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Approximate Analytical Solutions of the Damped Burgers and Boussinesq-Burgers Equations

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Abstract: In this paper, the Homotopy Analysis Method (HAM) is applied to the damped Burgers and Boussinesq-Burgers equations to obtain their approximate analytical solutions. The HAM solution includes an auxiliary parameter $\hbar$ which provides a convenient way to adjust and control the convergence region of the solution series. An appropriate choice of the auxiliary parameter in the model problems for increasing time is investigated.

Keywords: Homotopy analysis method, approximate analytical solution, damped Burgers equation, Boussinesq-Burgers equation.

1. Introduction

Mathematical modeling of many physical phenomena in various fields of physics and engineering generally leads to nonlinear ordinary or partial differential equations. It is known that investigating and constructing exact and numerical solutions of these equations are of great importance in applied mathematics. The HAM, which was first proposed by Liao [1, 2], is a powerful tool to find the approximate solutions of nonlinear evolution equations (NLEEs). Unlike perturbation techniques, the HAM is not limited to any small physical parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a powerful tool to analyze strongly nonlinear problems[3]. This method has been successfully applied in order to solve several nonlinear problems arising in science and engineering by many authors [1-17] and the references therein. In this paper, we will apply the HAM to the damped Burgers and Boussinesq-Burgers equations.

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2. Fundamentals of the HAM

In this manuscript, HAM has been applied to the problem discussed. In order to provide fundamentals of the method, let us consider the following differential equation

$$\mathcal{N}\left[u(x,t)\right]=0,$$

where $\mathcal{N}$ is a nonlinear operator, x and t denote independent variables, $u(x,t)$ is an unknown function, respectively. By generalizing the HAM, Liao [1, 2] has constructed the so-called zero-order deformation equation

$$(1-p)\mathcal{L}\left[\phi(x,t;p)-u_0(x,t)\right]=p\hbar\mathcal{N}\left[\phi(x,t;p)\right], \tag{1}$$

where $p\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear parameter, $u_0(x,t)$ is an initial guess of $u(x,t)$, $\phi(x,t;p)$ is an unknown function, respectively. One must note that one has great freedom to select auxiliary items in HAM. Obviously, if we choose $p=0$ and $p=1$ then we obtain

$$\phi(x,t;0)=u_0(x,t),\phi(x,t;1)=u(x,t),$$

respectively. Therefore, as the embedding parameter p increases from 0 to 1, the solutions $\phi(x,t;p)$ vary from the initial value $u_0(x,t)$ to the solution $u(x,t)$. If we expand $\phi(x,t;p)$ in the Taylor series with respect to the embedding parameter p, we get

$$\phi(x,t;p)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)p^m,$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{2}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are chosen in a proper way, the above series converges at $p=1$, and we obtain

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t),$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [2, 5]. According to (2), the governing equation can be reduced from the zero-order deformation equation (1). Define the vector

$$\mathbf{u}_n=\left\{u_0(x,t),u_1(x,t),...,u_n(x,t)\right\}.$$

If we differentiate Eq. (1) m times with respect to the embedding parameter p and then get $p=0$ and divide by $m!$, we obtain the mth-order deformation equation

$$\mathcal{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar R_m\left(\mathbf{u}_{m-1}\right), \tag{3}$$

where

$$R_m(\mathbf{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Finally, we emphasize that $u_m(x,t)$ for $m \ge 1$ is governed by the Eq. (3) with the boundary condition coming from the problem. It can be easily solved using symbolic computation software such as Mathematica.

3. Applications of the HAM

3.1. The damped Burgers equation. We first consider the damped Burgers equation [18-22] which describes the plane motion of a continuous medium, in the following form

$$u_t + uu_x - u_{xx} + \lambda u = 0, \tag{4}$$

with initial condition

$$u(x,0) = \lambda x. \tag{5}$$

To investigate the series solution of Eq. (4) with initial condition (5), we choose the linear operator

$$\mathscr{L}[\phi(x,t;p)] = \frac{\partial\phi(x,t;p)}{\partial t},$$

with the property

$$\mathscr{L}[c] = 0,$$

where c is constant. From Eq. (4), we now define a nonlinear operator as

$$\mathscr{N}[\phi(x,t;p)] = \frac{\partial\phi(x,t;p)}{\partial t} + \phi(x,t;p)\frac{\partial\phi(x,t;p)}{\partial x} - \frac{\partial^2\phi(x,t;p)}{\partial x^2} + \lambda\phi(x,t;p).$$

Therefore, we construct the zero-order deformation equation as

$$(1-p)\mathscr{L}[\phi(x,t;p) - u_0(x,t)] = p\hbar\mathscr{N}[\phi(x,t;p)]. \tag{6}$$

Obviously, if we choose $p = 0$ and $p = 1$ then we obtain

$$\phi(x,t;0) = u_0(x,t) = u(x,0), \phi(x,t;1) = u(x,t),$$

respectively. Thus, as the embedding parameter p increases from 0 to 1, the solutions $\phi(x,t;p)$ vary from the initial value $u_0(x,t)$ to the solution $u(x,t)$. By expanding $\phi(x,t;p)$ in the Taylor series with respect to the embedding parameter p, we get

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m, \tag{7}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{8}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are properly chosen, the above series converges at $p = 1$, and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [2, 5]. By differentiating Eq. (6) m times with respect to the embedding parameter p, we obtain the mth-order deformation equation

$$\mathscr{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar R_m\left(\mathbf{u}_{m-1}\right), \tag{9}$$

where

$$R_m\left(\mathbf{u}_{m-1}\right) = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2} + \lambda u_{m-1} + \sum_{n=0}^{m-1} u_n(x,t)\frac{\partial u_{m-1-n}}{\partial x}.$$

The solution of the mth-order deformation Eq. (9) for $m \geq 1$ leads to

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\mathscr{L}^{-1}\left[R_m\left(\mathbf{u}_{m-1}\right)\right]. \tag{10}$$

By using Eq.(10) with initial condition given by (5) we successively obtain

$$u_0(x,t) = 2\hbar t x\lambda^2,$$

$$u_1(x,t) = 2\hbar t x\lambda^2 + 2\hbar^2 t x\lambda^2 + 3\hbar^2 t^2 x\lambda^3,$$

$$u_2(x,t) = \frac{1}{3}\left(6\hbar t x\lambda^2 + 12\hbar^2 t x\lambda^2 + 6\hbar^3 t x\lambda^2 + 18\hbar^2 t^2 x\lambda^3 + 18\hbar^3 t^2 x\lambda^3 + 13\hbar^3 t^3 x\lambda^4\right),$$

$$u_3(x,t) = \frac{1}{4}\left(8htx\lambda^2 + 24h^2tx\lambda^2 + 24h^3tx\lambda^2 + 8h^4tx\lambda^2 + 36h^2t^2x\lambda^3 + 72h^3t^2x\lambda^3 + 36h^4t^2x\lambda^3 + 52h^3t^3x\lambda^4 + 52h^4t^3x\lambda^4 + 25h^4t^4x\lambda^5\right)$$

$$\vdots$$

Therefore, the series solutions expressed by HAM can be written in the form

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots. \tag{11}$$

To demonstrate the efficiency of the method, we compare the HAM solutions of Burgers equation with the/a linear damping term given by Eq. (4) with exact solutions [20]

$$u(x,t) = \frac{\lambda x}{2\exp(\lambda t) - 1}. \tag{12}$$

Note that our HAM solution series contains the auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence of the solution series. To obtain an appropriate range for $\hbar$, we consider the so-called $\hbar$-curve to choose a proper value of $\hbar$ which ensures that the solution series is convergent, as pointed by Liao [2], by discovering the valid region of $\hbar$, which corresponds to the line segments nearly parallel to the horizontal axis.

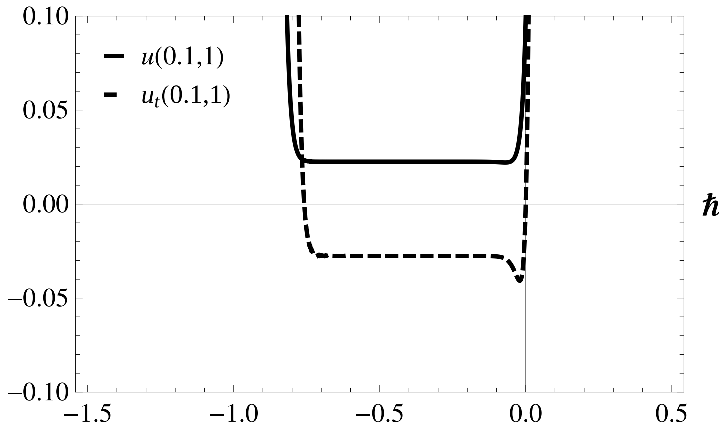

FIGURE 1. The $\hbar$-curves of 30th-order approximate solution obtained by the HAM for $\lambda = 1.0$.

In Fig.1, we demonstrate the $\hbar$-curves of $u(0.1,1)$ and $u_t(0.1,1)$ given by 30th-order HAM solution (11) for $\lambda = 1.0$. It can be seen from the figure that the valid range of $\hbar$ is approximately $-0.7 \leq \hbar \leq -0.1$.

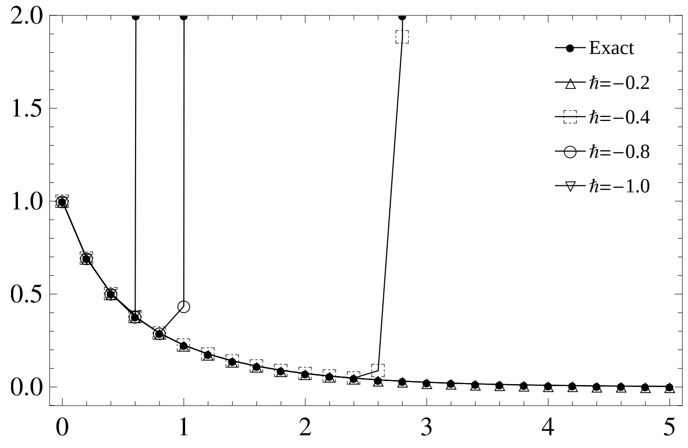

FIGURE 2. The results obtained by the HAM for various $\hbar$ by 30th-order approximate solution, in comparison with the exact solution at $x = 1$ for $\lambda = 1.0$.

Fig. 2 shows the numerical solutions of $u(x,t)$ at $x = 1$ during $0 \leq t \leq 5$ for $\hbar = -0.2$, -0.4, -0.8 and -1.0 obtained by 30th-order HAM and analytical solutions, respectively. Between $t = 0$ and $t = 5$, it can be seen from this figure that the choice of $\hbar = -0.2$ is a suitable value.

3.2. The Boussinesq-Burgers equation. We secondly consider the Boussinesq-Burgers equation [23-25]

$$\begin{aligned} & u_t - \tfrac{1}{2}v_x + 2uu_x = 0, \\ & v_t - \tfrac{1}{2}u_{xxx} + 2(uv)_x = 0, \end{aligned} \tag{13}$$

with initial conditions

$$\begin{aligned} u(x,0) &= -\tfrac{w}{2k} + \tfrac{k}{2}\tanh(kx), \\ v(x,0) &= -\tfrac{k^2}{2}\operatorname{sech}^2(kx), \end{aligned} \tag{14}$$

where k and w are arbitrary constants. To investigate the series solution of Eq. (13) with initial condition (14), we choose the linear operator

$$\mathscr{L}[\phi_i(x,t;p)] = \frac{\partial \phi_i(x,t;p)}{\partial t}, i = 1,2$$

with the property

$$\mathscr{L}[c_i] = 0,$$

where c_i $(i = 1,2)$ are integral constants. From (13), we now define a system of nonlinear operators as

$$\mathscr{N}_1[\phi_1(x,t;p),\phi_2(x,t;p)] = \frac{\partial \phi_1(x,t;p)}{\partial t} - \frac{1}{2}\frac{\partial \phi_2(x,t;p)}{\partial x} + 2\phi_1(x,t;p)\frac{\partial \phi_1(x,t;p)}{\partial x},$$

$$\mathscr{N}_2[\phi_1(x,t;p),\phi_2(x,t;p)] = \frac{\partial \phi_2(x,t;p)}{\partial t} - \frac{1}{2}\frac{\partial^3 \phi_1(x,t;p)}{\partial x^3} + 2\frac{\partial}{\partial x}(\phi_1(x,t;p)\phi_2(x,t;p)).$$

We construct the zero-order deformation equations

$$(1-p)\mathscr{L}[\phi_1(x,t;p) - u_0(x,t)] = p\hbar_1\mathscr{N}_1[\phi_1(x,t;p),\phi_2(x,t;p)], \tag{15}$$

$$(1-p)\mathscr{L}[\phi_2(x,t;p) - v_0(x,t)] = p\hbar_2\mathscr{N}_2[\phi_1(x,t;p),\phi_2(x,t;p)]. \tag{16}$$

Obviously, if we choose $p = 0$, then we obtain

$$\phi_1(x,t;0) = u_0(x,t) = u(x,0), \phi_2(x,t;0) = v_0(x,t) = v(x,0),$$

and when $p = 1$,

$$\phi_1(x,t;1) = u(x,t), \qquad \phi_2(x,t;1) = v(x,t).$$

Thus, as the embedding parameter p increases from 0 to 1, the solutions $\phi_i(x,t;p)$ (for $i = 1,2$) vary from the initial guess $u_0(x,t)$ and $v_0(x,t)$ to the solution $u(x,t)$ and $v(x,t)$, respectively. By expanding $\phi_i(x,t;p)$ for $i = 1,2$ in the Taylor series with respect to p, we get

$$\begin{aligned} \phi_1(x,t;p) &= u_0(x,t) + \textstyle\sum_{m=1}^{\infty} u_m(x,t)p^m, \\ \phi_2(x,t;p) &= v_0(x,t) + \textstyle\sum_{m=1}^{\infty} v_m(x,t)p^m, \end{aligned} \tag{17}$$

where

$$\begin{aligned} u_m(x,t) &= \tfrac{1}{m!}\tfrac{\partial^m \phi_1(x,t;p)}{\partial p^m}\Big|_{p=0}, \\ v_m(x,t) &= \tfrac{1}{m!}\tfrac{\partial^m \phi_2(x,t;p)}{\partial p^m}\Big|_{p=0}. \end{aligned} \tag{18}$$

When the auxiliary linear operator, the initial guess and the auxiliary parameters $\hbar_i$ are properly chosen, the above series converges at $p=1$, and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$$

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t),$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [2, 5]. By differentiating Eqs. (15) and (16) m-times with respect to the embedding parameter p, we obtain the mth-order deformation equations

$$\mathscr{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar_1 R_{1,m}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right), \tag{19}$$

$$\mathscr{L}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right] = \hbar_2 R_{2,m}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right), \tag{20}$$

where

$$R_{1,m}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right) = \frac{\partial u_{m-1}}{\partial t} - \frac{1}{2}\frac{\partial v_{m-1}}{\partial x} + 2\sum_{n=0}^{m-1} u_n(x,t)\frac{\partial u_{m-1-n}}{\partial x},$$

$$R_{2,m}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right) = \frac{\partial v_{m-1}}{\partial t} - \frac{1}{2}\frac{\partial^3 u_{m-1}}{\partial x^3} + 2\frac{\partial}{\partial x}\left(\sum_{n=0}^{m-1} u_n(x,t) v_{m-1-n}\right).$$

The solution of the mth-order deformation Eqs. (19) and (20) for $m \geq 1$ leads to

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar_1 \mathscr{L}^{-1}\left[R_{1,m}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right)\right], \tag{21}$$

$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar_2 \mathscr{L}^{-1}\left[R_{2,m}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right)\right]. \tag{22}$$

For simplicity, assuming that $\hbar_1 = \hbar_2 = \hbar$, by using Eqs.(21) and (22) with initial conditions given by (14) we successively obtain

$$\begin{aligned}
u_0(x,t) &= -\frac{w}{2k} + \frac{k}{2}\tanh(kx), \\
u_1(x,t) &= -\frac{1}{2}\hbar k t w \operatorname{sech}^2(kx), \\
u_2(x,t) &= -\frac{1}{2}\hbar k t w \operatorname{sech}^2(kx)\left(1 + \hbar + \hbar t w \tanh(kx)\right), \\
u_3(x,t) &= -\frac{1}{12}\hbar k t w \operatorname{sech}^4(kx)\left(3 + \hbar\left(6 + 3\hbar - 4\hbar t^2 w^2\right)\right. \\
&\left. + \left(3 + \hbar\left(6 + \hbar\left(3 + 2t^2 w^2\right)\right)\right)\cosh(2kx) + 6\hbar(1+\hbar) t w \sinh(2kx)\right) \\
&\vdots
\end{aligned}$$

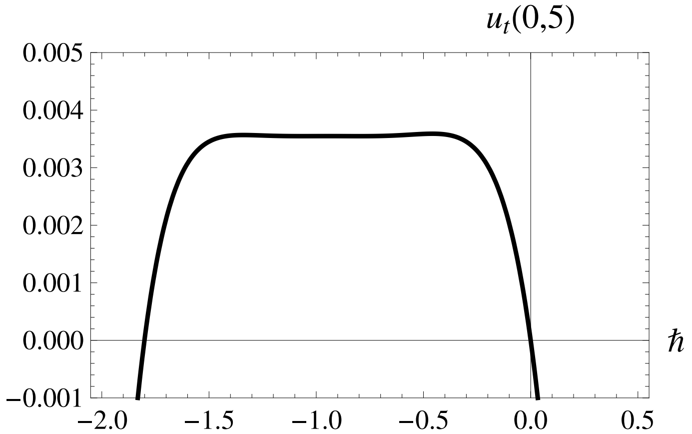

FIGURE 3. The $\hbar$-curve for 7th-order approximate solution of $u(x,t)$ obtained by the HAM.

and

$$\begin{aligned}
v_0(x,t) &= -\frac{k^2}{2}\operatorname{sech}^2(kx),\\
v_1(x,t) &= -\hbar k^2 tw \operatorname{sech}^2(kx)\tanh(kx),\\
v_2(x,t) &= -\frac{1}{2}\hbar k^2 tw \operatorname{sech}^4(kx)\left(\hbar tw(-2+\cosh(2kx)) + (1+\hbar)\sinh(2kx)\right),\\
v_3(x,t) &= -\frac{1}{6}\hbar k^2 tw \operatorname{sech}^4(kx)\left(6\hbar(1+h)tw\cosh(2kx)\right.\\
&\left.+\left(3+\hbar\left(6+3\hbar+2\hbar t^2w^2\right)\right)\sinh(2kx) - 12\hbar tw(1+\hbar+\hbar tw\tanh(kx))\right)\\
&\vdots
\end{aligned}$$

Therefore, the series solutions expressed by the HAM can be written in the form

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\ldots, \tag{23}$$

$$v(x,t) = v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\ldots. \tag{24}$$

To demonstrate the efficiency of the method, we compare the HAM solutions for Eq. (13) with exact solutions [25]

$$\begin{aligned}
u(x,t) &= -\tfrac{w}{2k}+\tfrac{k}{2}\tanh(kx+wt),\\
v(x,t) &= -\tfrac{k^2}{2}\operatorname{sech}^2(kx+wt).
\end{aligned} \tag{25}$$

For all calculations in the present paper, k and w are going to be taken as 1/8 and 1/16, respectively.

The $\hbar$-curves obtained based on the 7th-order HAM solutions for the Boussinesq-Burgers equation are presented in Figs. 3-4. As pointed above, the valid region of $\hbar$ is a nearly horizontal line

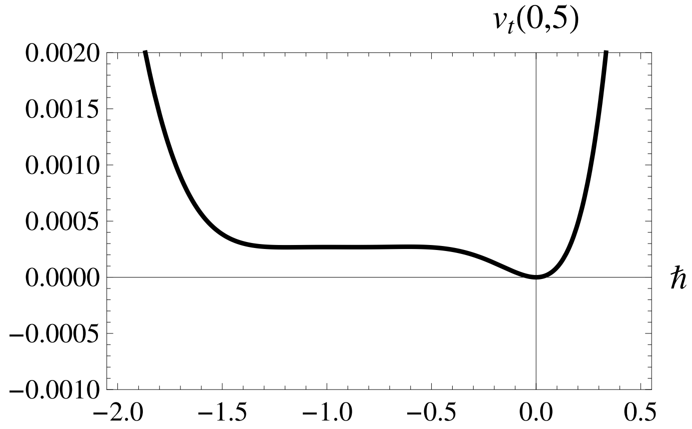

FIGURE 4. The $\hbar$-curve for 7th-order approximate solution of $v(x,t)$ obtained by the HAM.

segment. Therefore, it can be seen from the figure that the valid range of $\hbar$ is approximately $-1.5 \leq \hbar \leq -0.3$.

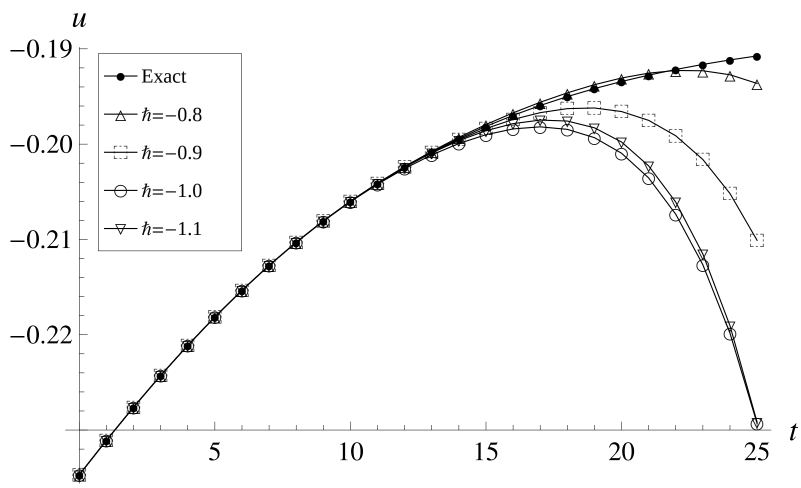

FIGURE 5. The results obtained by the HAM for various $\hbar$ by 7th-order approximate solution of $u(x,t)$, in comparison with the exact solution at $x = 2$.

Figs. 5-6 shows the numerical solutions of $u(x,t)$ and $v(x,t)$ at $x = 2$ during $0 \leq t \leq 25$ for $\hbar = -0.8$, -0.9, -1.0 and -1.1 obtained by the HAM and analytical solutions, respectively. Between $t = 0$ and $t = 25$, it can be seen from these figures that the choice of $\hbar = -0.8$ is a suitable value.

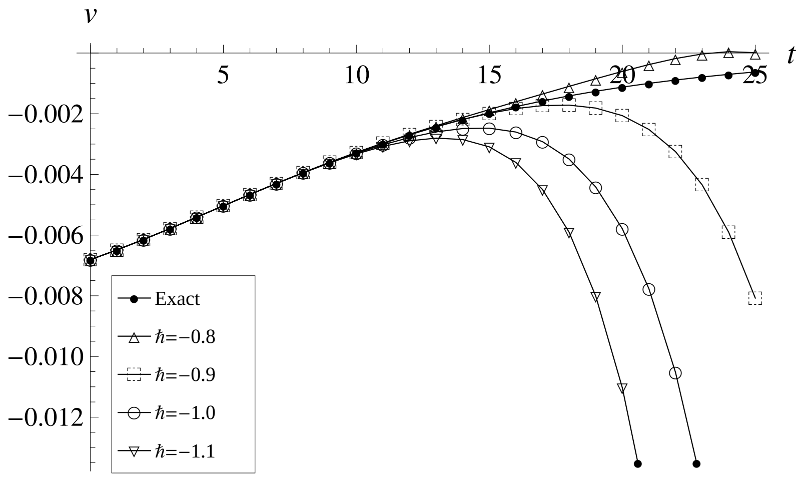

FIGURE 6. The results obtained by the HAM for various $\hbar$ by 7th-order approximate solution of $v(x,t)$, in comparison with the exact solution at $x=2$.

4. Conclusion

In this paper, the HAM has been successfully applied to obtain approximate analytical solution of the damped Burgers and Boussinesq-Burgers equations. It has been also seen that the HAM solution of the problem converges very rapidly to the exact one by choosing an appropriate auxiliary parameter $\hbar$. In conclusion, this study shows that the HAM is a powerful and efficient technique for finding the approximate analytical solution of the damped Burgers and Boussinesq-Burgers equations and also many other nonlinear evolution equations arising in science and engineering.

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