $F : I \times X \multimap K(X)$, in one case, has convex values and in another case has non-convex values ($K(X)$ denotes compact subsets of X). Moreover, some results on the existence of solutions for the extended version of BVP

$$\begin{cases} u''(t) \in Q(u) & t \in I, \\ u(0) = u(1) = 0, \end{cases}$$

are presented, where $Q : C(I,X) \multimap C(\mathscr{L}^2)$ is a multimap satisfying some appropriate conditions. Finally, we show how the results can be used to study periodic feedback control systems.

Keywords: Boundary Value Problems, Condensing map, Feedback control systems, Topological degree.

1. Introduction

During the last few years, the second-order boundary value problem for ordinary and functional differential equations with various conditions (periodic, nonlinear, integral conditions, etc..) have attracted the attention of many mathematicians and are still intensively studied. Indeed, these problems arise in different areas of physics, mechanics, and more generally in applied mathematics.

The first motivation of the study of the concept of differential inclusions comes from the development of some studies in control theory. Examples of such phenomena include mechanical systems with the Coulomb friction modeled as a force proportional to the sign of a velocity and systems

ISSN 1309 - 6788

whose control laws have discontinuities. For more information about the relation between differential inclusions and control theory, we refer the reader to [2, 7, 8, 16, 25] and the references therein.

The case of the second order boundary value problem for differential inclusions has been studied in [9] where the multi-function satisfies a Bernstein-Nagumo condition. Benchohra et al. [3] have studied some 3-point boundary value problems associated with a differential inclusion $x''(t) \in F(t,x(t))$ where F is a nonempty compact valued multi-valued mapping which is integrably bounded.

Very recently, C.S. Goodrich [11] demonstrated the existence of at least one positive solution to the differential inclusion $x''(t) \in F(t,x(t))$, equipped with the boundary conditions $x(1) = 0$ and $x(0) = H(\varphi(x))$, by imposing an asymptotic condition on H, where H is a nonlinear function and φ is linear functional realized as a Lebesgue-Stieltjes integral.

In 2015, R.P. Agarwal et al. discussed this inclusion problem in the Caputo fractional form $D^{\alpha}x(t) \in F(t,x(t))$ by utilising the α-ψ-Ciric generalized fixed point theorem for multifunctions where $1 < \alpha \leq 2$.

Hu and Papageorgiou [13, 14] proved the existence of periodic solutions for nonconvex differential inclusions in $\mathbb{R}^n$. The approach [13] was based on directionally continuous selectors for the orientor field and on a Nagumo type tangential condition. In [14], their approach was based on degree theory arguments. Some existence results for the periodic problems have been established by De Blasi et al. [6]. The method was based on the construction of the topological degree for the Poincaré maps and on a guiding potential condition. In [17], the approach is based on the LeraySchauder alternative theorem and the Schauder fixed point theorem where the orientor field (multivalued vector field) was nonconvex. It is worth mentioning that there are many papers about the existence of solutions for boundary value problems with differential inclusions in $\mathbb{R}^n$, the proofs of which are essentially based on the fixed point theorems for compact multi-maps (see for example [5, 23, 26, 27]).

Ravichandran and Baleanu [24] focused on establishing the existence result for a class of abstract fractional neutral functional integro-differential evolution systems involving the Caputo fractional derivative by using the properties of characteristic solution operators and Mönch's fixed point theorem via measures of noncompactness.

Motivated by the above, the goal of this paper is to investigate the existence of periodic solutions for systems governed by differential inclusions. Our approach is based on the method of the integral multioperator and the method of the translation multi operator along with the solutions of the inclusion, which were used in [15]. Three cases of boundary value problems with differential inclusions are considered here. In all of them we deal with the condensing multioperators. To the best of our knowledge, there are relatively many results on boundary value problems with differential inclusions of compact multimaps or J^c-multimaps [18, 19, 20] in finite dimension Banach spaces; however, this may be the first paper on the existence of solutions for boundary value problems with noncompact multivalued maps in Banach spaces.

The paper is organized such that the next section contains background materials and preliminaries from multivalued analysis which can be found in [15]. In Section 3, we investigate the existence of solutions for the BVP

$$\begin{cases} x''(t) \in F(t, x(t)) & t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{1}$$

where $I = [0,1]$, X is a real Banach space and the multifunction $F : I \times X \multimap K(X)$ in one case has convex values and in another case has nonconvex values. In section 4, some results on the existence of solutions for the extended version of BVP

$$\begin{cases} u''(t) \in Q(u) & t \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

are presented, where $Q : C(I,X) \multimap C(\mathscr{L}^2)$ is a multimap satisfying some appropriate conditions. In the last section, as an application we consider a feedback control system of the form

$$\begin{cases} x''(t) = f(t, x(t), u(t)) & t \in I, \\ u(t) \in U(t, x(t)) & t \in I \\ x(0) = x(1) = 0, \end{cases} \tag{3}$$

where $f : I \times X \times X_1 \to X$ and $U : I \times X_1 \to K(X_1)$, X, X_1 are Banach spaces. The first equation of the above system describes the dynamics of the system and the second inclusion represents the feedback.

2. Preliminaries

Let X be a metric space and Y be a norm space. $P(Y)$ denotes the collection of all nonempty subsets of Y, $K(Y)$ denotes the collection of all nonempty compact subsets of Y and $Kv(Y)$ denotes the collection of all $S \in K(Y)$ where S is convex.

Definition 1. (See, e.g., [10].) A multivalued map (multimap) $F : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set

$$F_+^{-1}(V) = \{x \in X : F(x) \subset V\}$$

is open in X. A u.s.c. multimap F is said to be completely u.s.c. if it maps every bounded subset $X_1 \subset X$ into a relatively compact subset $F(X_1)$ of Y.

A function $\gamma : \{B \subset X : B \text{ is bounded}\} \to [0,\infty)$ is said to be a measure of noncompactness (MNC), if it satisfies the invariance property under closure and convex hull i.e. $\gamma(\overline{co}B) = \gamma(B)$.
An MNC γ is called

(1) (regular): if $\gamma(B) = 0$ is equivalent to the relative compactness of B,
(2) (semi-additive): if $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$,
(3) (algebraic semi-additive): if $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$,
(4) (nonsingular): if $\gamma(\{a\} \cup B) = \gamma(B)$ for all $a \in X$.

Examples of an MNC satisfying all the above properties are the Kuratowski MNC defined by:

$$\alpha(B) = \inf\{r > 0 : B, \text{ which may be covered by finitely many sets of diameter } \leq r\},$$

and the Hausdorff MNC, defined by:

$$\beta(B) = \inf\{r > 0 : \text{there exists a finite } r\text{-net for } B \text{ in } X\},$$

Another example of an MNC which is defined on the space of continuous functions $C([0,T],X)$ with the values in a Banach space X is:

$$\phi(B) = \sup_{t\in[0,T]} \beta_E(B(t)),$$

where β_X is the Hausdorff MNC in X and $B(t) = \{y(t) : y \in B\}$. It is known ([15]) that for every $B \subset C([0,T],X)$, we have

$$\phi(B) \leq \beta_C(B),$$

where β_C is the Hausdorff MNC in $C([0,T],X)$. Another example of an MNC defined on the space of continuous functions $C([0,T],X)$ with the values in a naturally partially ordered $\mathbb{R}^2_+$ is

$$\upsilon(B) = \max_{D\subset\Delta(B)} (\eta(D), mod_C(D)), \tag{4}$$

where $\Delta(D)$ is the collection of all denumerable subsets of B, and

$$\eta(D) = \sup_{t\in[0,T]} e^{-bt}\beta(D(t))$$

given by $b > 0$ is large enough and $mod_C(D)$ is the modulus of equi-continuity of D defined as

$$mod_C(D) = \lim_{\delta\to 0} \sup_{y\in D} \max_{|t_1-t_2|\leq\delta} ||y(t_1) - y(t_2)||.$$

Definition 2. Let X be a Banach space. A multimap $F : X \to P(X)$ or a family of multimaps $G : [0,1] \times X \to P(X)$ is called condensing relative to an MNC γ if for every non-relatively compact set $B \subset X$

$$\gamma(F(B)) < \gamma(B) \quad or \quad \gamma(G([0,1] \times B)) < \gamma(B),$$

respectively.

Let $\omega \subset X$ be an open set, $K \subset X$ a closed convex subset, γ a monotone MNC in X, and $F : \overline{\omega}_K \to Kv(K)$ a u.s.c. multimap such that $x \notin F(x)$ for all $x \in \partial\omega_K$, where $\overline{\omega}_K$ and $\partial\omega_K$ denote, respectively, the closure and the boundary of the set $\omega_K = \omega \cap K$.
In such a setting, the relative topological degree $deg(F, \overline{\omega})$ is defined and satisfies the standard properties (see [15]).

3. Existence of Solutions

In this section we consider the following BVP

$$\begin{cases} x''(t) \in F(t, x(t)) & t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{5}$$

where $I = [0,1]$, X is a real Banach space and $F : I \times X \multimap X$ satisfies the following assumptions:

$(F0)$ F has nonempty, compact, and convex values,
$(F1)$ the multifunction $F(.,x) : [0,1] \to Kv(X)$ has a strongly measurable selection for every $x \in X$,
$(F2)$ the multimap $F(t,.) : X \to Kv(X)$ is u.s.c. for a.a. $t \in I$,
$(F3)$ there exists a function $q \in L^1([0,1])$ such that

$$||F(t,x)|| = \sup\{||z|| : z \in F(t,x)\} \le q(t)(1 + ||x||),$$

for a.a. $t \in I$,
$(F4)$ there exists a constant $k > 0$ such that

$$\beta(F(t,D)) \le k\beta(D) \ \text{ for a.a. } \ t \in I,$$

for every bounded set $D \subset X$, where β_C is the Hausdorff MNC.

Definition 3. (see [22, 15]) Let $F : I \times X \multimap X$ be a multimap satisfying assumptions $(F0) - (F3)$, then the superposition multioperator $\mathscr{P}_F : C([0,1], X) \multimap L^1([0,1], X)$ given by

$$\mathscr{P}_F(x) = \{f \in L^1([0,1], X) : f(t) \in F(t, x(t)) \ \ a.e \ \ t \in I\},$$

is correctly defined.

Definition 4. A function $x : I \to X$ is said to be a mild solution of (5) if there exists $f \in \mathscr{P}_F(x)$ such that x has the form

$$x(t) = \int_0^1 G(t,s) f(s) ds,$$

where

$$G(t,s) = \begin{cases} t(s-1) & 0 \le t \le s, \\ s(t-1) & s \le t \le 1. \end{cases} \tag{6}$$

Lemma 1. Every mild solution of (5), is a solution of it.

Proof. Let x be a mild solution, then there exists $f \in \mathscr{P}_F(x)$ such that

$$x(t) = \int_0^1 G(t,s)f(s)ds,$$

so we can write

$$x(t) = (t-1)\int_0^t sf(s)ds + t\int_t^1 (s-1)f(s)ds.$$

Therefore,

$$\begin{aligned} x'(t) = & \quad \textstyle\int_0^t sf(s)ds + t(t-1)f(t) + \int_t^1 (s-1)f(s)ds - t(t-1)f(t), \\ = & \quad \textstyle\int_0^t sf(s)ds + \int_t^1 (s-1)f(s)ds. \end{aligned}$$

Differentiating again

$$x''(t) = tf(t) - (t-1)f(t) = f(t),$$

so $x''(t) \in F(t,x(t))$ and $x(0) = x(1) = 0$ and the desired result is obtained. ■

In what follows, we need the following lemma.

Lemma 2. (See Theorems 5.1.2., 5.1.3. in [15].) Let $\mathscr{P}_F$ be a superposition multioperator generated by a multimap $F : [0,1] \times X \to Kv(X)$ satisfying properties $(F0)-(F4)$ and $S : L^1([0,1],X) \to C([0,1],X)$ be an operator satisfying:

$(S1)$ there exists $A \geq 0$ such that

$$||Sf(t) - Sg(t)||_X \leq A\int_0^t ||f(s) - g(s)||_X ds$$

for every $f,g \in L^1([0,1],X), t \in [0,1]$,

$(S2)$ for any compact set $K \subset X$ and sequence $\{f_n\}_1^\infty \subset L^1([0,1],X)$ such that $\{f_n\}_1^\infty \subset K$ for a.a. $t \in [0,1]$, the weak convergence $f_n \rightharpoonup f_0$ implies $Sf_n \to Sf_0$.

Then $S \circ \mathscr{P}_F$ is a u.s.c. closed multioperator with compact values and υ-condensing on bounded sets where υ is an MNC defined as (4).

Theorem 1. Under conditions $(F0)-(F4)$, the solution set of BVP (5) is nonempty.

Proof. By the assumptions $(F0)-(F3)$, the superposition multioperator $\mathscr{P}_F$ given by Definition 3 is correctly defined. Therefore, one can define the multioperator $A : C([0,1],X) \multimap C([0,1],X)$ by

$$Ax = \{y : y(t) = \int_0^t G(t,s)f(s)ds : f \in \mathscr{P}_F(x)\},$$

and in order to prove the theorem, it is sufficient to verify that the fixed point set $FixA$ is nonempty. Consider the closed set $K = \{x : x(0) = x(1) = 0\} \subset C([0,1],X)$ and the family of multioperators $\widetilde{A} : K \times [0,1] \multimap K$

$$\widetilde{A}(x,\lambda) = \{y : y(t) = (1-\lambda)x + \lambda \int_0^t G(t,s)f(s)ds : f \in \mathscr{P}_F(x)\}.$$

By Lemma 2, the multimap $\widetilde{A}$ is u.s.c. and υ-condensing on bounded sets of K in addition to having compact, convex values.
From $(F3)$ and the standard technique based on the Gronwall-type inequality, the solutions set of $x \in \widetilde{A}(x,\lambda)$ is a priori bounded in the norm by the constant

$$R = De^D,$$

where $D = \max_{t\in I} \int_0^t q(s)ds$. Therefore, if we take $B(0,r)$ as an open ball in $C([0,1],X)$ with $r > R$, then by the basic properties of the topological degree

$$deg_K(A,\overline{B}) = deg_K(\widetilde{A}(.,1),\overline{B}) = deg_K(\widetilde{A}(.,0),\overline{B}) = 1,$$

and the desired result is obtained. ∎

Now we want to consider the case that the multioperator $F : [0,1] \times X \multimap K(X)$ has nonconvex values, but instead of assumptions $(F1)$ and $(F2)$, it satisfies the almost lower semicontinuity assumption:

(F_L) there exists a sequence $\{I_n\}$ of disjoint compact sets $I_n \subset [0,1]$ such that
 (i) the restriction of F on each set $I_n \times X$ is l.s.c.
 (ii) meas$([0,1] \setminus I) = 0$, where $I = \cup_n I_n$.

Theorem 2. Let X be a separable Banach space and $D = \max_{t\in I} \int_0^t q(s)ds < 1$. Under assumptions $(F_L), (F3)$ and $(F4)$, the solution set of problem (5) is nonempty.

Proof. In this situation, the superposition multioperator $\mathscr{P}_F(x)$ defined by 3 is l.s.c. and by Fryszkowski-Bressan-Colombo theorem (see [4]), it admits a continuous selection $h(x)$. Now we want to show that there exists a compact convex subset of $C([0,1],X)$ invariant under the action of multioperator A. Consider the ball $\overline{B}_r(0) = \{x \in C([0,1],X) : ||x|| \leq r\}$ where $r > 0$ is chosen so that

$$r > D(1-D)^{-1}. \tag{7}$$

Let $x \in \overline{B}_r(0)$ and $y \in A(x)$, then by (7) we have

$$||y|| \leq D + rD \leq r.$$

Therefore, the multioperator A maps the ball $\overline{B}_r(0)$ into itself. It is clear that the continuous map $a : E \to E$ defined by

$$a(x)(t) = \int_0^1 G(t,s)h(x)(s)ds$$

is a continuous selection of A. Applying the Schauder fixed point theorem to the continuous map a, leads to the desired result. ■

4. Extended Version

Let X be a real Banach space, $I = [0,1]$ and $\mathscr{L}^2$ denotes the space of all square-integrable functions $L_2(I,X)$ with the norm

$$||f(t)||_2 = (\int_0^1 ||f(s)||^2 ds)^{\frac{1}{2}}.$$

Consider the BVP

$$\begin{cases} u''(t) \in Q(u) & t \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{8}$$

where $Q : C(I,X) \multimap C(\mathscr{L}^2)$ is a multimap satisfying the following conditions:

$(C1)$ for any operator $S : L^1(I,X) \to C(I,X)$ satisfying $(S1)$ and $(S2)$, the composition $S \circ Q$ is a condensing map on bounded sets.

$(C2)$ there are constants $a, b > 0$ such that

$$||Q(u)||_2 \le a(1 + ||u||_2^b),$$

for any $u \in C(I,X)$ where

$$||Q(u)||_2 = \sup\{||f||_2 : f \in Q(u)\}.$$

Note that the values of multimap Q are not necessarily convex.
Hereafter, we denote the space of all continuous mappings from $C(I,X)$ by $\mathscr{C}$. The symbol $\langle .,. \rangle$ denotes the inner product in $\mathscr{L}^2$, and $B_{\mathscr{C}}(0,r)$ denotes the ball in $\mathscr{C}$ of radius r centered at the origin.

Theorem 3. Let $(C1) - (C2)$ hold and there exists $N > 0$ such that for every $u \in \mathscr{C}$, $||u||_2 > N$, the relation

$$\langle f,u \rangle > 0 \quad \text{for all} \quad f \in Q(u),$$

holds, then Problem (8) has a solution.

Proof. Let $u(t) = \int_0^1 G(t,s)f(s)ds$ for $f(s) \in Q(u)$, then by the same argument in the proof of Lemma 1, $u(t)$ is a solution of Problem (8). Therefore, it is sufficient to show that the fixed point set of ϕ, where $\phi(u) = \{y : y(t) = \int_0^1 G(t,s)f(s)ds, f \in Q(u)\}$ is nonempty.

Assume that there exists $x \in \mathscr{C}$ such that $\phi(x) = x$ then $x(0) = x(1) = 0$ and there is $f^* \in Q(x)$ such that $x''(t) = f^*(t)$ for a.e $t \in I$, and hence

$$\langle f^*, x\rangle = \langle x'', x\rangle = -\langle x', x'\rangle \leq 0.$$

Therefore, $||x||_2 \leq N$.
By $(C2)$, for $t \in I$, we have

$$||x(t)|| \leq \int_0^1 |G(t,s)|||f^*(s)||ds \leq \int_0^1 ||f^*(s)||ds \leq ||f^*(t)||_2 \leq a(1+||x||_2^b),$$

thereupon , $||x||_{\mathscr{C}} = \max_{t\in I} ||x(t)|| \leq a(1+N^b)$. Consider the multimap $\varphi : B_{\mathscr{C}}(0,R) \times [0,1] \to P(\mathscr{C})$

$$\varphi(x,\lambda) = \int_0^1 G(t,s)((1-\lambda)\delta x(s) + \lambda f(s)ds,$$

where $0 < \delta < \frac{1}{N}$ is an arbitrary number and $R = aN^b + a + 1$. One can represent φ as $S \circ Q$, where $Sf = \int_0^1 G(t,s)((1-\lambda)\delta x(s) + \lambda f(s))ds$. It is clear that S satisfies $(S1)$ and condition $(S1)$ implies that S is a bounded linear operator from the space $L^1(I,X)$ into $\mathscr{C}$. Therefore it is continuous and by the standard argument in [15] on weak sequential convergence and the relative compactness of the sequence $\{Sf_n\}$, this convergence will be in the norm of the space $\mathscr{C}$. Therefore, S satisfies $(S2)$, then by $(C1)$, φ is condensing on bounded sets.
Now we will show that φ has no fixed points on $\partial B_{\mathscr{C}}(0,R) \times [0,1]$. On the contrary, let $(x_*, \lambda_*) \in \partial B_{\mathscr{C}}(0,R) \times [0,1]$, so we have

$$x_*(t) = \int_0^1 G(t,s)((1-\lambda_*)\delta x_*(s) + \lambda_* f^*(s))ds, \quad t \in I, \tag{9}$$

so,

$$\begin{cases} x_*''(t) = (1-\lambda_*)\delta x_*(s) + \lambda_* f^*(s) \quad t \in I, \\ x_*(0) = x_*(1) = 0, \end{cases} \tag{10}$$

If $||x_*||_2 \leq N$, then by (9)

$$||x_*(t)|| \leq \delta(1-\lambda_*)||x_*||_2 + \lambda_*||f^*(t)||_2 \leq \delta(1-\lambda_*)N + \lambda_* a(1+N^b) < R, \quad t \in I.$$

Hence $x_* \notin \partial B_{\mathscr{C}}(0,R)$, which is a contradiction.

Therefore $||x_*||_2 > N$ and from (10) it follows that

$$\langle x_*'', x_*\rangle = \delta(1-\lambda_*)\langle x_*, x_*\rangle + \lambda_*\langle x_*, f^*\rangle > 0,$$

which leads to a contradiction again.

Thus, φ is a homotopy joining δjoi and joQ, where $jf = \int_0^1 G(t,s)f(s)ds$. By the homotopic invariance property of topological degree, we have

$$deg(i - joQ, B_{\mathscr{C}}(0,R)) = deg(i - \delta joi, B_{\mathscr{C}}(0,R)).$$

If $\delta > 0$ is sufficiently small, then

$$||x-(x-\delta\, joi)||_{\mathscr{C}} = \delta||joi||_{\mathscr{C}} < ||x||_{\mathscr{C}}$$

for all $x \in \partial B_{\mathscr{C}}(0,R)$.

Therefore, the vector field i and $i-\delta\, joi$ are homotopic on $\partial B_{\mathscr{C}}(0,R)$ and

$$deg(i-A, B_{\mathscr{C}}(0,R)) = deg(i-\delta F, B_{\mathscr{C}}(0,R)) = deg(i, B_{\mathscr{C}}(0,R)) = 1.$$

Problem (8) thus has a solution in $B_{\mathscr{C}}(0,R)$.

■

5. Feedback Control Systems

As an application of Theorem 1, we consider a feedback control system of the form

$$\begin{cases} x''(t) = f(t,x(t),u(t)) \quad t \in I, \\ u(t) \in U(t,x(t)) \quad t \in I \\ x(0) = x(1) = 0, \end{cases} \tag{11}$$

where X, X_1 are separable Banach spaces, the map $f : I \times X \times X_1 \to X$ and the feedback multimap $U : I \times X_1 \to K(X_1)$ satisfy the following conditions:

- $(f1)$ the function $f(.,x,u) : I \times X \times X_1 \to X$ is measurable for any $(x,u) \in X \times X_1$,
- $(f2)$ $||f(t,x_1,u) - f(t,x_0,u)|| \leq k(t)||x_l - x_0||$ for any $x_0, x_1 \in X, u \in X_1$ where $k \in L^1(I)$,
- $(f3)$ the map $f(t,.,.) : X \times X_1 \to X$ is continuous for a.e. $t \in I$,
- $(U1)$ the multifunction $U(.,x) : I \to K(X_1)$ is measurable for every $x \in X$,
- $(U2)$ the multimap $U(t,.) : X \to K(X_1)$ is u.s.c. for a.e. $t \in I$,
- $(U3)$ the multimap $U : I \times X \to K(X_1)$ is superpositionally measurable, i.e., for every measurable multifunction $Q : I \to K(X)$, the multifunction $\phi : I \to P(X_1)$ with $\varphi(t) = U(t,Q(t))$ is measurable (see Proposition 1.3.1 in [15]),
- $(U4)$ the set $F(t,x) = f(t,x,U(t,x))$ is convex for all $(t,x) \in I \times X$,
- $(U5)$ the multimap F satisfies the boundedness condition $(F3)$,
- $(U6)$ for every $(t,x) \in I \times X$ the set $f(t,x,U(t,D))$ is relatively compact for any bounded set $D \subset X$.

In what follows, we need the following assertion.

Lemma 3. (See Proposition 2.2.2 of [15].) Let X_0, X_1 be Banach spaces, β_0, β_1 Hausdorff MNC in X_0 and X_1 respectively, $X \subset X_0$ and $\mathscr{M}$ a collection of bounded subsets of X. Suppose that the multimap $B : X \times X_0 \to K(X_1)$ satisfies the following conditions:

(i) for any $x \in X$ the multimap $B(x,.) : X_0 \to K(X_1)$ is k-Lipschitz with respect to the Hausdorff metric h on $K(X1)$, i.e.,

$$h(B(x,y_0), B(x,y_1)) \leq k||y_0 - y_1||,$$

for any $y_0, y_1 \in X_0$ where $k \in \mathbb{R}^+$ does not depend on x,

(ii) the set $B(\Omega \times \{y\})$ is relatively compact in X_1 for any $\Omega \in \mathscr{M}$ and $y \in X_0$.

Then the multimap $A : X \to K(X_1)$ defined as $A(x) = B(x,x)$ is (k, β_0, β_1)-bounded on $\mathscr{M}$, i.e.,

$$\beta_1(A(\Omega) \leq k\beta_0(\Omega),$$

for any $\Omega \in \mathscr{M}$.

Lemma 4. (See Theorem 1.3.4. of [15].) If a multimap $F : I \times X \to K(X)$ satisfies $(F1)$ and $(F2)$, then F is superpositionally measurable.

Theorem 4. Let conditions $(f1) - (f3)$ and $(U1) - (U6)$ hold. Then the feedback control system (11) has a solution.

Proof. We prove that the multimap $F(t,x) = f(t,x,U(t,x))$ defined on $I \times X$ besides condition $(F3)$ satisfies conditions $(F0), (F1), (F2)$ and $(F4)$. By conditions $(f3)$ and $(U4)$, F takes its values in $Kv(X)$, so it satisfies $(F0)$. From conditions $(f1), (f3), (U1)$ and Lemma 4, the multimap $F(.,x)$ is measurable for every $x \in X$, hence condition $(F1)$ is fulfilled. From assumptions $(f3), (U2)$ and Theorem 1.2.8 of [15], condition $(F2)$ follows. Now fix $t \in I$ and consider the multimap $B : X \times X \to K(X)$ defined as $B(x,y) = f(t,y,U(t,x))$. From $(U6)$ the set $B(D,y)$ is relatively compact for every $y \in X$ and bounded $D \subset X$. Fix $x \in X$ and let $y', y'' \in X$. If $b' \in B(x,y')$ then $b' = f(t,y',u)$, where $u \in U(t,x)$. Consider $b'' = f(t,y'',u) \in B(x,y'')$. From condition $(f2)$ it follows that

$$||b'' - b'|| \leq k(t)||y'' - y'||$$

i.e, the multimap $B(x,.)$ is $k(t)$-Lipschitz with respect to the Hausdorff metric on $K(X)$. Applying Lemma 3 to the multimap B, we confirm that condition $(F4)$ is satisfied for the multimap $F(t,x) = B(x,x)$. From Theorem 1, the solution set of

$$\begin{cases} x''(t) \in F(t,x) \quad t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{12}$$

is nonempty. Now using condition $(U3)$ and applying the Filippov Implicit Function Lemma (Theorem 1.3.3 of [15]), we conclude that for every solution function $x : I \to X$ with $x''(t) \in F(t,x)$, there exists a function $u(t) \in U(t,x)$ such that $x''(t) = f(t,x,u)$ and so x is a trajectory of the system (11). ■

References

[1] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina, B. N. Sadovskii, Measures of noncompactness and condensing operators, Operator Theory, *Advances and Applications*, **55**, (1992).

[2] J. P. Aubin and H. Frankowska, Set-Valued Analysis, *Birkhauser, Boston*, (1990).

[3] M. Benchohra, S. K. Ntouyas, A. Ouahab, A note on a nonlinear m-point boundary value problem for p-Laplacian differential inclusions, *Miskolc Math. Notes*, **1**, (2005), 19–26.

[4] A. Bressan, G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.*, **90**, (1988), 69–86.

[5] Z. Cai, L. Huang, Functional differential inclusions and dynamic behaviors for memristor-based BAM neural networks with time-varying delays, *Commun Nonlinear Sci Numer Simulat*. **19**, (2014), 1279–1300.

[6] F. S. De Blasi, L. Górniewicz, G. Pianigiani, Topological degree and periodic solutions of differential inclusions, *Nonlinear Anal.* **37**, (1999), 217–245.

[7] L. Erbe, R. Ma and C. C. Tisdell, On two point boundary value problems for second order differential inclusions, *Dynamic Systems and Applications*, **15**(1), (2006), 79–88.

[8] H. Frankowska, A priori estimates for operational differential inclusions, *J. Diff. Eqns*. **84**, (1990), 100–128.

[9] M. Frigon, Application de la Théorie de la Transversalité Topologique à des Problémes non Linéaires pour des Équations Différentielles Ordinaires, *Dissertationes Mathematicae Warszawa*, **CCXCVI**, (1990).

[10] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, *second edition, Springer, Dordrecht*, (2006).

[11] C. S. Goodrich, Positive solutions to differential inclusions with nonlocal, nonlinear boundary conditions, *Applied Mathematics and Computation*. **219**, (2013), 11071-11081.

[12] D. Guo, V. Lakshmikantham, Multiple solutions of two-point boundary value problems of ordinary differential equations in Banach spaces, *J. Math. Anal. Appl.*, **129**, (1988), 211–222.

[13] S. Hu, N. S. Papageorgiou, On the existence of periodic solutions for nonconvex valued differential inclusions in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* **123**, (1995), 3043–3050.

[14] S. Hu, N. S. Papageorgiou, Periodic solutions for nonconvex differential inclusions, *Proc. Amer. Math. Soc.* **127**, (1999), 89–94.

[15] M. Kamenskii, V. Obukhovskii, and P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, *de Gruyter Series in Nonlinear Analysis and Applications*, **7**, (2001).

[16] V. I. Korobov, Reduction of a controllability problem to a boundary value problem, *Different. Uranen*, **12**, (1976), 1310–1312.

[17] G. Li, X. Xue, On the existence of periodic solutions for differential inclusions, *J. Math. Anal. Appl.* **276**, (2002), 168–183.

[18] N. V. Loi, V. Obukhovskii, On global bifurcation of periodic solutions for functional differential inclusions, *Funct. Diff. Equat*, **17**(1-2), (2010), 157–168.

[19] N. V. Loi, Global behaviour of solutions to a class of feedback control systems, *Research and Communications in Mathematics and Mathematical Sciences*, **2**, (2013), 77–93.

[20] N. V. Loi, V. Obukhovskii, On the existence of solutions for a class of second-order differential inclusions and applications, *J. Math. Anal. Appl.* **385**, (2012), 517-533.

[21] D. O'Regan, Y. J. Cho, Y. Q. Chen, Topological degree theory and applications, *Serries in Mathematical Analysis and Applications*, **10**, 2006.

[22] V. Obukhovskii, P. Zecca, On boundary value problems for degenerate differential inclusions in Banach spaces, *Abstract and Applied Analysis*, **13**, (2003), 769–784.

[23] H. K. Pathak, R. P. Agarwal, Y. J. Chod, Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions, *Journal of Computational and Applied Mathematics*, **283**, (2015), 201–217.

[24] C. Ravichandran, D. Baleanu, Existence results for fractional neutral functional integro-differential evolution equations with infinite delay in Banach spaces, *Advances in Difference Equations*, (2013), 2013–215.

[25] G. V. Smirnov, Introduction to the Theory of Differential Inclusions, *Graduate Studies in Mathematics, American Mathematical Society, Providence*, (2002).

[26] S. Qin, X. Xue, Periodic solutions for nonlinear differential inclusions with multivalued perturbations, *J. Math. Anal. Appl*.**424**, (2015), 988–1005.

[27] J.-Z. Xiao, Y.-H. Cang, Q.-F. Liu, Existence of solutions for a class of boundary value problems of semilinear differential inclusions, *Mathematical and Computer Modelling*, **57**, (2013), 671–683.